LINEAR RECURRENCES FOR BERNOULLI POLYNOMIALS INVOLVING DIFFERENT KINDS OF SUMS

Takashi Agoh
Department of Mathematics, Tokyo University of Science, Noda, Chiba, Japan
agoh_takashi@ma.noda.tus.ac.jp

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Abstract

In this paper, we demonstrate a general method of producing special types of linear recurrence relations for Bernoulli polynomials that involve two different kinds of sums (precisely, an ordinary sum and a binomial sum) by applying a Mellin-Barnes integral representation. In addition, we show that the binomial theorem provides some of the most fundamental properties of these polynomials.

– *Dedicated to Paulo Ribenboim on the occasion of his 93rd birthday.*

1. Introduction

Let B_n and $B_n(x)$, $n = 0, 1, 2, \ldots$, be the Bernoulli numbers and polynomials defined by the generating functions

$$\begin{aligned} \mathbb{B}(t) &:= \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi; \\ \mathbb{B}(t, x) &:= \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi, \end{aligned} \tag{1.1}$$

respectively. It is easy to find that $B_n(0) = B_n$ and $B_n(x) = \sum_{i=0}^{n} \binom{n}{i} B_i x^{n-i}$.

As is widely known, these polynomials satisfy the fundamental properties

$$\begin{aligned} &\text{(i)} \quad B_n(x + 1) = B_n(x) + nx^{n-1}; \\ &\text{(ii)} \quad B_n(-x) = (-1)^n (B_n(x) + nx^{n-1}); \\ &\text{(iii)} \quad B_n(1 - x) = (-1)^n B_n(x). \end{aligned} \tag{1.2}$$

They can be easily shown by making use of the functional relations $\mathbb{B}(t, x)e^t = \mathbb{B}(t, x) + te^t$, $\mathbb{B}(t, -x) = \mathbb{B}(-t, x) - te^{-xt}$, and $\mathbb{B}(-t, x) = \mathbb{B}(t, 1 - x)$, respectively. As we will verify later, it is interesting to observe that (i) arises from the binomial theorem by applying a Mellin-Barnes integral representation for $B_n(x)$. As is clear, (ii) is derived from (i) by reflecting the fact that $B_1 = -\frac{1}{2}$ and $B_n = 0$ if $n \geq 3$ is odd. Further, we see that (iii) is an immediate consequence of (i) and (ii).

Various types of linear and non-linear recurrence formulas for these numbers and polynomials have been studied extensively over the years (for instance, see [13, 17]). Among them, the most basic linear ones are

$$\begin{aligned} &\text{(i)} \quad B_0 = 1, \quad \sum_{i=0}^{n-1} \binom{n}{i} B_i = 0 \quad (n \geq 2); \\ &\text{(ii)} \quad B_0(x) = 1, \quad \sum_{i=0}^{n-1} \binom{n}{i} B_i(x) = nx^{n-1} \quad (n \geq 2). \end{aligned} \tag{1.3}$$

These binomial recurrences are significant in their own right, but we now pay attention to extraordinary types of formulas that involve two different kinds of sums (precisely, an ordinary sum and a binomial sum). Such type formula was first discovered by Miki [15] in a convolution case for Bernoulli numbers. Indeed, applying a congruence related to the Fermat quotient $q_p(a) := (a^{p-1} - 1)/p$ of base a with $p \nmid a$ (p an odd prime), he established a remarkable formula such that

$$\sum_{i=2}^{n-2} \frac{B_i B_{n-i}}{i(n-i)} - \sum_{i=2}^{n-2} \binom{n}{i} \frac{B_i B_{n-i}}{i(n-i)} = 2H_n \frac{B_n}{n} \quad (n \geq 4), \tag{1.4}$$

where H_n is the nth harmonic number, namely

$$H_0 := 0, \quad H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n} \quad (n \geq 1).$$

After a while, Gessel [12] extended (1.4) to Bernoulli polynomials using the Stirling numbers of the second kind. For a brief historical development process of these types of formulas and more details, the reader is referred to, e.g., [2, 4, 10].

In our recent work on Miki-type formulas the following linear identity that is also constituted of two different kinds of sums was proved (see [3, (1.8)]):

$$\sum_{i=1}^{n-1} \frac{B_i}{i} m^{n-i} - \sum_{i=1}^{n-1} \binom{n}{i} \frac{B_i}{i} m^{n-i} = \sum_{j=1}^{m-1} \frac{(m-j)^n}{j} + m^n (H_n - H_m), \tag{1.5}$$

valid for all integers $n, m \geq 1$. In particular, when $m = 1$, it can be verified with easy manipulation that (1.5) reduces to Faulhaber's formula

$$\sum_{a=0}^{k-1} a^{n-1} = \frac{1}{n} \sum_{i=1}^{n} \binom{n}{i} k^i B_{n-i} \quad (n, k \geq 1).$$

As an application example of (1.5), taking $m = 1, 2, \ldots, p$ (with p an odd prime) and evaluating the sum of them modulo p^2 in consideration of the von Staudt-Clausen theorem, it is possible to reproduce Miki's formula without use of complicated tools. In that sense, we may regard that (1.5) is a linear version of (1.4).

It is very natural to search for polynomial analogues of (1.5). Based on such a motivation, in this paper we study special types of linear recurrences for Bernoulli polynomials involving two different kinds of sums using a Mellin-Barnes integral.

In Section 2, as preliminary, we introduce some known identities mainly related to harmonic numbers. In Section 3 we first develop a general method for how to produce special types of recurrences as desired above (see Theorem 3.1). Subsequently, applying this method to the identities stated in Section 2, we establish our targeted linear recurrence formulas for Bernoulli polynomials and numbers. We conclude this paper, in Section 4, with some additional remarks on the fundamental properties for Bernoulli polynomials as mentioned in (1.2).

2. Preliminary

Harmonic numbers have been studied in a wide range of diverse fields such as many branches of number theory, complex analysis, algorithmic analysis, elementary particle physics, quantum field theory, etc. The aim of this section is to introduce some identities mainly related to harmonic numbers involving two different kinds of sums (see Lemma 2.3 below). These identities will be used in the next section for further discussion on Bernoulli polynomials.

Lemma 2.1. *For an integer $n \geq 1$ we have*

$$\text{(i)} \quad \sum_{k=1}^{n} \frac{(-1)^k}{k}\binom{n}{k} = -H_n; \quad \text{(ii)} \quad \sum_{k=1}^{n} H_k = (n+1)H_n - n; \quad \text{(iii)} \quad \sum_{k=1}^{n}(-1)^k\binom{n}{k}H_k = -\frac{1}{n}. \tag{2.1}$$

Above (2.1) can be found as common formulas or conventional exercises in many textbooks. So we wish to omit proving them.

The following formula is not also uncommon, but we would like to give a full proof for convenience sake.

Lemma 2.2. *For an integer $n \geq 1$ we have*

$$\sum_{k=0}^{n-1}\binom{n}{k}\frac{x^k(1-x)^{n-k}}{n-k} = -\sum_{k=1}^{n}\binom{n}{k}H_k(x-1)^k. \tag{2.2}$$

Proof. By shifting x to $x+1$ in (2.2) we shall prove the equivalent identity

$$\sum_{k=0}^{n-1}\binom{n}{k}\frac{(x+1)^k(-x)^{n-k}}{n-k} = -\sum_{k=1}^{n}\binom{n}{k}H_k x^k. \tag{2.3}$$

Denoting by $U_n(x)$ the left-hand side of (2.3), we write it in standard polynomial form. That is,

$$U_n(x) = \sum_{k=0}^{n} a_k x^k \quad \text{with} \quad a_k \in \mathbb{Q}.$$

Since $U_n(0) = 0$, it is trivial that $a_0 = \binom{n}{0} H_0 = 0$. When $k \geq 1$, expanding the factor $(x+1)^k$ in the numerator by the binomial theorem and gathering all the coefficients of x^k in a double summation, we find from (2.1) (i) that

$$\begin{aligned} a_k &= \sum_{j=1}^{k} \frac{(-1)^j}{j} \binom{n}{n-j} \binom{n-j}{n-k} = \frac{n!}{(n-k)!} \sum_{j=1}^{k} \frac{(-1)^j}{j} \frac{1}{j!(k-j)!} \\ &= \binom{n}{k} \sum_{j=1}^{k} \frac{(-1)^j}{j} \binom{k}{j} = -\binom{n}{k} H_k, \end{aligned}$$

which proves (2.3) (and hence (2.2)). □

It is not so difficult to explore various types of polynomial identities constituted of two different kinds of sums. From these identities, we especially choose the following examples, which are relatively easy to understand and accept without any prior knowledge.

Lemma 2.3. *For an integer $n \geq 1$ we have*

$$\begin{aligned} &\text{(i)} \quad \sum_{k=1}^{n} x^k - \sum_{k=1}^{n} \binom{n+1}{k+1} (x-1)^k = n; \\ &\text{(ii)} \quad \sum_{k=1}^{n} \frac{x^k}{k} - \sum_{k=1}^{n} \binom{n}{k} \frac{(x-1)^k}{k} = H_n; \\ &\text{(iii)} \quad \sum_{k=0}^{n-1} \frac{x^k}{n-k} + \sum_{k=1}^{n} \binom{n}{k} H_k (x-1)^k = H_n x^n; \\ &\text{(iv)} \quad \sum_{k=0}^{n-1} (H_n - H_{n-1-k}) x^k - \sum_{k=1}^{n} \binom{n}{k} H_k (x-1)^{k-1} = 0. \end{aligned} \tag{2.4}$$

Proof. If $x = 1$, then (i)–(iii) trivially follow. Further, we observe from (2.1) (ii) that (iv) is also valid for $x = 1$. Hence, in what follows we assume that $x \neq 1$.

(i) By direct calculation we can easily show that

$$\begin{aligned} \sum_{k=1}^{n} x^k &= \frac{((x-1)+1)^{n+1} - 1}{x-1} - 1 = \sum_{k=1}^{n+1} \binom{n+1}{k} (x-1)^{k-1} - 1 \\ &= n + \sum_{k=1}^{n} \binom{n+1}{k+1} (x-1)^k. \end{aligned}$$

(ii) We denote the left-hand side of (ii) by $f(x)$. Several methods of proof of (ii) are available, but perhaps the simplest being to confirm the fact that $f(x)$ is a polynomial function with vanishing derivative. Actually, as was already mentioned in [3, Lemma 3.1], it follows that

$$\begin{aligned}\frac{d}{dx}f(x) &= \sum_{k=1}^{n} x^{k-1} - \sum_{k=1}^{n}\binom{n}{k}(x-1)^{k-1}\\ &= \frac{x^n-1}{x-1} - \frac{1}{x-1}\sum_{k=1}^{n}\binom{n}{k}(x-1)^{k}\\ &= \frac{x^n-1}{x-1} - \frac{((x-1)+1)^n-1}{x-1} = 0.\end{aligned}$$

Since $f(1) = H_n$, the proof of (ii) is complete. Besides [3], the same identity as (ii) can be also found in [21] and [7, (17)].

(iii) If $x = 0$, then (iii) obviously follows in view of (2.1) (iii). So assuming that $x \neq 0$, we replace x by x^{-1} in (ii) and multiply both sides by x^n. After changing the order of summation, we obtain

$$H_n x^n = \sum_{k=0}^{n-1}\frac{x^k}{n-k} - \sum_{k=1}^{n}\binom{n}{k}\frac{x^{n-k}(1-x)^k}{k},$$

which leads to (iii) by using (2.2). Note that an equivalent identity to (iii) has been proved in [6, (36)] and [7, (11)] in quite a different way from ours.

(iv) Recently, Frontczak [11, (3)] proved the following formula by referring to Boyadzhiev's identity [8, (15)] for harmonic numbers obtained by using Euler's transform. For any $a, b \in \mathbb{C}$ and for all integers $n \geq 1$ it follows that

$$\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}H_k = ((a+b)^n - b^n)H_n - a\sum_{k=0}^{n-1}(a+b)^k b^{n-1-k}H_{n-1-k}. \quad (2.5)$$

By setting here $a = x - 1$ and $b = 1$ we have

$$\sum_{k=0}^{n}\binom{n}{k}H_k(x-1)^k = H_n(x^n-1) - (x-1)\sum_{k=0}^{n-1}H_{n-1-k}x^k.$$

Noticing that $H_0 = 0$ and dividing both sides by $x - 1$ ($\neq 0$), this yields

$$\sum_{k=1}^{n}\binom{n}{k}H_k(x-1)^{k-1} = H_n\sum_{k=0}^{n-1}x^k - \sum_{k=0}^{n-1}H_{n-1-k}x^k = \sum_{k=0}^{n-1}(H_n - H_{n-1-k})x^k,$$

which is just the one as we desired. We do not mention in detail here, but (iv) can be also proved by making use of (iii) independently of (2.5). □

3. Main Results

To begin with, we present some crucial requirements for our later discussion.

(a) Let f and g be arbitrary polynomials in x over $\mathbb{C}$ of degree n and m, respectively. We write them in standard polynomial form, that is,

$$f(x) = \sum_{k=0}^{n} a_k x^k \quad \text{and} \quad g(x) = \sum_{k=0}^{m} b_k x^k \quad (a_k, b_k \in \mathbb{C}).$$

Without loss of generality, we assume that $n = \max\{n, m\}$ and $b_k = 0$ for $k > m$. Let $q \geq 0$ be an integer and $h_q(x)$ be the sum of $f(x+q)$ and $g(x)$. Similar to the above, we write $h_q(x)$ in standard polynomial form as follows:

$$h_q(x) := f(x+q) + g(x) = \sum_{k=0}^{n} c_k(q) x^k \quad (c_k(q) \in \mathbb{C}). \tag{3.1}$$

Since $f(x+q)$ is calculated as

$$f(x+q) = \sum_{k=0}^{n} a_k (x+q)^k = \sum_{k=0}^{n} a_k \sum_{j=0}^{k} \binom{k}{j} q^{k-j} x^j,$$

by gathering all the coefficients of x^k in one place we see that each coefficient of h_q is determined by

$$c_k(q) = \sum_{j=0}^{n-k} \binom{k+j}{k} a_{k+j} q^j + b_k \quad (0 \leq k \leq n).$$

In particular, when $q = 0$, we have $c_k(0) = a_k + b_k$. As will be revealed later, the polynomial h_q defined in (3.1) plays an important role in deriving special types of identities for Bernoulli polynomials involving two different kinds of sums.

(b) For a polynomial $f \in \mathbb{C}[x]$ with the same form as above, we define the notation $f(B(x))$ by

$$f(B(x)) := \sum_{k=0}^{n} a_k B_k(x).$$

To explain in other words, let $\mathcal{F}$ be a linear mapping $\mathcal{F} : \mathbb{C}[x] \to \mathbb{C}[x]$ defined by $x^k \mapsto B_k(x)$ for any integer $k \geq 0$. Then the above notation means that $\mathcal{F}(f) = f(B(x))$. As is clear, for any $f, g \in \mathbb{C}[x]$ and $c \in \mathbb{C}$ it follows that

$$\mathcal{F}(f+g) = \mathcal{F}(f) + \mathcal{F}(g) = (f+g)(B(x))$$

and $\mathcal{F}(cf) = c\mathcal{F}(f) = cf(B(x))$. Based on the fact that the set $\{B_k(x) \mid 0 \leq k \leq n\}$ forms a basis for the vector space $V_n := \{f \in \mathbb{C}[x] \mid \deg f \leq n\}$ with $\dim V_n = n+1$, any polynomial $f \in \mathbb{C}[x]$ with $\deg f = n$ can be uniquely represented by a linear combination of $B_k(x)$, $k = 0, 1, \ldots, n$, over $\mathbb{C}$. Thus, the above $\mathcal{F}$ is bijective.

(c) Various kinds of integral representations for $B_n(x)$ are known in the literature. Among them, we now employ a Mellin-Barnes integral representation such that

$$B_n(x) = \frac{1}{2\pi i} \int_{-c-i\infty}^{-c+i\infty} (x+z)^n \left(\frac{\pi}{\sin(\pi z)}\right)^2 dz \quad (0 < c < 1), \tag{3.2}$$

which is valid for all $n \geq 0$, where the integration path is a vertical line in the complex z-plane, lying within the given strip. For details on this representation, see, e.g., [18], [19, § 2.3–§ 2.6], and Equation 24.7.11 in [9]. In addition, see also the paper [24] by Touchard, in which a Mellin-Barnes integral representation for Bernoulli numbers has been mentioned in detail including a careful proof of

$$B_n = -\frac{\pi i}{2} \int_{-c-i\infty}^{-c+i\infty} \frac{z^n}{\sin^2(\pi z)} dz \quad \text{(cf. Equation (48) in [24])},$$

which explicitly coincides with (3.2) in the case for $x = 0$. We will use (3.2) to prove our main theorem, as stated below.

Theorem 3.1. *Let f, g, and h_q be the polynomials as indicated in* (a). *With the notation as defined in* (b) *we have*

$$f(B(x)) + g(B(x)) = h_q(B(x)) - \sum_{i=0}^{q-1} \frac{d}{dx} f(x+i), \tag{3.3}$$

where the sum on the right-hand side should be interpreted to be zero if $q = 0$.

Proof. A proof of (3.3) is quite easy. Using (1.2) (i) repeatedly q times, we obtain

$$B_n(x+q) = B_n(x) + \sum_{i=0}^{q-1} \frac{d}{dx}(x+i)^n \quad (n \geq 0). \tag{3.4}$$

Applying the integral representation (3.2), it follows from (3.1) and (3.4) that

$$\begin{aligned} h_q(B(x)) &= \frac{1}{2\pi i} \int_{-c-i\infty}^{-c+i\infty} \left(\sum_{k=0}^{n} c_k(q)(x+z)^k\right) \left(\frac{\pi}{\sin(\pi z)}\right)^2 dz \\ &= \frac{1}{2\pi i} \int_{-c-i\infty}^{-c+i\infty} \left(\sum_{k=0}^{n} a_k(x+q+z)^k\right) \left(\frac{\pi}{\sin(\pi z)}\right)^2 dz \\ &\qquad + \frac{1}{2\pi i} \int_{-c-i\infty}^{-c+i\infty} \left(\sum_{k=0}^{n} b_k(x+z)^k\right) \left(\frac{\pi}{\sin(\pi z)}\right)^2 dz \\ &= \sum_{k=0}^{n} a_k B_k(x+q) + \sum_{k=0}^{n} b_k B_k(x) \end{aligned}$$

$$= f(B(x)) + \sum_{i=0}^{q-1} \frac{d}{dx} f(x+i) + g(B(x)),$$

which is the same as (3.3). Thus, the proof of Theorem 3.1 is complete. □

We are now ready in virtue of Lemma 2.3 to derive some recurrence formulas for Bernoulli polynomials involving two different kinds of sums, as initially desired.

Corollary 3.2. *For $n \geq 1$ we have*

$$\begin{aligned}
&\text{(i)} \quad \sum_{k=1}^{n} B_k(x) - \sum_{k=1}^{n} \binom{n+1}{k+1} B_k(x) = -\sum_{k=1}^{n} kx^{k-1} + n;\\
&\text{(ii)} \quad \sum_{k=1}^{n} \frac{B_k(x)}{k} - \sum_{k=1}^{n} \binom{n}{k} \frac{B_k(x)}{k} = -\sum_{k=1}^{n} x^{k-1} + H_n;\\
&\text{(iii)} \quad \sum_{k=0}^{n-1} \frac{B_k(x)}{n-k} + \sum_{k=1}^{n-1} \binom{n}{k} H_k B_k(x) = -\sum_{k=1}^{n-1} \frac{kx^{k-1}}{n-k} + nH_n x^{n-1};\\
&\text{(iv)} \quad \sum_{k=0}^{n-1} (H_n - H_{n-1-k}) B_k(x) - \sum_{k=1}^{n} \binom{n}{k} H_k B_{k-1}(x)\\
&\qquad = -\sum_{k=1}^{n-1} (H_n - H_{n-1-k}) kx^{k-1}.
\end{aligned} \tag{3.5}$$

Proof. All proofs of (i)–(iv) are almost the same as in the basic context; so we wish to give below only a proof of (ii) as a representative one. In Theorem 3.1 we set

$$f(x) = \sum_{k=1}^{n} \frac{x^k}{k} \quad \text{and} \quad g(x) = -\sum_{k=1}^{n} \binom{n}{k} \frac{x^k}{k}$$

and observe the special case where $q = 1$, i.e., $h_1(x) = f(x+1) + g(x)$. Since (2.4) (ii) yields $h_1(x) = H_n$, it can be deduced from (3.3) that

$$f(B(x)) + g(B(x)) = h_1(B(x)) - \frac{d}{dx} f(x) = H_n - \sum_{k=1}^{n} x^{k-1},$$

which is what we wanted in (ii). □

The right-hand sides of the formulas in (3.5) can be written in various equivalent forms taking advantage of the characteristics of harmonic numbers. For example, we may rewrite the right-hand side of (iii) as in the form

$$nx^{n-1} - \sum_{k=1}^{n-1} \frac{1}{k} \binom{n}{k+1} (1-x)^k x^{n-1-k}.$$

It is quite easy to find such a variation. Indeed, we have only to combine (2.2) and the identity obtained by differentiating the whole of (2.4) (iii) with respect to x, namely

$$\sum_{k=1}^{n-1} \frac{kx^{k-1}}{n-k} + \sum_{k=1}^{n} \binom{n}{k} kH_k(x-1)^{k-1} = nH_n x^{n-1}.$$

As an immediate consequence of Corollary 3.2, setting $x = 0$ in (3.5) yields the following recurrence formulas for Bernoulli numbers.

Corollary 3.3. *For $n \geq 1$ we have*

(i) $$\sum_{k=1}^{n} B_k - \sum_{k=1}^{n} \binom{n+1}{k+1} B_k = n-1;$$

(ii) $$\sum_{k=1}^{n} \frac{B_k}{k} - \sum_{k=1}^{n} \binom{n}{k} \frac{B_k}{k} = H_n - 1;$$

(iii) $$\sum_{k=0}^{n-1} \frac{B_k}{n-k} + \sum_{k=1}^{n-1} \binom{n}{k} H_k B_k = \begin{cases} 1 & \textit{for} \quad n=1, \\ -\dfrac{1}{n-1} & \textit{otherwise}; \end{cases}$$

(iv) $$\sum_{k=0}^{n-1} (H_n - H_{n-1-k}) B_k - \sum_{k=1}^{n} \binom{n}{k} H_k B_{k-1} = \begin{cases} 0 & \textit{for} \quad n=1, \\ \dfrac{1-2n}{n(n-1)} & \textit{otherwise.} \end{cases}$$

Note that at least the above (ii) and (iii) are not new. In fact, identity (ii) has been obtained in [3, (1.9)] as a special case of (1.5). Meanwhile, Qin and Lu's formula proved in [20, (2.17)] is essentially the same as (iii), although their appearance is slightly different.

4. Additional Remarks on (1.2)

It is worthwhile observing the fundamental properties for Bernoulli polynomials mentioned in (1.2) from the point of view of a Mellin-Barnes integral representation. As a result, it turns out that the binomial theorem works effectively.

Let x and y be independent variables and consider the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} y^{n-k} x^k.$$

Applying the integral representation (3.2) to this expansion formula by regarding y as a fixed parameter, we obtain

$$B_n(x+y) = \frac{1}{2\pi i} \int_{-c-i\infty}^{-c+i\infty} (x+y+z)^n \left(\frac{\pi}{\sin(\pi z)} \right)^2 dz$$

$$= \frac{1}{2\pi i}\int_{-c-i\infty}^{-c+i\infty}\left(\sum_{k=0}^{n}\binom{n}{k}y^{n-k}(x+z)^k\right)\left(\frac{\pi}{\sin(\pi z)}\right)^2 dz$$
$$= \sum_{k=0}^{n}\binom{n}{k}y^{n-k}\left(\frac{1}{2\pi i}\int_{-c-i\infty}^{-c+i\infty}(x+z)^k\left(\frac{\pi}{\sin(\pi z)}\right)^2 dz\right)$$
$$= \sum_{k=0}^{n}\binom{n}{k}y^{n-k}B_k(x) \quad (0 < c < 1),$$

or, equivalently,

$$B_n(x+y) - B_n(x) = \sum_{k=0}^{n-1}\binom{n}{k}y^{n-k}B_k(x). \tag{4.1}$$

Observing the special case for $y = 1$ allows us to conclude that (1.2) (i) holds if and only if the sum on the right-hand side of (4.1) with $y = 1$ equals nx^{n-1}. Therefore, it was shown that (1.2) (i) is equivalent to (1.3) (ii).

It should be mentioned here that (4.1) (or the one before) tells us the fact that a family of Bernoulli polynomials forms an Appell sequence. Indeed, (4.1) is tantamount to the Appell condition, i.e., $\frac{d}{dx}B_n(x) = nB_{n-1}(x)$ $(n \geq 1)$ with $B_0(x)$ a non-zero constant. For general and specific properties of Appell sequences, see, e.g., [1, 5, 14, 16, 22, 23] and the references therein. More generally, considering the case when $y = q$ (a positive integer) in (4.1) and referring to (3.4), one is able to assert that the following two equations are actually equivalent:

$$\text{(i)} \quad B_n(x+q) - B_n(x) = n\sum_{i=0}^{q-1}(x+i)^{n-1};$$
$$\text{(ii)} \quad \sum_{k=0}^{n-1}\binom{n}{k}q^{n-k}B_k(x) = \sum_{i=0}^{q-1}\frac{d}{dx}(x+i)^n.$$

Needless to say, these equations reflect the characteristics of Appell sequences in the background.

Next we replace x by $-x$ in (4.1) and set $y = 1$ to obtain

$$B_n(1-x) - B_n(-x) = \sum_{k=0}^{n-1}\binom{n}{k}B_k(-x) = n(-x)^{n-1}. \tag{4.2}$$

Since $-B_1 = B_1 + 1$ and $B_k = 0$ if $k \geq 3$ is odd, we may rewrite $B_n(-x)$ as

$$B_n(-x) = \sum_{k=0}^{n}\binom{n}{k}B_k\cdot(-x)^{n-k} = (-1)^n\left(\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k} + \binom{n}{1}x^{n-1}\right)$$
$$= (-1)^n\left(B_n(x) + nx^{n-1}\right),$$

which is just (1.2) (ii). Obviously, combining this with (4.2) leads to (1.2) (iii).

In conclusion of this section, we were able to depart from the binomial theorem and safely arrive at the fundamental properties in (1.2).

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