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ON THE EQUATION $\phi(\mathbf{N}+\mathbf{K}) = 2\phi(\mathbf{N})$

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Abstract

In this paper we prove various results (conditional and unconditional) about the solvability of the equation $\phi(n+k) = 2\phi(n)$.

1. Introduction

W. Sierpinski [9] proved that the equation $\phi(n+k) = \phi(n)$ has at least one solution for every fixed positive integer k, where $\phi(n)$ denotes Euler's phi-function. A. Makowski [5] proved the analogous result for the equation

$$\phi(n+k) = 2\phi(n). \tag{1}$$

The proof is easy: if $(k,6) = 1$, we can take $n = 2k$; if k is even, $k = 2^l u$ ($l \geq 1$, u odd), we may take $n = 2^l u$; if k is odd and $3|k$, $k = 6l+3$, take $n = 2l+1$.

In 1959, A. Schinzel and A. Wakulicz [8] showed that for each fixed $k \leq 2 \cdot 10^{58}$ there are at least two solutions to $\phi(n+k) = \phi(n)$. The next theorem establishes the result of the same spirit for Equation (1).

Theorem 1. *(a) For all integers $k \leq 4 \cdot 10^{58}$, Equation* (1) *has at least two solutions. (b) Moreover, for at least $15 \cdot 10^{55}$ odd integers $k \leq 4 \cdot 10^{58}$, Equation* (1) *has at least three solutions.*

Following K. Ford [3], Hypothesis $\mathcal{P}(a,b)$ will refer to the statement that there are infinitely many $n \in \mathbb{N}$ such that both $an+1$ and $bn+1$ are primes. Note that $\mathcal{P}(a,b)$ is just the particular case of Dickson's prime k-tuple conjecture.

There is an intimate connection between Hypothesis $\mathcal{P}(a,b)$ and the following conjecture for certain integers k.

Hypothesis $\mathbf{\Phi}_{\boldsymbol{k}}$. Equation (1) has infinitely many solutions $n \in \mathbb{N}$.

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If k is odd and divisible by 3, and both $2n+1$ and $3n+1$ are primes greater than k, then

$$\phi(3(2n+1)k) = \phi(2(3n+1)k+k) = 2\phi(2(3n+1)k).$$

Thus Hypothesis $\mathcal{P}(2,3)$ implies Φ_k for every $3|k$, k odd.

Our next lemma generalizes this observation.

Lemma 1. *Let k be an odd integer, divisible by 3. Suppose j and $2j+k$ have the same prime divisors (thus j is odd), $g=(j,2j+k)$ and for some positive integer r, $\frac{2j}{g}r+1$ and $\frac{2j+k}{g}r+1$ are primes that do not divide j. Then $n=2j(\frac{2j+k}{g}r+1)$ is a solution of (1). In other words, $\mathcal{P}\Big(\frac{2j}{(j,2j+k)}, \frac{2j+k}{(j,2j+k)}\Big)$ implies Φ_k for $3|k$, k odd.*

Proof. We get

$$\phi(n) = \phi\Big(2j\Big(\frac{2j+k}{g}r+1\Big)\Big) = \frac{2j+k}{g}r\phi(j)$$

and

$$\phi(n+k) = \phi\Big((2j+k)\Big(\frac{2j}{g}r+1\Big)\Big) = \frac{2j}{g}r\phi(2j+k).$$

Since by assumption j and $2j+k$ have the same prime factors, it follows that

$$\phi(j)(2j+k) = j\phi(2j+k),$$

which yields the desired result. □

In order to proceed further we need some terminology. We say that a collection of distinct linear forms

$$L_i(x) = a_i x + b_i,\ (1 \le i \le k),\ a_i, b_i \in \mathbb{Z},\ a_i > 0$$

is admissible if for every prime p there exists $x_p \in \mathbb{Z}$ such that $p \nmid \prod_{i=1}^{k} L_i(x_p)$. We say that $DHL^*(k;m)$ holds if, for any admissible set of k linear forms $(a_1n+b_1,...,a_kn+b_k)$ there exist distinct $i_1,...,i_m \in \{1,...,k\}$ such that there are infinitely many r with m numbers $a_{i_1}r+b_{i_1},...,a_{i_m}r+b_{i_m}$ simultaneously prime (see [3]).

Assuming the Elliot-Halberstam conjecture, Maynard [6] showed that $DHL^*(5;2)$. A generalized version of the Elliot-Halberstam conjecture implies $DHL^*(3;2)$ holds.

Theorem 2. *(a) If $DHL^*(5;2)$ holds then Φ_k is true for all odd k with $3255|k$.*

(b) If $DHL^*(4;2)$ holds then Φ_k is true for all odd k with $105|k$.

(c) If $DHL^*(3;2)$ holds then Φ_k is true for all odd k with $21|k$.

2. Proofs

Proof of Theorem 1. (a) We consider two cases: for even values of k and for odd values of k.

For even values of k, we will apply a modified version of Schinzel's argument [7]. Let $k = 2^l r$, r is odd, $l \geq 1$. We use the notation $a|^*b$ to indicate that each prime divisor of a is a prime divisor of b. Assume that the sequence of primes $3 = p_1 < p_2 < ... < p_m$ satisfies the conditions

1. $(p_i - 2)|p_1p_2...p_{i-1}$, $(2 \leq i \leq m)$,
2. $(p_i - 1)|^*2p_1p_2...p_{i-1}$, $(2 \leq i \leq m)$.

Suppose that $p_1p_2...p_m \nmid r$ and let p_j be the smallest prime in the sequence that does not divide r. Then $(r, p_j) = (p_j, 2) = 1$ and either $p_j = 3$ or $p_1p_2...p_{j-1}|r$ $(j \geq 2)$. From (1), (2), $p_j - 2|r$ and $p_j - 1|^*2r$. Since $(p_j - 1, p_j - 2) = 1$, then $p_j - 1|^*\frac{2r}{p_j-2}$ and

$$\phi\Big(\frac{2^{l+1}r}{p_j - 2}(p_j - 1)\Big) = (p_j - 1)\phi\Big(\frac{2^{l+1}r}{p_j - 2}\Big) = 2^l\phi(p_j)\phi\Big(\frac{r}{p_j - 2}\Big) = 2\phi\Big(\frac{2^l p_j r}{p_j - 2}\Big).$$

As $\frac{p_j}{p_j-2} > 1$, $n = \frac{2^l r p_j}{p_j-2}$ is a solution of (1) distinct from that given by Makowski's result. The sequence of primes 3, 5, 7, 17, 19, 37, 97, 113, 257, 401, 487, 631, 971, 1297, 1801, 19457, 22051, 28817, 65537, 157303, 160001 satisfies the required conditions and $3 \cdot 5 \cdot 7 \cdots 160001 > 2 \cdot 10^{58}$. This argument works unless r is a multiple of $p_1 \cdots p_m$, and so for all $r < 2 \cdot 10^{58}$.

Let k be an odd integer. Observe that if $F_n = 2^{2^n} + 1$ is the Fermat's prime number such that $(2^{2^n} + 1, k) = 1$, then $x = 2^{2^n}k$ is a new solution of the equation $\phi(x+k) = 2\phi(x)$. There are only five known Fermat's primes: $F_0 = 3, F_1 = 5, F_2 = 17, F_3 = 257, F_4 = 65537$. Now every odd integer k is either coprime to all F_n's or divisible by exactly m distinct F_n's $(0 \leq n \leq 4,\ 1 \leq m \leq 5)$. Clearly this defines the partition of all odd integers. If $(k, F_0F_1F_2F_3F_4) = 1$, then we get four new solutions: $x_j = 2^{2^j}k$, $1 \leq j \leq 4$. Otherwise, suppose $i_1, i_2, ..., i_m \in \{0, 1, 2, 3, 4\}$ are distinct such that $F_{i_1}F_{i_2}...F_{i_m}|k$. Choose $i_t \neq 0$, $i_t \notin \{i_1, i_2, ..., i_m\}$. This gives a solution $x = 2^{2^{i_t}}k$ in addition to Makowski's result. There are two exceptional cases when such a choice does not yield a new solution of this form. If $5 \cdot 17 \cdot 257 \cdot 65537|k$, then choose $l \in \mathbb{N}$ such that $4l + 1$ and $5l + 1$ are both primes relatively prime to k. Then $x = 4k(5l + 1)$ is a new solution. If $3 \cdot 5 \cdot 17 \cdot 257 \cdot 65537|k$, then choose $m \in \mathbb{N}$ such that $2m + 1$ and $3m + 1$ are both primes relatively prime to k. Then $x = 2k(3m + 1)$ is an additional solution.

Maple computations show that we can take, for example,

$$l = 2^3 \cdot 3^2 \cdot 5^2 \cdot 229 \cdot 13643373661321 \cdot 1214436755047 \cdot 1464182919280833707$$
$$m = 2 \cdot 7 \cdot 17 \cdot 37 \cdot 148140750636881 \cdot 260630774746636705884081906766 9.$$

For such choice of l and m, we have $5 \cdot 17 \cdot 257 \cdot 65537 \cdot (5l+1) > 4 \cdot 10^{58}$ and $3 \cdot 5 \cdot 17 \cdot 257 \cdot 65537 \cdot (3m+1) > 4 \cdot 10^{58}$. Hence for all odd $k \le 4 \cdot 10^{58}$, Equation (1) has at least two solutions.

(b) If k is odd, $k \cdot 2^n + 1$ is prime for some $n \ge 1$, then

$$\phi(k^2 \cdot 2^n) = 2^{n-1}\phi(k^2) = 2^{n-1}k\phi(k),$$
$$\phi(k^2 \cdot 2^n + k) = \phi(k(k \cdot 2^n + 1)) = 2^n k\phi(k).$$

Let $p \equiv 3 \pmod 4$ and $k = \frac{p-1}{2}$. Then $k \cdot 2 + 1$ is prime. If $p \equiv 13 \pmod{280}$ and $k = \frac{p-1}{4}$, then $k \cdot 2^2 + 1$ is prime while $k \cdot 2 + 1$ is composite (see [4]). Using explicit bounds for $\pi(x; q, a)$ (Bennett et al. [1]) we get

$$\pi(4 \cdot 10^{58}; 4, 3) > \frac{4 \cdot 10^{58}}{\phi(4)\log(4 \cdot 10^{58})},$$
$$\pi(4 \cdot 10^{58}; 280, 13) > \frac{4 \cdot 10^{58}}{\phi(280)\log(4 \cdot 10^{58})}.$$

From the above estimates we infer that for at least $\frac{(2+\frac{1}{24}) \cdot 10^{58}}{\log(4 \cdot 10^{58})} > 15 \cdot 10^{55}$ of all odd $k \le 4 \cdot 10^{58}$, Equation (1) has at least three solutions. □

Remark. Note that if the hypotheses $\mathcal{P}(2,3)$ and $\mathcal{P}(4,5)$ are true, then for a positive proportion of all odd $k \in \mathbb{N}$, Equation (1) has at least three solutions. The result follows from the theorem of Erdös and Odlyzko [2]: there exist a positive, effectively computable constant c_1 such that if $N(x)$ is the number of odd positive integers $k \le x$ such that $k \cdot 2^n + 1$ is prime for some positive n, then $N(x) \ge c_1 x$ for $x \ge 1$.

Let $a < b$ be integers such that

$$a = 2^l u,\ b = 2^s t,\ l > s, s \ge 0,\ u, t \text{ odd}. \tag{2}$$

Define

$$a_1 = \frac{a}{2^{k_1}(a,b)},\ b_1 = \frac{b}{(a,b)},\ k_1 = l - s,\ \kappa(a,b) = (b_1 - 2^{k_1}a_1)\prod_{p \mid a_1 b_1} p. \tag{3}$$

The next lemma is a modified version of [3, Lemma 3].

Lemma 2. *Asssume $\mathcal{P}(a,b)$, where a, b are of the form (2). Then Φ_k holds for every $k = v\kappa(a,b)$, v odd.*

Proof. Observe that $\mathcal{P}(a,b) \Rightarrow \mathcal{P}(2^{k_1}a_1, b_1)$. Let $s_1 = \prod_{p \mid a_1 b_1} p$ and suppose $r > max(2^{k_1}a_1, b_1)$ such that $2^{k_1}a_1 r + 1$ and $b_1 r + 1$ are both primes. Let $v \in \mathbb{N}$ be odd. Note that we can choose r so large that $(2^{k_1}a_1 r + 1, v) = (b_1 r + 1, v) = 1$. Set

$$m_1 = b_1 v s_1(2^{k_1}a_1 r + 1),\ m_2 = 2^{k_1}a_1 v s_1(b_1 r + 1).$$

Then

$$\phi(m_1) = \phi(b_1 v s_1(2^{k_1} a_1 r + 1)) = 2^{k_1} a_1 r b_1 \phi(v s_1),$$
$$\phi(m_2) = \phi(2^{k_1} a_1 v s_1(b_1 r + 1)) = 2^{k_1 - 1} b_1 r a_1 \phi(v s_1).$$

It follows that $\phi(m_1) = 2\phi(m_2)$ and $m_1 - m_2 = (b_1 - 2^{k_1} a_1) v s_1 = v\kappa(a, b)$. □

Proof of Theorem 1.3. (a) Let $\{a_1, a_2, a_3, a_4, a_5\} = \{2^4, 2^3 \cdot 3, 2^2 \cdot 7, 2 \cdot 3 \cdot 5, 31\}$. If $DHL^*(5; 2)$ holds, then for some i, j with $1 \le i, j \le 5$, $\mathcal{P}(a_i, a_j)$ is true. We compute

$$lcm\{\kappa(a_i, a_j) : 1 \le i, j \le 5\} = 3 \cdot 5 \cdot 7 \cdot 31.$$

Thus by Lemma 2, Φ_k is true for all odd multiples of 3255.

(b) This is the same as the proof of (a), but take $\{a_1, a_2, a_3, a_4\} = \{2^3, 2^2 \cdot 3, 2 \cdot 7, 3 \cdot 5\}$ if $DHL^*(4; 2)$ holds.

(c) Take $\{a_1, a_2, a_3\} = \{2^2, 2 \cdot 3, 7\}$ if $DHL^*(3; 2)$ holds. □

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