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A BIJECTION BETWEEN TWO DIFFERENT CLASSES OF PARTITIONS ENUMERATED BY $p_\nu(n)$

A.S. Andersen
Plainedge High School, North Massapequa, New York
asandersen1@gmail.com

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Abstract
In this paper, we give a purely bijective proof that two different partition classes that are both combinatorial interpretations of the partition function $p_\nu(n)$, a partition function related to the third order mock theta function $\nu(q)$, are equinumerous. In doing so, we give a partial solution to a combinatorial problem proposed in a paper by Andrews.

1. Introduction and Notation

Consider the third order mock theta function $\nu(q)$, which was first defined by Watson [**4**] and may be defined as follows:

$$\nu(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}, \tag{1}$$

where the q-Pochhammer symbol $(a;q)_n$ is defined as usual

$$(a;q)_n := \prod_{k=0}^{n-1} (1 - aq^k). \tag{2}$$

The partition function $p_\nu(n)$ may be defined as the partition function for which $\nu(-q)$ is the generating function, and a number of combinatorial interpretations have been given for this partition function. Among these is the number of self-conjugate odd Ferrers graphs of $2n{+}1$ and the number of self-conjugate partitions of $4n{+}1$ into odd parts [**2**, **3**]. Odd Ferrers graphs, introduced by Andrews in [**1**], may be defined as Ferrers graphs in which a 2 is placed in every box, except the surrounding border, where 1s are placed in each box. For example, the following odd Ferrers graph represents the partition 7 + 7 + 3 + 1:

1	1	1	1	1	1	1
1	2	2	2			
1	2					
1						

.

Let $\mathcal{O}_{2n+1}$ be the set of self-conjugate odd Ferrers graphs for $2n+1$, let $\mathcal{S}_{4n+1}$ be the set of self-conjugate partitions of $4n+1$ into odd parts, let $\mathcal{O} = \cup_{n>0}\mathcal{O}_{2n+1}$ and let $\mathcal{S} = \cup_{n>0}\mathcal{S}_{4n+1}$. The following theorem has previously been proven through non-bijective means [**2**].

Theorem 1. *For all n, the number of partitions in the class $\mathcal{O}_{2n+1}$ is equal to the number of partitions in the class $\mathcal{S}_{4n+1}$.*

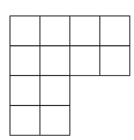
We will give a purely bijective proof of this theorem by describing a bijection ϕ such that $\phi(\lambda) = \mu$, where λ and μ are both partitions, $\lambda \in \mathcal{O}_{2n+1}$, and $\mu \in \mathcal{S}_{4n+1}$, and use the case where $\lambda = 3 + 5 + 3$, representable as the following odd Ferrers graph

1	1	1
1	2	2
1	2	

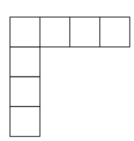
as an example (note that in this example case, $\lambda \in \mathcal{O}_{11}$, and that $\mu \in \mathcal{S}_{21}$). In doing so, we give a partial solution to the combinatorial challenge proposed by Andrews [**2**] asking for bijections between the various classes of partitions enumerated by $p_\nu(n)$.

2. A Bijection Between $\mathcal{O}_{2n+1}$ and $\mathcal{S}_{4n+1}$

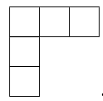
Consider the fact that the Ferrers diagrams of self-conjugate partitions may be thought of as being made up of "hooks" of other self-conjugate partitions in which every part other than the greatest part is equal to 1. For example, the Ferrers digram of the self-conjugate partition 4 + 4 + 2 + 2



can be thought of as consisting of the following"hooks":



and



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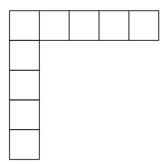
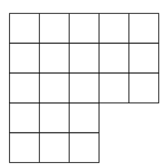
Let h_i denote the ith"hook" in a self-conjugate partition π, where $i > 0$. Note that, where $|\pi|$ may denote the sum of the parts of π, where $|h_i|$ may denote the sum of the parts in each hook in the Ferrers diagram of π, and where n may denote the number of hooks in π, that $\sum_{i=1}^{n} |h_i| = |\pi|$. Additionally, for $\lambda \in \mathcal{O}$, let $t = \sum_{i=2}^{n} |h_i|$, or the sum of the 2s in the odd Ferrers diagram. We will distinguish between the hooks in λ and the hooks in μ by using h_i to denote the ith hook in the former and η_i to denote the ith hook in the latter. The map $\phi(\lambda) = \mu$ may be described as follows.

Step 1: Create η_1 by creating a hook with the largest part equal to $|h_1|$. Note that $|\eta_1| = 2|h_1| - 1$. For the example case for λ given above, η_1 would be the following.



Step 2: For each h_i where $i > 1$, create η_{2i-2} such that $|\eta_{2i-2}| = |h_i| + 1$, and η_{2i-1} such that $|\eta_{2i-1}| = |h_i| - 1$. For example, in the example case of λ given above, $|h_2|$ $= 6$, so we create η_2 and η_3 such that $|\eta_2| = 7$ and $|\eta_3| = 5$, and since the number of hooks in λ is equal to 2, the creation of these hooks completes the bijection resulting in the following partition, that being $5 + 5 + 5 + 3 + 3$.



The map described evidently always results in a self-conjugate partition. The map described also always results in a partition of $4n+1$, because in creating η_1 we create a partition of size $2h_1 - 1$, and in adding every η_i such that $i > 1$, we add $2t$ to this partition, thus making a partition of size $2(h_1 + t) - 1$. We know that $h_1 + t = |\lambda| = 2n + 1$, so substituting $2n + 1$ for $h_1 + t$ in the previous expression reveals that the sum of the parts in the newly created partition is always equal to $4n+1$. Additionally, we know that the newly created partition is always a partition into odd parts because it always creates a partition in which the greatest part of η_1 is odd, the number of hooks is odd, and in which the greatest part of each hook alternates in parity, where the greatest part of η_{2i-2} is always one greater than the greatest part of η_{2i-1}. The inverse map is obvious, so ϕ is a bijection, and thus $|\mathcal{O}_{2n+1}| = |\mathcal{S}_{4n+1}|$ for all n.

3. Further Remarks

Recall the natural bijection that exists between the class of self-conjugate partitions of n and the class of partitions of n into distinct odd parts that maps a self-conjugate partition onto a partition into distinct odd parts by making the sum of the parts in each of the hooks in the self conjugate partition into a part in the newly created partition. Where $\mathcal{D}_{2n+1}$ may denote the set of partitions of $2n+1$ into distinct parts in which there is one odd part which is greater than half the greatest even part and every other part is even and is of the form $4k+2$ where $k \in \mathbb{N}$, and where $\mathcal{DO}_{4n+1}$ may denote the set of partitions of $4n+1$ into an odd number of distinct odd parts such that, when ordered from largest to smallest, the parts alternate between being of the form $4k+1$ and being of the form $4k+3$ where again $k \in \mathbb{N}$, an analogous bijection exists between $\mathcal{O}_{2n+1}$ and $\mathcal{D}_{2n+1}$ and between $\mathcal{S}_{4n+1}$ and $\mathcal{DO}_{4n+1}$. Thus, the bijection given above induces one between $\mathcal{D}_{2n+1}$ and $\mathcal{DO}_{4n+1}$.

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