BALANCING AND LUCAS-BALANCING NUMBERS EXPRESSIBLE AS SUMS OF TWO REPDIGITS

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Abstract

Repdigits are natural numbers formed by the repetition of a single digit. In this paper, we find all numbers in the balancing and Lucas-balancing sequence which are expressible as sums of two repdigits.

1. Introduction

Behera and Panda [4] defined balancing numbers n as solutions of the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r),$$

where r is called the balancer corresponding to the balancing number n. The definition of balancing numbers coincides with the definition of numerical centers defined in [8]. The sequence of balancing numbers is denoted by $\{B_n\}_{n\geq 1}$ and satisfy the binary recurrence $B_{n+1} = 6B_n - B_{n-1},\ n \geq 1$, with initial terms $B_0 = 0, B_1 = 1$. For every balancing number B_n, $8B_n^2 + 1$ is always a square and its positive square root is called the Lucas-balancing number denoted by C_n [20]. The sequence of Lucas-balancing numbers $\{C_n\}_{n\geq 1}$ satisfies a similar recurrence as that of the balancing numbers with initial terms $C_0 = 1, C_1 = 3$. Further, the n^{th} balancing and Lucas-balancing number can be obtained using their respective Binet formulas given by

$$B_n = \frac{\alpha^n - \beta^n}{4\sqrt{2}},\ C_n = \frac{\alpha^n + \beta^n}{2}\ ; \quad n \geq 1, \tag{1}$$

where $\alpha = 3+\sqrt{8}$ and $\beta = 3-\sqrt{8}$ are roots of the characteristic equation $x^2-6x+1 = 0$.

For any given positive integer $g \geq 2$, a natural number N of the form

$$N = a\left(\frac{g^m-1}{g-1}\right), \quad \text{for some } m \geq 1, \text{ where } a \in \{1, 2, \cdots, g-1\}$$

is called a *base g repdigit* and when $g = 10$, N is simply called a repdigit. In recent years, there have been several studies concerning Diophantine equations involving repdigits, terms of binary recurrence sequences, their sums and their products. The identification of repdigits in Fibonacci, Lucas, Pell and Pell-Lucas sequences has been studied in [7, 11]. Further, the authors investigated the existence of repdigits in balancing and Lucas-balancing sequences [21]. Similar studies have been carried out by replacing Fibonacci, Lucas, balancing and Lucas-balancing numbers by their respective consecutive products (see [9, 14, 21]). In [22], the authors studied the existance of repdigits that are expressible as products of balancing and Lucas-balancing numbers with their indices in arithmetic progressions. Fibonacci, Lucas, Pell and Pell-Lucas numbers which are expressible as sums of two repdigits have been studied in [1, 2, 3]. Repdigits which are sums of three Fibonacci or Lucas numbers have been investigated in [12, 18]. Subsequently, repdigits that are sums of four Fibonacci or Lucas or Pell numbers have been investigated in [13, 17]. Repdigits in the base b expansion as sums of four balancing numbers can be found in [10]. Repdigits that are sums of two Fibonacci and two Lucas numbers appear in [19], where as those which are products of two Pell or Pell-Lucas numbers can be seen in [24]. In [25], Şiar and Keskin searched for the repdigits that are sums of two Lucas numbers. Subsequently, the authors [23] searched for all repdigits that are sums of two associated Pell numbers. In addition, they obtained the corresponding results for Pell-Lucas and Lucas-balancing numbers.

The objective of this paper is to extend this study by exploring the balancing and Lucas-balancing numbers expressible as sums of two repdigits. In particular, we prove the following results.

Theorem 1. *The only balancing numbers which are sums of two repdigits are*

$$B_2 = 6 = 1+5 = 2+4 = 3+3 \quad \text{and} \quad B_3 = 35 = 2+33.$$

Theorem 2. *The only Lucas-balancing numbers which are sums of two repdigits are*

$$C_1 = 3 = 1+2, \quad C_2 = 17 = 8+9 = 6+11,$$

$$C_3 = 99 = 11+88 = 22+77 = 33+66 = 44+55$$

and

$$C_4 = 577 = 22+555.$$

2. Preliminaries

The theory of lower bounds for nonzero linear forms in logarithms of algebraic numbers serves as one of the main tools for the proof of our main results. A modified version of a result of Matveev [16] appears in [5, Theorem 9.4]. Let $\mathbb{L}$ be an algebraic number field of degree $d_{\mathbb{L}}$. Let $\eta_1, \eta_2, \ldots, \eta_l \in \mathbb{L}$ not 0 or 1 and $d_1, d_2, \ldots, d_l$ be nonzero integers. Let

$$D = \max\{|d_1|, \ldots, |d_l|\}, \quad \text{and} \quad \Gamma = \prod_{i=1}^{l} \eta_i^{d_i} - 1.$$

Let $A_1, \ldots, A_l$ be positive integers such that

$$A_j \geq h^{'}(\eta_j) := \max\{d_{\mathbb{L}} h(\eta_j), |\log \eta_j|, 0.16\}, \quad j = 1, \ldots, l,$$

where η is an algebraic number having the minimal polynomial

$$f(X) = a_0(X - \eta^{(1)}) \cdots (X - \eta^{(k)}) \in \mathbb{Z}[X]$$

over the integers with $a_0 > 0$. The Weil height of η is given by

$$h(\eta) = \frac{1}{k}\left(\log\, a_0 + \sum_{j=1}^{k} \max\{0, \log|\eta^{(j)}|\}\right).$$

The following properties [26, Property 3.3] of logarithmic height will be used with or without further reference as and when needed.

$$\begin{aligned} h(\eta \pm \gamma) &\leq h(\eta) + h(\gamma) + \log 2, \\ h(\eta\gamma^{\pm 1}) &\leq h(\eta) + h(\gamma), \\ h(\eta^s) &= |s| h(\eta),\ s \in \mathbb{Z}. \end{aligned}$$

Theorem 3. ([5]) *If* $\Gamma \neq 0$ *and* $\mathbb{L} \subseteq \mathbb{R}$, *then*

$$\log|\Gamma| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1, \cdots, A_l.$$

Another main tool for the proof of our main results is a variant of the Baker and Davenport reduction method due to de Weger [15].

Let $\vartheta_1, \vartheta_2, \beta \in \mathbb{R}$ and let $x_1, x_2 \in \mathbb{Z}$ be unknowns. Let

$$\Gamma = \beta + x_1\vartheta_1 + x_2\vartheta_2. \tag{2}$$

Let c, δ be positive constants. Set $X = \max\{|x_1|, |x_2|\}$. Let X_0, Y be positive. Assume that

$$|\Gamma| < c \cdot \exp(-\delta \cdot Y), \tag{3}$$

$$Y \leq X \leq X_0. \tag{4}$$

When $\beta = 0$ in Equation (2), we get

$$\Gamma = x_1\vartheta_1 + x_2\vartheta_2.$$

Let $\vartheta = -\vartheta_1/\vartheta_2$, where $\vartheta_2 \neq 0$. We assume that x_1 and x_2 are coprime. Let the continued fraction expansion of ϑ be given by

$$[a_0, a_1, a_2, \ldots],$$

and let the k^{th} convergent of ϑ be p_k/q_k for $k = 0, 1, 2, \ldots$. We may assume without loss of generality that $|\vartheta_1| < |\vartheta_2|$ and that $x_1 > 0$. We have the following results.

Lemma 1. ([15]) *Let*

$$A = \max_{0 \leq k \leq Y_0} a_{k+1}.$$

If Equation (3) *and Equation* (4) *hold for* x_1, x_2 *and* $\beta = 0$*, then*

$$Y < \frac{1}{\delta}\log\left(\frac{c(A+2)X_0}{|\vartheta_2|}\right).$$

When $\beta \neq 0$ in Equation (2), let $\vartheta = -\vartheta_1/\vartheta_2$ and $\psi = \beta/\vartheta_2$, where $\vartheta_2 \neq 0$. Then, we have

$$\frac{\Gamma}{\vartheta_2} = \psi - x_1\vartheta + x_2.$$

Let p/q be a convergent of ϑ with $q > X_0$. For a real number x, we let $||x|| = \min\{|x - n|, n \in \mathbb{Z}\}$ be the distance from x to the nearest integer. We have the following result.

Lemma 2. (see [15]) *Suppose that*

$$||q\psi|| > \frac{2X_0}{q}.$$

Then, the solutions of Equation (3) *and Equation* (4) *satisfy*

$$Y < \frac{1}{\delta}\log\left(\frac{q^2c}{|\vartheta_2|X_0}\right).$$

3. Main Results

3.1. Proof of Theorem 1

Assume that

$$B_n = d_1\left(\frac{10^{m_1} - 1}{9}\right) + d_2\left(\frac{10^{m_2} - 1}{9}\right) \tag{5}$$

for some positive integers $m_1 \leq m_2$ and $d_1, d_2 \in \{1, 2, \ldots, 9\}$. A quick computer search reveals that there is no solution in the interval $[4, 55]$. For this, we first note that B_{55} contains 42 digits. Thus, we list all the repdigits with maximum length 42; let $\mathcal{A}$ be the set of these repdigits. Then, for every $n \in [4, 55]$, we compute and check whether $B_n - d(10^m - 1)/9$ is a member of $\mathcal{A}$, for some digit $1 \leq d \leq 9$ and some $1 \leq m \leq M$, where M is the number of digits of B_n. So from now on, we assume that $n > 55$.

Lemma 3. *All solutions of Equation* (5) *satisfy*

$$m_2 \log 10 - 2.31 < n \log \alpha < m_2 \log 10 + 2.46.$$

Proof. Since the balancing numbers satisfy $\alpha^{n-1} < B_n < \alpha^n$ for $n > 1$ and from Equation (5) we have $10^{m_2 - 1} < B_n < 2 \cdot 10^{m_2}$, it follows that

$$\alpha^{n-1} < B_n < 2 \cdot 10^{m_2} \quad \text{and} \quad 10^{m_2-1} < B_n < \alpha^n.$$

Taking the logarithm of all sides, we get

$$(n-1)\log\alpha < \log 2 + m_2 \log 10 \quad \text{and} \quad (m_2 - 1)\log 10 < n \log\alpha$$

yielding

$$n\log\alpha < m_2\log 10 + 2.46 \quad \text{and} \quad m_2 \log 10 - 2.31 < n\log\alpha.$$

□

Using Equation (1) in Equation (5), we get

$$\frac{\alpha^n - \beta^n}{4\sqrt{2}} = d_1\Big(\frac{10^{m_1} - 1}{9}\Big) + d_2\Big(\frac{10^{m_2} - 1}{9}\Big),$$

i.e.,

$$\frac{9}{4\sqrt{2}}(\alpha^n - \beta^n) - d_1 10^{m_1} - d_2 10^{m_2} = -(d_1 + d_2). \tag{6}$$

We study Equation (6) in two different steps.

`Step 1:` We rewrite Equation (6) as

$$\frac{9}{4\sqrt{2}}\alpha^n - d_2 10^{m_2} = d_1 10^{m_1} + \frac{9}{4\sqrt{2}}\beta^n - (d_1 + d_2),$$

which implies

$$\left|\frac{9}{4\sqrt{2}}\alpha^n - d_2 10^{m_2}\right| = \left|d_1 10^{m_1} + \frac{9}{4\sqrt{2}}\beta^n - (d_1 + d_2)\right| < 29 \cdot 10^{m_1}.$$

Dividing both sides of the above inequality by $d_2 10^{m_2}$ and taking the absolute value, we get

$$\left|\left(\frac{9}{4\sqrt{2}d_2}\right)\alpha^n 10^{-m_2} - 1\right|| < \frac{29}{10^{m_2 - m_1}}. \tag{7}$$

Let

$$\Gamma := \left(\frac{9}{4\sqrt{2}d_2}\right)\alpha^n 10^{-m_2} - 1. \tag{8}$$

If $\Gamma = 0$, then $\sqrt{2} = q\alpha^n$, where $q \in \mathbb{Q}$ and hence, $\alpha^{2n} = 2q^{-2} \in \mathbb{Q}$, which is not possible for any $n > 0$. Therefore $\Gamma \neq 0$. Let

$$\eta_1 = \frac{9}{4\sqrt{2}d_2},\ \eta_2 = \alpha,\ \eta_3 = 10,\ d_1 = 1,\ d_2 = n,\ d_3 = -m_2,$$

where $\eta_1, \eta_2, \eta_3 \in \mathbb{Q}(\sqrt{2})$ and $d_1, d_2, d_3 \in \mathbb{Z}$. The degree of $\mathbb{L} := \mathbb{Q}(\sqrt{2})$ is $d_{\mathbb{L}} = 2$.

Since $10^{m_2-1} < B_n < \alpha^n$, we have $m_2 < n+1$. Therefore, we take $D = n+1$. Conjugating η_1, η_2, η_3, we get $\eta_1' = -\eta_1$, $\eta_2' = \beta$, $\eta_3' = \eta_3$. The minimal polynomial of η_1 over $\mathbb{Q}$ is

$$(X - \eta_1)(X - \eta_1') = X^2 - \frac{81}{32d_2^2}.$$

Using the properties of logarithmic height, we get

$$h(\eta_1) \leq h(9) + h(4\sqrt{2}d_2) \leq h(9) + h(36) + h(\sqrt{2}),$$

which implies that

$$2h(\eta_1) \leq 12.3.$$

Thus, we take

$$A_1 := 12.3.$$

Further, since

$$h(\eta_2) = \frac{1}{2}\log\alpha, \quad h(\eta_3) = \log 10,$$

we have,

$$\max\{2h(\eta_2), |\log\eta_2|, 0.16\} = \log\alpha < 1.8 := A_2,$$

$$\max\{2h(\eta_3), |\log\eta_3|, 0.16\} = \log\alpha < 4.7 := A_3.$$

In view of Theorem 3 and Equation (7), we have

$$(m_2 - m_1)\log(10) < \log(29) + 1.1 \cdot 10^{14}(1 + \log(n+1))$$

giving

$$(m_2 - m_1)\log(10) < 1.2 \cdot 10^{14}(1 + \log(n+1)).$$

Thus, we obtain

$$m_2 - m_1 < 0.6 \cdot 10^{13}(1 + \log(n+1)). \tag{9}$$

`Step 2:` We rewrite Equation (6) as

$$\frac{\alpha^n}{4\sqrt{2}} - \frac{d_1 10^{m_1} + d_2 10^{m_2}}{9} = \frac{\beta^n}{4\sqrt{2}} - \frac{d_1 + d_2}{9},$$

which implies

$$\left|\frac{\alpha^n}{4\sqrt{2}} - 10^{m_2}\left(\frac{d_1 10^{m_1-m_2}+d_2}{9}\right)\right| = \left|\frac{\beta^n}{4\sqrt{2}} - \frac{d_1+d_2}{9}\right| < 2.2.$$

Dividing both sides of the above inequality by $\alpha^n/4\sqrt{2}$ and taking the absolute value, we get

$$\left|1-\alpha^{-n}10^{m_2}\left(\frac{4\sqrt{2}(d_1 10^{m_1-m_2}+d_2)}{9}\right)\right| < \frac{2.2\alpha}{\alpha^n} < \frac{1}{\alpha^{n-1.45}}. \tag{10}$$

Let

$$\Gamma' := 1-\alpha^{-n}10^{m_2}\left(\frac{4\sqrt{2}(d_1 10^{m_1-m_2}+d_2)}{9}\right). \tag{11}$$

If $\Gamma' = 0$, then

$$\alpha^n = 4\sqrt{2}\left(\frac{d_1 10^{m_1}}{9} + \frac{d_2 10^{m_2}}{9}\right).$$

Conjugating α^n in $\mathbb{Q}(\sqrt{2})$, we get

$$\beta^n = -4\sqrt{2}\left(\frac{d_1 10^{m_1}}{9} + \frac{d_2 10^{m_2}}{9}\right)$$

and consequently,

$$\frac{8\sqrt{2}\cdot 10^{m_1}}{9} \leq 4\sqrt{2}\left(\frac{d_1 10^{m_1}}{9} + \frac{d_2 10^{m_2}}{9}\right) = |\beta|^n < 1,$$

which is not possible for any m_1 no less than 1. Therefore, $\Gamma' \neq 0$. Let

$$\eta_1 = \left(\frac{4\sqrt{2}(d_1 10^{m_1-m_2}+d_2)}{9}\right),\ \eta_2 = \alpha,\ \eta_3 = 10,\ d_1 = 1,\ d_2 = -n,\ d_3 = m_2,$$

where $\eta_1, \eta_2, \eta_3 \in \mathbb{Q}(\sqrt{2})$ and $d_1, d_2, d_3 \in \mathbb{Z}$. The degree of $\mathbb{L} := \mathbb{Q}(\sqrt{2})$ is $d_{\mathbb{L}} = 2$.

Since $1 \leq m_1 \leq m_2$ and $m_2 < n+1$, we take $D = n+1$. Now,

$$\begin{aligned}
h(\eta_1) &\leq h\left(4\sqrt{2}\left(\frac{d_1 10^{m_1-m_2}+d_2}{9}\right)\right)\\
&\leq h(\sqrt{2}) + h(36) + h(d_1 10^{m_1-m_2}+d_2)\\
&\leq \frac{1}{2}\log 2 + h(36) + h(d_1) + h(d_2) + (m_2-m_1)h(10) + \log 2\\
&\leq 9.02 + 2.31(m_2-m_1),
\end{aligned}$$

which implies that

$$2h(\eta_1) \leq 18.04 + 4.62(m_2-m_1).$$

Thus, we take

$$A_1 := 18.04 + 4.62(m_2 - m_1).$$

Further, since

$$h(\eta_2) = \frac{1}{2}\log\alpha, \quad h(\eta_3) = \log 10,$$

we have

$$\max\{2h(\eta_2), |\log\eta_2|, 0.16\} = \log\alpha < 1.8 := A_2,$$

$$\max\{2h(\eta_3), |\log\eta_3|, 0.16\} = \log\alpha < 4.7 := A_3.$$

In view of Theorem 3 and Equation (9), we have

$$(n-1.45)\log(\alpha) < 8.21\cdot 10^{12}(18.04+4.62(0.6\cdot 10^{13}(1+\log(n+1))))(1+\log(n+1))$$

giving

$$n < 6.26\cdot 10^{29}.$$

Thus, Lemma 3 implies

$$m_1 \le m_2 < 4.8\cdot 10^{29}.$$

We record this in the following lemma.

Lemma 4. *All solutions of Equation* (5) *satisfy*

$$m_1 \le m_2 < 4.8\cdot 10^{29} \quad \text{and} \quad n < 6.26\cdot 10^{29}.$$

To lower the above bounds, we rewrite Equation (5) as

$$B_n = \frac{d_2 10^{m_2}}{9} + \left(d_1\frac{10^{m_1}-1}{9} - \frac{d_2}{9}\right).$$

One can observe that the term in the parenthesis is always positive and is zero only when $d_1 = m_1 = 1$ and $d_2 = 9$. The case zero implies $B_n = 10^{m_2}$, which is not possible for any $n > 50$, because by the primitive divisor theorem (see [6]), the balancing number B_n has a prime factor not less than $n-1$ for all values $n > 12$. Thus, the number appearing in the parentheses is at least $\frac{1}{9}$. Hence,

$$\frac{\alpha^n}{4\sqrt{2}} - \frac{d_2 10^{m_2}}{9} = \left(d_1\frac{10^{m_1-1}}{9} - \frac{d_2}{9}\right) + \frac{\beta^n}{4\sqrt{2}} \ge \frac{1}{9} + \frac{1}{4\sqrt{2}\alpha^n} > 0.$$

In view of Equation (7) and Equation (8), we have

$$0 < \eta_1\eta_2^n\eta_3^{-m_2} - 1 < \frac{29}{10^{m_2-m_1}}.$$

Let

$$\Lambda = -m_2\log 10 + n\log\alpha + \log\left(\frac{9}{4\sqrt{2}d_2}\right).$$

We obtain that
$$\frac{\alpha^n}{4\sqrt{2}} - \frac{d_2 10^{m_2}}{9} = \frac{d_2 10^{m_2}}{9}(e^{\Lambda} - 1) > 0,$$
so
$$0 < \Lambda < e^{\Lambda} - 1 = \Gamma < \frac{29}{10^{m_2 - m_1}},$$
which implies that
$$0 < \log\left(\frac{9}{4\sqrt{2}d_2}\right) + m_2(-\log 10) + n\log\alpha < \frac{29}{10^{m_2-m_1}}$$
$$< 10^{1.47}\exp(-2.3 \cdot (m_2 - m_1)).$$

Thus,
$$\Lambda < 10^{1.47}\exp(-2.3 \cdot Y)$$
holds with
$$Y := m_2 - m_1 < m_2 < 4.8 \cdot 10^{29}.$$
We also have
$$\frac{\Lambda}{\log 10} = \frac{\log(9/(4\sqrt{2}d_2))}{\log 10} + n\frac{\log\alpha}{\log 10} - m_2.$$
Thus, we take
$$c = 10^{1.47},\ \delta = 2.3,\ X_0 = 4.8 \cdot 10^{29},\ \psi = \frac{\log(9/(4\sqrt{2}d_2))}{\log 10},$$
$$\vartheta = -\frac{\log\alpha}{\log 10},\ \vartheta_1 = \log\alpha,\ \vartheta_2 = \log 10,\ \beta = \log(9/(4\sqrt{2}d_2)).$$
The smallest value of $q > X_0$ is
$$q_{57} = 808643106803003389273254071835.$$
We find that
$$q_{62} = 36828997285703648759419496796098$$
satisfies the hypothesis of Lemma 2 for $1 \le d_2 \le 9$. Applying it, we get $m_2 - m_1 = Y < 34.6$. Now, we take $0 \le m_2 - m_1 \le 34$.

Let
$$\Lambda' = -n\log\eta_2 + m_2\log\eta_3 + \log\eta_1.$$
Using Equation (6), we have
$$\frac{\alpha^n}{4\sqrt{2}}(1 - e^{\Lambda'}) = \frac{\beta^n}{4\sqrt{2}} - \frac{d_1 + d_2}{9} = -\left(\frac{d_1 + d_2}{9} - \frac{1}{4\sqrt{2}\alpha^n}\right).$$
Further, since
$$\frac{d_1 + d_2}{9} - \frac{1}{4\sqrt{2}\alpha^n} > \frac{2}{9} - \frac{1}{4\sqrt{2}\alpha^n} > 0,$$

we get

$$e^{\Lambda'} - 1 > 0$$

and consequently $\Lambda' > 0$. In view of Equation (10), we have

$$0 < \Lambda' < e^{\Lambda'} - 1 = |\Gamma'| < \frac{1}{\alpha^{n-1.45}},$$

which implies that

$$0 < \log\left(\frac{4\sqrt{2}(d_1 10^{m_1-m_2} + d_2)}{9}\right) + m_2\log 10 + n(-\log\alpha) < \frac{1}{\alpha^{n-1.45}}$$
$$< \alpha^{1.45}\exp(-1.76 \cdot n).$$

We consider

$$X_0 = 6.26 \cdot 10^{29},\ \psi' = \frac{\log(4\sqrt{2}(d_1 10^{m_1-m_2} + d_2)/9)}{\log 10},\ c = \alpha^{1.45},\ \delta = 1.76,$$

$$\vartheta = \frac{\log\alpha}{\log 10},\ \vartheta_1 = -\log\alpha,\ \vartheta_2 = \log 10,\ \beta = \log(4\sqrt{2}(d_1 10^{m_1-m_2} + d_2)/9).$$

We find that

$$q_{57} = 808643106803003389273254071835 > X_0$$

is the smallest value and then applying Lemma 2 with

$$q_{69} = 1282560142280424961584126590701 0799,$$

we obtain $n < 51.25$, i.e., $n \leq 51$, which is a contradiction to our assumption that $n > 55$. This completes the proof.

3.2. Proof of Theorem 2

The proof is similar to that of Theorem 1. Assume that

$$C_n = d_1\left(\frac{10^{m_1} - 1}{9}\right) + d_2\left(\frac{10^{m_2} - 1}{9}\right) \tag{12}$$

for some positive integers $m_1 \leq m_2$ and $d_1, d_2 \in \{1, 2, \ldots, 9\}$. A quick computer search reveals that there is no solution in the interval $n \in [5, 50]$. For this, we first note that C_{50} contains 38 digits. Thus, we list all the repdigits with maximum length 38; let $\mathcal{A}$ be the set of these repdigits. Then, for every $n \in [5, 50]$, we compute and check whether $C_n - d(10^m - 1)/9$ is a member of $\mathcal{A}$, for some digit $1 \leq d \leq 9$ and some $1 \leq m \leq M$, where M is the number of digits of C_n. So from now on, we assume that $n > 50$.

Lemma 5. *All solutions of Equation* (12) *satisfy*

$$m_2\log 10 - 3.38 < n\log\alpha < m_2\log 10 + 1.39.$$

Proof. Since the Lucas-balancing numbers satisfy $\alpha^n < 2C_n < \alpha^{n+1}$ for $n \geq 1$ and from Equation (12) we have $10^{m_2-1} < C_n < 2 \cdot 10^{m_2}$, it follows that

$$\alpha^n < 2C_n < 4 \cdot 10^{m_2} \quad \text{and} \quad 10^{m_2-1} < C_n < \alpha^{n+1}/2.$$

Taking the logarithm of all sides, we get

$$n\log\alpha < \log 4 + m_2\log 10 \quad \text{and} \quad (m_2 - 1)\log 10 < (n+1)\log\alpha - \log 2$$

yielding

$$n\log\alpha < m_2\log 10 + 1.39 \quad \text{and} \quad m_2\log 10 - 3.38 < n\log\alpha.$$

□

Using Equation (1) in Equation (12), we get

$$\frac{\alpha^n + \beta^n}{2} = d_1\Big(\frac{10^{m_1} - 1}{9}\Big) + d_2\Big(\frac{10^{m_2} - 1}{9}\Big),$$

i.e.,

$$\frac{9}{2}(\alpha^n + \beta^n) - d_1 10^{m_1} - d_2 10^{m_2} = -(d_1 + d_2). \tag{13}$$

We study Equation (13) in two different steps.
Step 1: We rewrite Equation (13) as

$$\frac{9}{2}\alpha^n - d_2 10^{m_2} = d_1 10^{m_1} - \frac{9}{2}\beta^n - (d_1 + d_2),$$

which implies

$$\left|\frac{9}{2}\alpha^n - d_2 10^{m_2}\right| = \left|d_1 10^{m_1} - \frac{9}{2}\beta^n - (d_1 + d_2)\right| < 32 \cdot 10^{m_1}.$$

Dividing both sides of the above inequality by $d_2 10^{m_2}$ and taking the absolute value, we get

$$\left|\left(\frac{9}{2d_2}\right)\alpha^n 10^{-m_2} - 1\right|| < \frac{32}{10^{m_2-m_1}}. \tag{14}$$

Let

$$\Gamma_1 := \left(\frac{9}{2d_2}\right)\alpha^n 10^{-m_2} - 1. \tag{15}$$

If $\Gamma_1 = 0$, then $\frac{2d_2 10^{m_2}}{9} = \alpha^n \in \mathbb{Q}$, which is not possible for any $n > 0$. Therefore $\Gamma_1 \neq 0$. Let

$$\eta_1 = \frac{9}{2d_2},\ \eta_2 = \alpha,\ \eta_3 = 10,\ d_1 = 1,\ d_2 = n,\ d_3 = -m_2,$$

where η_1, η_2, $\eta_3 \in \mathbb{Q}(\alpha)$ and d_1, d_2, $d_3 \in \mathbb{Z}$. The degree of $\mathbb{L} := \mathbb{Q}(\alpha)$ is $d_{\mathbb{L}} = 2$.

Since $10^{m_2-1} < C_n < \alpha^{n+1}/2$, we have $m_2 < n+2$. Therefore, we take $D = n+2$. Conjugating η_1, η_2, η_3, we get $\eta_1' = \eta_1$, $\eta_2' = \beta$, $\eta_3' = \eta_3$. The minimal polynomial of η_2 over $\mathbb{Q}$ is

$$(X - \eta_2)(X - \eta_2') = X^2 - 6X + 1.$$

Using the properties of logarithmic height, we get

$$h(\eta_1) \leq h(9) + h(2d_2) \leq 2h(9) + h(2),$$

which implies that

$$2h(\eta_1) < 10.18.$$

Thus, proceeding as in the previous subsection, we have

$$A_1 := 10.18, \quad A_2 = 1.8, \quad A_3 = 4.7.$$

In view of Theorem 3 and Equation (14), we have

$$(m_2 - m_1)\log(10) < \log(32) + 8.36 \cdot 10^{13}(1 + \log(n+2))$$

giving

$$m_2 - m_1 < 3.64 \cdot 10^{13}(1 + \log(n+2)). \tag{16}$$

Step 2: We rewrite Equation (13) as

$$\frac{\alpha^n}{2} - \frac{d_1 10^{m_1} + d_2 10^{m_2}}{9} = -\frac{\beta^n}{2} - \frac{d_1 + d_2}{9},$$

which implies

$$\left|\frac{\alpha^n}{2} - 10^{m_2}\left(\frac{d_1 10^{m_1-m_2} + d_2}{9}\right)\right| = \left| -\frac{\beta^n}{2} - \frac{d_1 + d_2}{9}\right| < 2.5.$$

Dividing both sides of the above inequality by $\alpha^n/2$ and taking the absolute value, we get

$$\left|1 - \alpha^{-n} 10^{m_2}\left(\frac{2(d_1 10^{m_1-m_2} + d_2)}{9}\right)\right| < \frac{4.5}{\alpha^n} < \frac{1}{\alpha^{n-0.9}}. \tag{17}$$

Let

$$\Gamma_1' := 1 - \alpha^{-n} 10^{m_2}\left(\frac{2(d_1 10^{m_1-m_2} + d_2)}{9}\right). \tag{18}$$

If $\Gamma_1' = 0$, then

$$\frac{\alpha^n}{2} = \frac{d_1 10^{m_1}}{9} + \frac{d_2 10^{m_2}}{9}.$$

Conjugating the above equation in $\mathbb{Q}(\alpha)$, we get

$$\frac{\beta^n}{2} = \frac{d_1 10^{m_1}}{9} + \frac{d_2 10^{m_2}}{9}$$

and consequently,

$$\frac{4\cdot 10^{m_1}}{9} \le 2\left(\frac{d_1 10^{m_1}}{9} + \frac{d_2 10^{m_2}}{9}\right) = \beta^n < 1$$

which is not possible for any m_1 no less than 1. Therefore, $\Gamma_1' \neq 0$. Let

$$\eta_1 = \frac{2(d_1 10^{m_1-m_2} + d_2)}{9},\ \eta_2 = \alpha,\ \eta_3 = 10,\ d_1 = 1,\ d_2 = -n,\ d_3 = m_2,$$

where $\eta_1, \eta_2, \eta_3 \in \mathbb{Q}(\alpha)$ and $d_1, d_2, d_3 \in \mathbb{Z}$. The degree of $\mathbb{L} := \mathbb{Q}(\alpha)$ is $d_{\mathbb{L}} = 2$.

Since $1 \le m_1 \le m_2$ and $m_2 < n+2$, we take $D = n+2$. Now,

$$\begin{aligned} h(\eta_1) &\le h\left(\frac{2(d_1 10^{m_1-m_2} + d_2)}{9}\right) \\ &\le h(2) + h(9) + h(d_1 10^{m_1-m_2} + d_2) \\ &\le h(2) + +h(9) + h(d_1) + h(d_2) + (m_2 - m_1)h(10) + \log 2 \\ &\le 7.98 + 2.31(m_2 - m_1), \end{aligned}$$

which implies that

$$2h(\eta_1) < 15.96 + 4.62(m_2 - m_1).$$

Thus, as before we take

$$A_1 := 15.96 + 4.62(m_2 - m_1), \quad A_2 = 1.8, \quad A_3 = 4.7.$$

In view of Theorem 3 and Equation (16), we have

$$(n-0.9)\log(\alpha) < 8.21\cdot 10^{12}(15.96 + 4.62(3.64\cdot 10^{13}(1+\log(n+2))))(1+\log(n+2))$$

giving

$$n < 4.01\cdot 10^{30}.$$

Thus, Lemma 5 implies

$$m_1 \le m_2 < 3.07\cdot 10^{30}.$$

We record this in the following lemma.

Lemma 6. *All solutions of Equation* (12) *satisfy*

$$m_1 \le m_2 < 3.07\cdot 10^{30} \quad \text{and} \quad n < 4.01\cdot 10^{30}.$$

To lower the above bounds, we rewrite Equation (12) as

$$C_n = \frac{d_2 10^{m_2}}{9} + \left(d_1 \frac{10^{m_1} - 1}{9} - \frac{d_2}{9}\right).$$

One can observe that the term in the parenthesis is always positive and is zero only when $d_1 = m_1 = 1$ and $d_2 = 9$. The case zero implies $C_n = 10^{m_2}$, which is not possible for any $n > 50$, because by the primitive divisor theorem (see [6]), the Lucas-balancing C_n has a prime factor not less than $n-1$ for all values $n > 12$. Thus, the number appearing in the parentheses is at least $\frac{1}{9}$. Hence,

$$\frac{\alpha^n}{2} - \frac{d_2 10^{m_2}}{9} = \left(d_1 \frac{10^{m_1-1}}{9} - \frac{d_2}{9}\right) + \frac{\beta^n}{2} \geq \frac{1}{9} - \frac{1}{2\alpha^n} > 0.$$

In view of Equation (14) and Equation (15), we have

$$0 < \eta_1 \eta_2^n \eta_3^{-m_2} - 1 < \frac{32}{10^{m_2-m_1}}.$$

Let

$$\Lambda_1 = -m_2 \log 10 + n \log\alpha + \log\left(\frac{9}{2d_2}\right).$$

We obtain that

$$\frac{\alpha^n}{2} - \frac{d_2 10^{m_2}}{9} = \frac{d_2 10^{m_2}}{9}(e^{\Lambda_1} - 1) > 0,$$

so

$$0 < \Lambda_1 < e^{\Lambda_1} - 1 = \Gamma_1 < \frac{32}{10^{m_2-m_1}},$$

which implies that

$$\begin{aligned} 0 < \log\left(\frac{9}{2d_2}\right) + m_2(-\log 10) + n\log\alpha &< \frac{32}{10^{m_2-m_1}} \\ &< 10^{1.51}\exp(-2.3 \cdot (m_2 - m_1)). \end{aligned}$$

Thus,

$$\Lambda_1 < 10^{1.51}\exp(-2.3 \cdot Y)$$

holds with

$$Y := m_2 - m_1 < m_2 < 4.01 \cdot 10^{30}.$$

We also have

$$\frac{\Lambda_1}{\log 10} = \frac{\log(9/(2d_2))}{\log 10} + n\frac{\log\alpha}{\log 10} - m_2.$$

Thus, we take

$$c = 10^{1.51},\ \delta = 2.3,\ X_0 = 4.01 \cdot 10^{30},\ \psi = \frac{\log(9/(2d_2))}{\log 10},$$

$$\vartheta = -\frac{\log\alpha}{\log 10},\ \vartheta_1 = \log\alpha,\ \vartheta_2 = \log 10,\ \beta = \log(9/(2d_2)).$$

The smallest value of $q > X_0$ is

$$q_{61} = 3431695068347591447908964370918 9$$

We find that

$$q_{66} = 650361718131527006267901676895219$$

satisfies the hypothesis of Lemma 2 for $1 \leq d_2 \leq 9$. Applying it, we get $m_2 - m_1 = Y < 36.22$, i.e., $Y \leq 36$. Now, we take $0 \leq m_2 - m_1 \leq 36$.

Let

$$\Lambda_1' = -n\log\eta_2 + m_2\log\eta_3 + \log\eta_1.$$

Using Equation (13), we have

$$\frac{\alpha^n}{2}(1 - e^{\Lambda_1'}) = -\frac{\beta^n}{2} - \frac{d_1 + d_2}{9} = -\left(\frac{d_1 + d_2}{9} + \frac{1}{2\alpha^n}\right).$$

Further, since

$$\frac{d_1 + d_2}{9} + \frac{1}{2\alpha^n} > \frac{2}{9} + \frac{1}{2\alpha^n} > 0,$$

we get

$$e^{\Lambda_1'} - 1 > 0$$

and consequently $\Lambda_1' > 0$. In view of Equation (17), we have

$$0 < \Lambda_1' < e^{\Lambda_1'} - 1 = |\Gamma_1'| < \frac{1}{\alpha^{n-0.9}},$$

which implies that

$$\begin{aligned} 0 &< \log\left(\frac{2(d_1 10^{m_1 - m_2} + d_2)}{9}\right) + m_2\log 10 + n(-\log\alpha) < \frac{1}{\alpha^{n-0.9}} \\ &< \alpha^{0.9}\exp(-1.76 \cdot n). \end{aligned}$$

We consider the same X_0 and the other values as

$$\psi' = \frac{\log(2(d_1 10^{m_1 - m_2} + d_2)/9)}{\log 10},\ c = \alpha^{0.9},\ \delta = 1.76,$$

$$\vartheta = \frac{\log\alpha}{\log 10},\ \vartheta_1 = -\log\alpha,\ \vartheta_2 = \log 10,\ \beta = \log(2(d_1 10^{m_1 - m_2} + d_2)/9).$$

Clearly, $\beta \neq 0$ and evidently $\psi' \neq 0$ except when $a = 5$, $b = 4$, $m_2 - m_1 = 1$. Thus, for $\psi' \neq 0$, we find that

$$q_{70} = 1658296778905282479269132783428463 0 > X_0$$

satisfies the hypothesis of Lemma 2 and hence the application of Lemma 2 gives $n < 49.93$, i.e., $n \leq 49$, which is a contradiction to our assumption that $n > 50$.

Now, when $a = 5$, $b = 4$, $m_2 - m_1 = 1$, we have $\beta = 0$. The largest partial quotient a_k for $0 \leq k \leq 148$ is $a_{122} = 280$. Applying Lemma 1, we get

$$n < \frac{1}{1.76} \cdot \log\left(\frac{\alpha^{0.9}(280 + 2) \cdot 4.01 \cdot 10^{30}}{|\log 10|}\right).$$

Thus, we obtain $n < 43.68$, i.e., $n \leq 43$, which again contradicts the assumption that $n > 50$. This completes the proof.

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