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A COMBINATORIAL VIEWPOINT ON PRESERVING NOTIONS OF LARGENESS AND AN ABSTRACT RADO THEOREM

Sayan Goswami
Department of Mathematics, University of Kalyani, Kalyani, West Bengal, India
sayan92m@gmail.com

Subhajit Jana
Department of Mathematics, University of Kalyani, Kalyani, West Bengal, India
suja12345@gmail.com

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Abstract

In this article we study two different types of problems. In the first part of this article we study the preservation of some notions of "largeness" for subsets of semigroups under taking arbitrary semigroup homomorphisms, and in the second part we generalize a recent result of Leader and Russell concerning the partition regularity of inhomogeneous systems of linear equations over the integers to Abelian groups, dispensing with the ring structure entirely.

1. Introduction

Ramsey theoretic study consists of studying "large subsets" of semigroups which helps to explore the field's main results. There are various methods of studying these large sets including dynamical systems and the study of algebra in the Stone-Čech compactification of semigroups. In some recent works [1, 5, 7, 8], studies have been made on how these large sets behave under homomorphisms and embeddings into the difference group. It has been shown that under some restrictions, these large sets are well preserved under homomorphism. But those proofs use the algebra of the Stone-Čech compactification. We search for combinatorial proofs because using only arithmetic arguments is aesthetically desirable.

In the second part, we will prove an abstract version of Rado's theorem which generalizes a recent result of Leader and Russell's result [9] concerning the partition regularity of inhomogeneous systems of linear equations over the integers. In doing so, we dispense with the ring structure in [9] and replace this with a group homomorphism.

To state the results, we need some definitions and properties of different large sets which are described below. For details the readers are invited to read [6].

For any set X, let $\mathcal{P}_f(X)$ be the set of finite nonempty subsets of X. Given a subset A of a semigroup $(S,\cdot)$, and $x \in S$, let $x^{-1}A = \{y \in S : x \cdot y \in A\}$.

Definition 1.1 ([6])**.** Let $(S,\cdot)$ be a semigroup and $A \subseteq S$, then we have the following definitions.

1. The set A is *thick* if and only if for any finite subset F of S, there exists an element $x \in S$ such that $F \cdot x \subset A$.

2. The set A is *piecewise syndetic* if and only if there exists $G \in \mathcal{P}_f(S)$ such that for every $F \in \mathcal{P}_f(S)$, there exists $x \in S$ such that $F \cdot x \subseteq \bigcup_{t \in G} t^{-1}A$.

3. $\mathcal{T} = {}^{\mathbb{N}}S$.

4. For $m \in \mathbb{N}$, $\mathcal{J}_m = \{(t(1), t(2) \ldots, t(m)) \in \mathbb{N}^m : t(1) < t(2) < \ldots < t(m)\}$

5. Given $m \in \mathbb{N}$, $a \in S^{m+1}$, $t \in \mathcal{J}_m$ and $f \in F$,
$$x(m,a,t,f) = \left(\prod_{j=1}^{m} \left(a(j) \cdot f(t(j))\right)\right) \cdot a(m+1)$$
where the terms in the product $\prod$ are arranged in increasing order.

6. $A \subseteq S$ is called a *J-set* if and only if for each $F \in \mathcal{P}_f(\mathcal{T})$, there exists $m \in \mathbb{N}$, $a \in S^{m+1}$, $t \in \mathcal{J}_m$ such that, for each $f \in \mathcal{T}$,
$$x(m,a,t,f) \in A.$$

The readers can find more around the definition of J-set in [6, Definition 14.14.1].

The following definition is quite complicated but is the key to keeping the proofs combinatorial.

Definition 1.2 ([6, Definition 14.19])**.** Let $(S,\cdot)$ be a semigroup and $\mathcal{A} \subseteq \mathcal{P}(S)$. Then $\mathcal{A}$ is *collection-wise piecewise syndetic* if and only if there exist functions $K : \mathcal{P}_f(\mathcal{A}) \to \mathcal{P}_f(S)$ and $x : \mathcal{P}_f(\mathcal{A}) \times \mathcal{P}_f(S) \to S$ such that for all $F \in \mathcal{P}_f(S)$ and all $\mathcal{F}$ and $\mathcal{H}$ in $\mathcal{P}_f(\mathcal{A})$ with $\mathcal{F} \subseteq \mathcal{H}$ one has $F \cdot x(\mathcal{H}, F) \subseteq \bigcup_{t \in K(\mathcal{F})} t^{-1}\left(\bigcap \mathcal{F}\right)$, where $\bigcap \mathcal{F} = \bigcap_{F \in \mathcal{F}} F$.

When we write that $\langle C_F \rangle_{F \in \mathcal{I}}$ is a downward directed family, we mean that $(\mathcal{I}, \geq)$ is a directed set and when $F, G \in \mathcal{I}$ with $F \geq G$, one has $C_F \subseteq C_G$.

Definition 1.3. Let $(S,\cdot)$ be an infinite semigroup and $A \subseteq S$.
(a) ([3]) The set A is said to be *quasi-central* if and only if there is a downward directed family $\langle C_F \rangle_{F \in \mathcal{I}}$ of subsets of A such that,

1. for each $F \in \mathcal{I}$ and each $x \in C_F$, there exists $G \in \mathcal{I}$ such that $C_G \subseteq x^{-1}C_F$ and

2. for each $F \in \mathcal{I}$, C_F is piecewise syndetic.

(b) ([6]) The set A is said to be *central* if and only if there is a downward directed family $\langle C_F \rangle_{F \in \mathcal{I}}$ of subsets of A such that,

1. for each $F \in \mathcal{I}$ and each $x \in C_F$, there exists $G \in \mathcal{I}$ such that $C_G \subseteq x^{-1}C_F$ and

2. $\{C_F : F \in \mathcal{I}\}$ is collectionwise piecewise syndetic.

(c) ([6]) The set A is said to be *C-set* if and only if there is a downward directed family $\langle C_F \rangle_{F \in \mathcal{I}}$ of subsets of A such that,

1. for each $F \in \mathcal{I}$ and each $x \in C_F$, there exists $G \in \mathcal{I}$ such that $C_G \subseteq x^{-1}C_F$ and

2. for each $\mathcal{F} \in \mathcal{P}_f(\mathcal{I})$, $\bigcap_{F \in \mathcal{F}} C_F$ is a J-set.

Note that thick implies central implies piecewise syndetic [3], piecewise syndetic set implies J-set [6, Theorem 14.8.3], central implies quasi-central [4] and central sets implies C-sets [6, Corollary 14.14.10]. We will show that large sets in a commutative semigroup $(S,+)$ are also large in the difference group $(S-S,+)$. To prove these results we will use the combinatorial characterizations of these sets instead of using the structure of the Stone-Čech compactification of $(S,+)$ and $(S-S,+)$. So we will not recall the structure of the Stone-Čech compactification of $(S,+)$.

For a commutative cancellative semigroup $(S,+)$, the difference group $(S-S,+)$ consists of elements of the form $\{a-b : a,b \in S\}$ where $a-b$ is defined to be that element for which $b+(a-b)=a$. It is easy to check that $(S-S,+)$ is a commutative group with identity 0, where $a+0=a$. There are many embedding homomorphisms from S to $S-S$ but here we fix one of them as, an element $a \in S$ can be embedded in $S-S$ as $a \to (a+b)-b$ for some fixed $b \in S$. We see that $S \subseteq S-S$. For construction one can see [6, Exercise 1.1.8].

In [1, 5] it was proved that, if $\Phi : (S,\cdot) \to (T,\cdot)$ is a semigroup homomorphism and $\Phi(S)$ is large in some sense in T, then for any large set A, $\Phi(A)$ is large in T. In this paper we will prove those results using combinatorial tools and we will prove a new result that if $\Phi(S)$ is a J-set in T, then for any J-set set A in S, $\Phi(A)$ is a J-set in T.

The paper is arranged as follows: section 2 is devoted to the study of preserving large sets under homomorphism and section 3 is devoted to the abstract version of Rado's theorem regarding inhomogeneous partition regularity.

2. Largeness Under Homomorphism

In this section we will prove that, if $\Phi : S \to T$ is a semigroup homomorphism such that $\Phi(S)$ is a large set in T, then $A \subseteq S$ is a large set in S implies that $\Phi(A)$ is a large set in T. As an application, we will see the notions of largeness are preserved under the group of differences of a cancellative commutative semigroup. Some related works in this direction can be found in [7, 8].

Theorem 2.1 ([1, Lemma 4.6])**.** *Let $\Phi : (S, \cdot) \to (T, \cdot)$ be a semigroup homomorphism such that $\Phi(S)$ is a piecewise syndetic set in T. Then $A \subseteq S$ is a piecewise syndetic set in S implies that $\Phi(A)$ is also a piecewise syndetic set in T.*

Proof. Let $A \subseteq S$ be a piecewise syndetic set in S. So, there exists $H \in \mathcal{P}_f(S)$ such that, for all $F \in \mathcal{P}_f(S)$, there exists $x \in S$ satisfying $F \cdot x \subseteq \bigcup_{t\in H} t^{-1}A$. Hence $\bigcup_{t\in H} \Phi(t)^{-1}\Phi(A)$ is a thick set in $\Phi(S)$, as $\Phi(\bigcup_{t\in H} t^{-1}A) \subseteq \bigcup_{t\in H} \Phi(t)^{-1}\Phi(A)$. As $\Phi(S)$ is a piecewise syndetic set in T, there exists $K \in \mathcal{P}_f(T)$ such that, for all $G \in \mathcal{P}_f(T)$, there exists $y \in T$ satisfying $G \cdot y \subseteq \bigcup_{t\in K} t^{-1}\Phi(S)$.

Now we will show that $\Phi(A)$ is a piecewise syndetic set in T. Let $F \in \mathcal{P}_f(T)$ and $z \in T$ be such that $F \cdot z \subset \bigcup_{t\in K} t^{-1}\Phi(S)$. Let $F = \{a_1, a_2, \ldots, a_n\}$, and this implies $\{a_1z, a_2z, \ldots, a_nz\} \subseteq \bigcup_{t\in K} t^{-1}\Phi(S)$. Hence $\{t_1a_1z, t_2a_2z, \ldots, t_na_nz\} \subseteq \Phi(S)$, where $t_1, t_2, \ldots, t_n \in K$, not necessarily being distinct. So, there exists $y \in \Phi(S)$ such that

$$\{t_1a_1z, t_2a_2z, \ldots, t_na_nz\} \cdot y \subset \bigcup_{t'\in H} \Phi(t')^{-1}\Phi(A),$$

implies

$$F \cdot zy \subseteq \bigcup_{t\in K} \bigcup_{t'\in H} t^{-1}\Phi(t')^{-1}\Phi(A) = \bigcup_{t'\in H, t\in K} \left(\Phi\left(t'\right)t\right)^{-1}\Phi(A).$$

Hence $\Phi(A)$ is piecewise syndetic in T. □

Example 2.2. Consider the homomorphism $h : \mathbb{N} \to \mathbb{Z}$, is defined by $h(n) = 2n$. Now, note that $\cup_{j\in\{0,1\}} -j + 2\mathbb{N} = \mathbb{N}$ is a thick set in $\mathbb{Z}$, and so $2\mathbb{N}$ is a piecewise syndetic set in $\mathbb{Z}$. Now from the above theorem, it follows that, if $A \subseteq \mathbb{N}$ is a piecewise syndetic set in $\mathbb{N}$, then $2 \cdot A = \{2x : x \in A\}$ is a piecewise syndetic set in $\mathbb{Z}$.

Corollary 2.3. *Let $(S, +)$ be a commutative cancellative semigroup. If $A \subseteq S$ is a piecewise syndetic set in S, then A is also a piecewise syndetic set in $S - S$.*

Proof. First we will check that, $i(S) = S$ is a thick set in $S - S$. Consider any finite subset of $S - S$, say $F \in \mathcal{P}_f(S - S)$. Then $F = \{x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\}$, where $x_i, y_i \in S$, for $i = 1, 2, \ldots, n$. Hence,

$$G = F + (y_1 + y_2 + \ldots + y_n) = \left\{ x_1 + \sum_{i=2}^{n} y_i, \ldots, x_j + \sum_{\substack{i=1 \\ i \neq j}}^{n} y_i, \ldots, x_n + \sum_{i=1}^{n-1} y_i \right\} \subset S.$$

Thus, $i(S) = S$ is a thick set in $S - S$. So, S is a piecewise syndetic set in $S - S$. Hence it follows that, if A is a piecewise syndetic set in S, $i(A) = A$ is also a piecewise syndetic set in $S - S$. □

Theorem 2.4. *Let $\Phi : (S, \cdot) \to (T, \cdot)$ be a semigroup homomorphism such that $\Phi(S)$ is a piecewise syndetic set in T. Then $A \subseteq S$ is a quasi-central set in S implies $\Phi(A)$ is also a quasi-central set in T.*

Proof. It follows from the above theorem and the definition 1.3. So we leave it to the reader. □

Corollary 2.5. *Let $(S, +)$ be a commutative cancellative semigroup. If $A \subseteq S$ is a quasi-central set in S, then A is also a quasi-central set in $S - S$.*

Proof. From the proof of Corollary 2.3, we have $i(S) = S$ is a piecewise syndetic set in $S - S$. Hence if A is a quasi-central set in S, $i(A) = A$ is also a quasi-central set in $S - S$. □

In [1, Lemma 4.6], one can find an algebraic proof of the following theorem. Here we provide a combinatorial proof. This proof demonstrates the utility of the combinatorial characterization of central sets.

Theorem 2.6. *Let $\Phi : (S, \cdot) \to (T, \cdot)$ be a semigroup homomorphism such that $\Phi(S)$ is a piecewise syndetic set in T. Then $A \subseteq S$ is a central set implies $\Phi(A)$ is also a central set in T.*

Proof. Let $\Phi : (S, \cdot) \to (T, \cdot)$ be a semigroup homomorphism satisfying the given condition. As we have already proved that the quasi-central sets are preserved under the homomorphism Φ, we have to prove only that if $\mathcal{A}$ is also collectionwise piecewise syndetic in S, then $\Phi(\mathcal{A}) = \{\Phi(B) : B \in \mathcal{A}\}$ is collectionwise piecewise syndetic in T.

Let K and x be the functions guaranteed by the definition 1.2. As $\Phi(S)$ is a piecewise syndetic set in T, there exists a finite subset G of T, such that $\bigcup_{t \in G} t^{-1}\Phi(S)$ is a thick set in T. Therefore for each $F \in \mathcal{P}_f(T)$ we can fix an element $y \in T$, and $\Psi_F \in \mathcal{P}_f(\Phi(S))$, such that $F \cdot y \subseteq \bigcup_{t \in G} t^{-1}\Psi_F$.

Before getting into the proof that, $\Phi(\mathcal{A})$ is collectionwise piecewise syndetic in T, let us define:

1. $\eta : \mathcal{P}_f(T) \to \mathcal{P}_f(\Phi(S))$ by $\eta(F) = \Psi_F$
2. $\kappa : \mathcal{P}_f(T) \to T$ by $\kappa(F) = y$
3. For each $B \in \mathcal{P}_f(\Phi(S))$ fix $B' \in \mathcal{P}_f(S)$ such that $\Phi(B') = B$, where $\Phi(B') = \{\Phi(x) : x \in B\}$
4. For each $\mathcal{B} \in \mathcal{P}_f(\Phi(\mathcal{A}))$ fix $\theta_{\mathcal{B}} \in \mathcal{P}_f(\mathcal{A})$ such that $\Phi(\theta_{\mathcal{B}}) = \mathcal{B}$
5. $K_1 : \mathcal{P}_f(\Phi(\mathcal{A})) \to \mathcal{P}_f(T)$ by $K_1(\mathcal{B}) = \Phi(K(\theta_{\mathcal{B}})) \cdot G$
6. $z : \mathcal{P}_f(\Phi(\mathcal{A})) \times \mathcal{P}_f(T) \to T$ by, $z(\mathcal{H}, F) = \kappa(F) \cdot \Phi(x(\theta_{\mathcal{H}}, \Psi'_F))$.

Now we will show that $\Phi(\mathcal{A})$ is collectionwise piecewise syndetic in T. Let $F \in \mathcal{P}_f(T)$, and $\mathcal{F}, \mathcal{H} \in \mathcal{P}_f(\Phi(\mathcal{A}))$ with $\mathcal{F} \subseteq \mathcal{H}$. Note that $\theta_{\mathcal{F}} \subseteq \theta_{\mathcal{H}}$. As $\mathcal{A}$ is collectionwise piecewise syndetic in S, it follows that,

$$\begin{aligned}
\Psi'_F \cdot x(\theta_{\mathcal{H}}, \Psi'_F) &\subseteq \bigcup_{t \in K(\theta_{\mathcal{F}})} t^{-1}\left(\bigcap \theta_{\mathcal{F}}\right), \\
i.e.\ \Phi(\Psi'_F) \cdot \Phi(x(\theta_{\mathcal{H}}, \Psi'_F)) &\subseteq \bigcup_{t \in K(\theta_{\mathcal{F}})} \Phi(t)^{-1}\Phi\left(\bigcap \theta_{\mathcal{F}}\right), \\
i.e.\ \Phi(\Psi'_F) \cdot \Phi(x(\theta_{\mathcal{H}}, \Psi'_F)) &\subseteq \bigcup_{t \in K(\theta_{\mathcal{F}})} \Phi(t)^{-1}\left(\bigcap \mathcal{F}\right), \\
i.e.\ \Psi_F \cdot \Phi(x(\theta_{\mathcal{H}}, \Psi'_F)) &\subseteq \bigcup_{t \in K(\theta_{\mathcal{F}})} \Phi(t)^{-1}\left(\bigcap \mathcal{F}\right).
\end{aligned}$$

But $F \cdot y \subseteq \bigcup_{s \in G} s^{-1}\Psi_F$. Therefore we have,

$$\begin{aligned}
F \cdot y \cdot \Phi(x(\theta_{\mathcal{H}}, \Psi'_F)) &\subseteq \bigcup_{s \in G} s^{-1}\Psi_F \cdot \Phi(x(\theta_{\mathcal{H}}, \Psi'_F)) \\
&\subseteq \bigcup_{s \in G, t \in K(\theta_{\mathcal{F}})} s^{-1}\Phi(t)^{-1}\left(\bigcap \mathcal{F}\right) \\
&= \bigcup_{s \in G, t \in K(\theta_{\mathcal{F}})} (\Phi(t) \cdot s)^{-1}\left(\bigcap \mathcal{F}\right) \\
&\subseteq \bigcup_{l \in K_1(\mathcal{F})} l^{-1}\left(\bigcap \mathcal{F}\right).
\end{aligned}$$

Hence $F \cdot z(\mathcal{H}, F) \subseteq \bigcup_{l \in K_1(\mathcal{F})} l^{-1}\left(\bigcap \mathcal{F}\right)$, and it follows that $\Phi(\mathcal{A})$ is collectionwise piecewise syndetic in T. Therefore $\Phi(A)$ is a central set in T. □

Example 2.7. Take the function field $\mathbb{F}_q[x]$, where q is prime. As $x \cdot \mathbb{F}_q[x]$ is a piecewise syndetic set in $\mathbb{F}_q[x]$, it follows that for any central set C of $\mathbb{F}_q[x]$, $x \cdot C$ is a central set in $\mathbb{F}_q[x]$

Remark 2.8. The above theorem is not applicable for $\mathbb{Z}[x]$, as $x \cdot \mathbb{Z}[x]$ is not a piecewise syndetic set in $\mathbb{Z}[x]$.

Corollary 2.9. *Let $(S, +)$ be a commutative cancellative semigroup. If $A \subseteq S$ is a central set in S, then A is also a central set in $S - S$.*

Proof. From the proof of Corollary 2.3, we have $i(S) = S$ is a piecewise syndetic set in $S - S$. Hence if A is a central set in S, $i(A) = A$ is a central set in $S - S$. □

J-sets are intimately connected with van der Waerden's theorem and they play a central role in proving the Central Sets theorem. It is natural to ask whether they are preserved under homorphisms. Here we give a proof.

Theorem 2.10. *Let $\Phi : (S, \cdot) \to (T, \cdot)$ be a semigroup homomorphism such that $\Phi(S)$ is a J-set in T. Then $A \subseteq S$ is a J-set in S implies that $\Phi(A)$ is also a J-set in T.*

Proof. As $A \subseteq S$ is a J-set in S, $\Phi(A)$ is also a J-set in $\Phi(S)$. Let $F \in \mathcal{P}_f\left({}^{\mathbb{N}}T\right)$ and $m \in \mathbb{N}$, $a \in T^{m+1}$, $t \in \mathcal{J}_m$, i.e., $t(1) < t(2) < \cdots < t(m)$ in $\mathbb{N}$ such that for each $f \in F$, $x(m, a, t, f) \in \Phi(S)$. Let $t' = \max\{t(1), t(2), \ldots, t(m)\} = t(m)$ since the sequence t is increasing. For each $f \in F$, let $g_f \in {}^{\mathbb{N}}T$ such that $g_f(n) = f(t' + n)$, for all $n \in \mathbb{N}$. Define $G \in \mathcal{P}_f\left({}^{\mathbb{N}}T\right)$ by $G = \{g_f : f \in F\}$. Then again, as $\Phi(S)$ is J-set in T and $G \in \mathcal{P}_f\left({}^{\mathbb{N}}T\right)$, we can apply the above argument on G. Using the above argument repeatedly we obtain $\{m_n\}_{n=1}^{\infty}$, $\{a_n\}_{n=1}^{\infty}$, and $\{t_n\}_{n=1}^{\infty}$ with $m_n \in \mathbb{N}, a_n \in T^{m_n+1}$, and $t_n \in \mathcal{J}_{m_n}$ for all $n \in \mathbb{N}$ such that $x(m_n, a_n, t_n, f) \in \Phi(S)$ for all $f \in F$ and $\max\{t_n\} < \min\{t_{n+1}\}$.

Define $F' \in \mathcal{P}_f\left({}^{\mathbb{N}}\Phi(S)\right)$, as $F' = \{\{x(m_n, a_n, t_n, f)\}_{n=1}^{\infty} : f \in F\}$. Now, as $\Phi(A)$ is a J-set in $\Phi(S)$, there exists $m' \in \mathbb{N}$, $a' \in \Phi(S)^{m+1}$, $t' \in \mathcal{J}_m$, i.e., $t'(1) < t'(2) < \cdots < t'(m)$ in $\mathbb{N}$, such that for each $g \in F'$, $x(m', a', t', g) \in \Phi(A)$. Now,

$$x(m', a', t', g) = \left(\prod_{i=1}^{m'} a'(i) g(t'(i))\right) \cdot a'(m' + 1)$$

$$= \left(\prod_{i=1}^{m'} a'(i).x(m_{t(i)}, a_{t(i)}, t_{t'(i)}, f)\right) \cdot a'(m' + 1)$$

$$= \left(\prod_{i=1}^{n} b(i).f\left(t'(i)\right)\right) \cdot b(n + 1),$$

where $n = m_{t'(1)} + \cdots + m_{t'(m')}$, $b = (b(1), b(2), \ldots, b(n)) \in T^{n+1}$, $t \in \mathcal{J}_n$ with some $b(n + 1) \in T$. The last line can be found by expanding the previous one. Hence, $\Phi(A)$ is a J-set in T. □

Corollary 2.11. *Let $(S,+)$ be a commutative cancellative semigroup. If $A \subseteq S$ is a J-set in S, then A is also a J-set in $S-S$.*

Proof. As $i(S) = S$ is thick in $S-S$, from [6, Theorem 14.8.3], it follows that S is a J-set in $S-S$. Hence, if A is a J-set in S, then $i(A) = A$ is a J-set in $S-S$. □

Theorem 2.12. *Let $\Phi : (S,\cdot) \to (T,\cdot)$ be a semigroup homomorphism such that $\Phi(S)$ is a J-set in T. Then $A \subseteq S$ is a C-set in S implies that $\Phi(A)$ is also a C-set in T.*

Proof. The proof is similar to that of Theorem 2.4 and is left to the reader. □

Corollary 2.13. *Let $(S,+)$ be a commutative cancellative semigroup. If $A \subseteq S$ is a C-set in S, then A is also a C-set in $S-S$.*

Proof. From the proof of Corollary 2.11, it follows that S is a J-set in $S-S$. Hence if A is a C-set in S, then $i(A) = A$ is also a C-set in $S-S$. □

Here we present two interesting applications of the above theorems.

Example 2.14. Recall that any piecewise syndetic set is a J-set. In fact, if $A \subseteq (\mathbb{N},+)$ is a J-set, then it is also a J-set in $(\mathbb{Z},+)$. Now, if we consider the sequences $f_1 = (-1,-2,\ldots,-n,\ldots)$, and $f_2 = (1,2,\ldots,n,\ldots)$, then there exists a number $a \in \mathbb{Z}$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that

$$\left\{a - \sum_{t\in H} t, a + \sum_{t\in H} t\right\} \subset A.$$

Example 2.15. As $(\mathbb{Q}^+,\cdot)$ is the group of differences of $(\mathbb{N},\cdot)$, it follows that if $A \subseteq (\mathbb{N},\cdot)$ is a J-set, then A is also a J-set in $(\mathbb{Q}^+,\cdot)$. If $f_1 = (1, 1/2, \ldots, 1/n, \ldots)$, and $f_2 = (1/2, \ldots, 1/n, \ldots)$, then $a \in \mathbb{Q}$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that

$$\left\{a/\prod_{t\in H} t, a/\prod_{t\in H}(t+1)\right\} \subset A.$$

3. Abstract Rado Theorem

A system of equations $Ax = b$ is said to be partition regular over $\mathbb{Z}$, if given any finite coloring of $\mathbb{Z}$, there exists a solution x to the equations $Ax = b$, all of whose coordinates are of the same color. In [10], Rado proved that the system of equations $Ax = b$ where $b \neq 0$ is partition regular over $\mathbb{Z}$ if and only if it has a constant solution. Recently in [9], it has been established that the result is still true if one considers the entries of the matrix from a ring and considers the equation over any ring R. Their result is the following.

Theorem 3.1. *Let A be an $m \times n$ matrix with entries in a ring R, and let $b \in R^m$ be non-zero. Then the system of equations $Ax = b$ is partition regular over R if and only if it has a constant solution.*

Since multiplication by a matrix with ring entries is a homomorphism from the Abelian group $(R^n, +)$ to the Abelian group $(R^m, +)$, this motivates us to generalize the above theorem for groups. But we need an analogous notion of matrix for groups. Recently in [2, Definition 7.3] the authors, removing the matrix structure, have studied some results concerning partition regularity using the group homomorphisms. Now we will proceed using the similar ideas of [2, Definition 7.3] . The proof of the following version is similar to the proof of [9, Theorem 2].

Theorem 3.2. *Let G and L be two Abelian groups with identity 0. Let $k, n \in \mathbb{N}$ and let $A : G^n \to L$ be a homomorphism and $b \in L$ be non zero. Then the equation $Ax = b$, where $x \in G^n$, is partition regular over G if and only if it has a constant solution.*

Proof. Let G be a Abelian group with identity 0. Let $n \in \mathbb{N}$ and let $A : G^n \to L$ be a homomorphism. For each $i = 1, 2, \ldots, n$ let $c_i : G \to L$ be the map defined by $c_i(y) = A(0, 0, \ldots, y, 0, \ldots, 0)$, where y appears in the i-th position.

Let $H = \{\sum_{i=1}^{n} c_i(y) : y \in G\}$. One can easily check that H is a subgroup of L.

Suppose that there is no constant solution (i.e. entries of x are not constant) : this means that b does not belongs to the subgroup H of L.

Let K be the maximal subgroup of L which contains H but not the element b. Then the quotient map θ from L to L/K has $\theta(b)$ non-zero and also every non trivial subgroup of L/K must contain $\theta(b)$ (by the maximality of K). It follows from this that L/K is either a cyclic group of prime power order or else the group $\mathbb{Z}_{p^\infty}$ of all p^k-th root of unity for any k , for some fixed prime p [9]. In each case this is isomorphic to a subgroup of the circle $\mathbb{T}$. Hence there is a group homomorphism ϕ from L to the circle $\mathbb{T}$ such that $\phi(H) = 0$ and $\phi(b) \neq 0$. Now, define a coloring of G with d^n colors, where d is large enough positive integer, by coloring $t \in G$ with the n tuple $(f(\phi(c_1(t))), f(\phi(c_2(t))), \ldots, f(\phi(c_n(t))))$, where f is the map that sends interval of the circle with arguments in $[2\pi j/d, 2\pi(j+1)/d]$ to j, for each $0 \leq j \leq d - 1$. Suppose that for this coloring we have a monochromatic vector x in G^n, such that $Ax = b$. We have that

$$\begin{aligned}
\phi\left(\sum_{i=1}^{n} c_i(x_i)\right) &= \phi\left(\sum_{i=1}^{n} A(0, 0, \ldots, x_i, 0, \ldots, 0)\right) \\
&= \phi(Ax) \\
&= \phi(b),
\end{aligned}$$

where $x = (x_1, \ldots, x_n)$. But, $\phi\left(\sum_{i=1}^{n} c_i(x_1)\right) = 0$ as $\sum_{i=1}^{n} c_i(x_1) \in H$.

So we have,

$$\sum_{i=1}^{n}\left(\phi\left(c_i\left(x_i\right)\right)-\phi\left(c_i\left(x_1\right)\right)\right)=\phi\left(\sum_{i=1}^{n} c_i\left(x_i\right)\right)-\phi\left(\sum_{i=1}^{n} c_i\left(x_1\right)\right)=\phi(b).$$

But this is a contradiction for d large, as each term in the sum on the left-hand side has argument in $[0, 4\pi/d]$, by the definition of coloring. □

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