

**ALGEBRAIC PROPERTIES OF A HYPERGRAPH LIFTING MAP****Mark Budden**

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Abstract

Recent work in hypergraph Ramsey theory has involved the introduction of a “lifting map” that associates a certain 3-uniform hypergraph to a given graph, bounding cliques in a predictable way. In this paper, we interpret the lifting map as a linear transformation. This interpretation allows us to use algebraic techniques to prove several structural properties of the lifting map. We conclude by providing new lower bounds for certain t -color 3-uniform hypergraph Ramsey numbers.

1. Introduction

In [1], a lifting map $\varphi : \mathcal{G}_2 \rightarrow \mathcal{G}_3$ was described that assigned to each graph a unique 3-uniform hypergraph. Here, \mathcal{G}_2 denotes the set of all graphs of order at least 3 and \mathcal{G}_3 is the set of all 3-uniform hypergraphs of order at least 3. Both a graph G and its image $\varphi(G)$ share the same vertex set, and an unordered 3-tuple abc forms a hyperedge in $\varphi(G)$ if and only if the subgraph of G induced by $\{a, b, c\}$ contains an odd number of edges.

Also in [1], the lifting φ was shown to preserve complements (i.e., $\varphi(\overline{G}) = \overline{\varphi(G)}$) and the way in which φ lifted to complete graphs was analyzed. Specifically, it was shown that if $\varphi(G)$ contains a complete hypergraph with vertices x_1, x_2, \dots, x_n , then the subgraph of G induced by $\{x_1, x_2, \dots, x_n\}$ is the disjoint union of at most 2 complete graphs (including the possibility that it is complete). The preservation of complements allows one to consider the lifting map as a function on 2-colored complete graphs. As a result, these properties were then used to provide new lower bounds for certain 3-uniform hypergraph Ramsey numbers.

In [3], a generalization of φ was described that allowed graphs to be lifted to r -uniform hypergraphs. In this variation, denoted $\varphi^{(r)}$, a hyperedge $x_1 x_2 \cdots x_r$ is formed in the image of $\varphi^{(r)}$ if and only if the subgraph of G induced by $\{x_1, x_2, \dots, x_r\}$ is the disjoint union of at most $r - 1$ complete graphs. Like φ , it was shown that $\varphi^{(r)}$ lifted to complete subhypergraphs in a predictable way, but unfortunately, complements were no longer preserved when $r > 3$ making it ineffective as a tool in Ramsey theory. Rather, [3] included an application involving Turán numbers.

The present paper sets out to gain a better understanding of φ by recognizing it as a linear transformation between certain vector spaces over the finite field of order 2. This construction is described in Section 2 and allows one to extend the lifting map to arbitrary uniformities (in a different way than $\varphi^{(r)}$). It also provides a framework for employing standard algebraic techniques to prove structural properties of φ . Section 3 continues this investigation for the special case of lifting graphs to r -uniform hypergraphs.

In [2], another generalization of φ was considered in which more than 2 colors were used. The problem of deciding how a rainbow triangle should lift led to a focus on Gallai colorings (those lacking rainbow triangles). In Section 4, we describe a method for using φ to obtain lower bounds for certain t -color Ramsey numbers and discuss the limitations of further generalizing the lifting map as a linear transformation.

2. The Lifting Map as a Linear Transformation

In order to establish the lifting map as a linear transformation, we must first formalize the terminology and background surrounding the objects to be studied. An r -uniform hypergraph $H = (V(H), E(H))$ consists of a nonempty set of vertices $V(H)$ and a set of hyperedges $E(H)$, whose elements are different unordered r -tuples of distinct vertices from $V(H)$. The complete r -uniform hypergraph of order n is denoted by $K_n^{(r)}$ and consists of n vertices, every r -element subset of which forms a hyperedge. When $r = 2$, we simplify the notation $K_n^{(2)}$ and just write K_n .

Throughout this paper, denote by \mathbb{F} the finite field of order 2 (whose elements we identify with 0 and 1). An \mathbb{F} -hyperedge coloring of an r -uniform hypergraph H

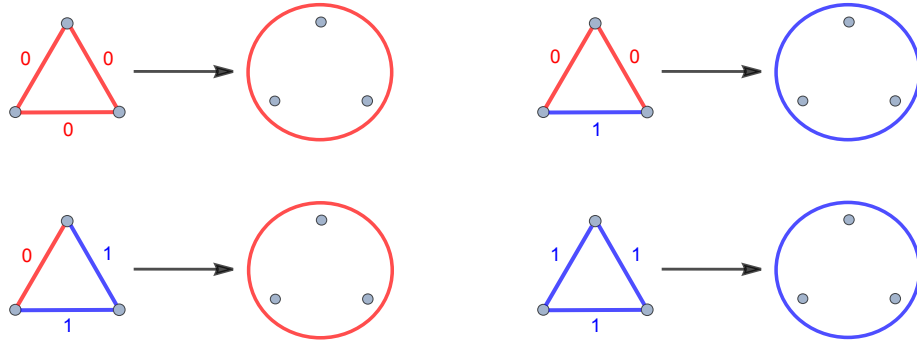


Figure 1: The lifting map as the linear transformation $\Psi_3^{(2,3)}$. Here, we identify the color red with 0 and the color blue with 1.

is a map $f : E(H) \rightarrow \mathbb{F}$. Denote the set of all \mathbb{F} -hyperedge colorings of $K_n^{(r)}$ by $\mathcal{H}_n^{(r)}$ and observe that it forms a vector space over \mathbb{F} under the operations

$$(f + g)(e) = f(e) + g(e) \quad \text{and} \quad (\alpha f)(e) = \alpha f(e),$$

where $e \in E(K_n^{(r)})$, $\alpha \in \mathbb{F}$, and $f, g \in \mathcal{H}_n^{(r)}$. A basis for $\mathcal{H}_n^{(r)}$ can be formed using the \mathbb{F} -hyperedge colorings

$$f_{e'}(e) = \begin{cases} 1 & \text{if } e = e' \\ 0 & \text{if } e \neq e', \end{cases}$$

where $e' \in E(K_n^{(r)})$. It follows that $\dim_{\mathbb{F}}(\mathcal{H}_n^{(r)}) = \binom{n}{r}$.

If T is a set, then denote by T^r the set of all r -element subsets of T . For $2 \leq s < r \leq n$, define the lifting $\Psi_n^{(s,r)} : \mathcal{H}_n^{(s)} \rightarrow \mathcal{H}_n^{(r)}$ by

$$(\Psi_n^{(s,r)} f)(e) = \sum_{e' \in e^s} f(e'),$$

where $f \in \mathcal{H}_n^{(s)}$ and $e \in E(K_n^{(r)})$. It can be seen in Figure 1 that $\Psi_n^{(2,3)}$ corresponds with the lifting described in [1] and we leave it as an exercise for the reader to check that $\Psi_n^{(s,r)}$ is a linear transformation in general. Realizing this map as a linear transformation elucidates some of its properties, the first of which involves the time required to determine if a given hyperedge coloring is in the image of such a map.

Proposition 1. For $g \in \mathcal{H}_n^{(r)}$, there exists a polynomial time algorithm to determine whether or not $g \in \text{Im}(\Psi_n^{(s,r)})$.

Proof. Apply Gaussian Elimination to solve the system $\Psi_n^{(s,r)} f = g$. □

In the next theorem, we consider the sum

$$S_n^{(r)}(g) := \sum_{e \in K_n^{(r)}} g(e), \quad g \in \mathcal{H}_n^{(r)},$$

which is the linear functional that determines the parity (mod 2) of the number of hyperedges in the coloring g that are assigned color 1.

Theorem 1. *If $n \geq r > s \geq 2$ and $g = \Psi_n^{(s,r)} f$, then*

$$S_n^{(r)}(g) = \begin{cases} 0 & \text{if } \binom{n-s}{r-s} \text{ is even} \\ S_n^{(s)}(f) & \text{if } \binom{n-s}{r-s} \text{ is odd.} \end{cases}$$

Proof. Let $g = \Psi_n^{(s,r)} f$. By definition, the sum $S_n^{(r)}(g)$ in Theorem 1 becomes

$$\sum_{e \in E(K_n^{(r)})} g(e) = \sum_{e \in E(K_n^{(r)})} \sum_{e' \in e^s} f(e').$$

Within this sum, $f(e')$ occurs $\binom{n-s}{r-s}$ times, corresponding to the number of hyperedges $e \in E(K_n^{(r)})$ that contain e' . The assumption the $\binom{n-s}{r-s}$ is even implies that the sum is 0. In the case where $\binom{n-s}{r-s}$ is odd, observe that

$$S_n^{(r)}(g) = \sum_{e \in E(K_n^{(r)})} \sum_{e' \in e^s} f(e'),$$

with $f(e')$ occurring $\binom{n-s}{r-s}$ times. It follows that the sum simplifies to

$$\sum_{e' \in E(K_n^{(s)})} f(e')$$

in this case. □

Define the *complement* of an \mathbb{F} -hyperedge coloring $f : E(H) \rightarrow \mathbb{F}$ to be the \mathbb{F} -hyperedge coloring $\bar{f} : E(H) \rightarrow \mathbb{F}$ such that for all $e \in E(H)$,

$$\bar{f}(e) = \begin{cases} 0 & \text{if } f(e) = 1 \\ 1 & \text{if } f(e) = 0. \end{cases}$$

Theorem 2. *Let $f \in \mathcal{H}_n^{(s)}$. Then*

$$\Psi_n^{(s,r)} \bar{f} = \begin{cases} \overline{\Psi_n^{(s,r)} f} & \text{if } \binom{r}{s} \text{ is odd} \\ \Psi_n^{(s,r)} f & \text{if } \binom{r}{s} \text{ is even.} \end{cases}$$

Proof. Let $f \in \mathcal{H}_n^{(s)}$. Note that each r -uniform hyperedge $e \in E(K_n^{(r)})$ corresponds with a selection of r vertices in $K_n^{(s)}$. Let H be the subhypergraph induced by e in $K_n^{(s)}$ and observe that H has $\binom{r}{s}$ s -uniform hyperedges. Denote the subhypergraph of H spanned by all hyperedges of color 0 in f by H_0 and the subhypergraph of H spanned by all hyperedges of color 1 in f by H_1 . Define the subhypergraphs $\overline{H_0}$ and $\overline{H_1}$ similarly under \overline{f} and note that $\overline{H_0} \cong H_1$ and $\overline{H_1} \cong H_0$. By definition, the r -uniform hyperedge e will be given color 1 in $\Psi_{2,n}^{(s,r)} f$ if $|E(H_1)|$ is odd and color 0 if $|E(H_1)|$ is even. In the case where $\binom{r}{s}$ is odd, exactly one of $|E(H_1)|$ and $|E(H_0)|$ must be odd. Without loss of generality, assume $|E(H_1)|$ is odd. This means that $|E(\overline{H_1})|$ is even, and while the hyperedge e is assigned color 1 in $\Psi_n^{(s,r)} \overline{f}$, e is assigned color 0 in $\Psi_n^{(s,r)} f$. It follows that

$$\Psi_n^{(s,r)} \overline{f} = \overline{\Psi_n^{(s,r)} f}$$

when $\binom{r}{s}$ is odd. In the case where $\binom{r}{s}$ is even, either $|E(H_1)|$ and $|E(H_0)|$ are both odd or they are both even. Likewise, either $|E(H_1)|$ and $|E(\overline{H_1})|$ are both odd or they are both even. This means that the hyperedge e receives the same color in $\Psi_n^{(s,r)} \overline{f}$ as it receives in $\Psi_n^{(s,r)} f$, proving that

$$\Psi_n^{(s,r)} \overline{f} = \Psi_n^{(s,r)} f$$

whenever $\binom{r}{s}$ is even. □

Theorem 2 shows that the property of preserving complements that was so useful in the original lifting also holds for all cases in which $\binom{r}{s}$ is odd. In the following three theorems of this section, we now focus on the case where $s = r - 1$.

Theorem 3. *Let $r \geq 3$ and assume that $\Psi_n^{(r-1,r)} f = \Psi_n^{(r-1,r)} g$. If $f \neq g$, then f and g differ by at least $n - r + 2$ hyperedge colors.*

Proof. Assume that $\Psi_n^{(r-1,r)} f = \Psi_n^{(r-1,r)} g$ and $f \neq g$. Then some hyperedge $x_1 x_2 \cdots x_{r-1}$ in $K_n^{(r-1)}$ receives a different color under g than it does under f . The hyperedge $x_1 x_2 \cdots x_{r-1}$ is contained in exactly $n - (r - 1)$ r -tuples, each of which contains a distinct vertex from the set

$$V(K_n^{(r)}) - \{x_1, x_2, \dots, x_{r-1}\} = \{y_1, y_2, \dots, y_{n-(r-1)}\}.$$

Retaining the colors of these r -tuples under $\Psi_n^{(r-1,r)}$ requires at least $n - (r - 1)$ additional hyperedges in $K_n^{(r-1)}$ (each of which includes a single element from $\{y_1, y_2, \dots, y_{n-(r-1)}\}$ and some selection of $r - 2$ vertices from $\{x_1, x_2, \dots, x_{r-1}\}$) be colored differently under g than under f . Hence, f and g differ by at least $n - r + 2$ hyperedge colors. □

To see that this theorem is optimal, consider the case of $\Psi_5^{(2,3)}$. By Theorem 4 of [1], a $K_1 \dot{\cup} K_4$ and a $K_2 \dot{\cup} K_3$ in color 1 both map to a $K_5^{(3)}$ in color 1. It can be checked that these two preimages differ by exactly 4 edge colors, the minimum number implied by Theorem 3. The following theorem was motivated by Theorem 2.1 of [3].

Theorem 4. *Let $r \geq 3$ be odd and consider a coloring $g = \Psi_n^{(r-1,r)} f$. If the image of a subhypergraph K of $K_n^{(r-1)}$ under g is complete in some color, then K consists of at most $r - 1$ connected components in that color.*

Proof. Suppose false, then K consists of at least r components. Select a single vertex from r of the components and consider the resulting hyperedge. In the case where the components have color 1, all hyperedges induced by these vertices will have color 0 and the resultant hyperedge will be given color 0. By our assumption that K is monochromatic, all other resultant hyperedges must receive color 0 as well. Now consider selecting $r - 1$ adjacent vertices from one of the components and a single vertex from one other. These vertices will induce a single $r - 1$ -uniform hyperedge in color 1, resulting in an r -uniform hyperedge in color 1. This gives us a contradiction. In the case where the components have color 0, all edges induced by these vertices will have color 1. Since r is odd, we have an odd number of $r - 1$ -uniform hyperedges in color 1, which will result in an r -uniform hyperedge in color 1. As before, select $r - 1$ vertices joined by a hyperedge in color 0 and a single vertex from one other component. These vertices induce $r - 1$ hyperedges in color 1 which, because $r - 1$ is even, result in an r -uniform hyperedge in color 0. Once again, this contradicts the assumption that K is monochromatic, proving that K consists of at most $r - 1$ connected components. \square

The following theorem is a generalization of Theorem 7 of [1].

Theorem 5. *Let $r \geq 3$ be odd and $n \geq r + 1$. If $m > r$ is an integer and $g = \Psi_n^{(r-1,r)} f$, then g does not contain $K_m^{(r)} - e$ as an induced subhypergraph.*

Proof. Suppose that $K_m^{(r)} - e$ does occur as an induced subhypergraph in some hyperedge coloring $\Psi_n^{(r-1,r)} f$. Then the coloring necessarily contains an induced subhypergraph isomorphic to $K_{r+1}^{(r)} - e$. First, consider the case where the $K_{r+1}^{(r)} - e$ is in color 1 (so that e has color 0). There must necessarily be an even number of $(r - 1)$ -element subsets of e that are colored 1 in f and an odd number that are colored 0 (since r is odd). For any other hyperedge in the $K_{r+1}^{(r)} - e$ other than e , there are an odd number of $(r - 1)$ -element subsets that are colored 1 and an even number that are colored 0. Summing over all r -element subsets, we obtain an odd number of $(r - 1)$ -elements subsets in color 1. However, note that each $(r - 1)$ -element subset occurs in exactly two r -element subsets. It follows that this sum should be 0, giving a contradiction. The case where $K_{r+1}^{(r)} - e$ is in color 0 is the same, but with 0 and 1 switched. \square

3. The Lifting $\Psi_n^{(2,r)}$

Throughout this section, we consider the lifting of graphs to r -uniform hypergraphs (the case $s = 2$). Our investigation focuses on the graphs that lift to complete and empty subhypergraphs (in color 1), so to simplify the statements of our theorems, we introduce some new terminology. Let G be a graph of order n and let G' be a subgraph of G induced by a selection of r vertices such that $2 < r \leq n$. If G' always has an odd number of edges, regardless of the specific vertices chosen, G is called r -complete. If G' always has an even number of edges, regardless of the specific vertices chosen, G is called r -void. If G is neither r -complete nor r -void, it is called r -neutral. Under $\Psi_n^{(2,r)}$, an r -complete graph in color 1 will always lift to a complete r -uniform hypergraph in color 1, while an r -void graph will always lift to a complete r -uniform hypergraph in color 0.

Theorem 6. *Given an odd $r \geq 3$, a complete bipartite graph of order at least r is always r -void.*

Proof. Let $r \geq 3$ be odd and $n \geq r$ such that $K_{s,t}$ is a complete bipartite graph of order n . The vertex set of $K_{s,t}$ is the disjoint union of partite sets V_1 and V_2 , having cardinalities $|V_1| = s$ and $|V_2| = t$, where $n = s + t$. Any selection of r vertices from $K_{s,t}$ results in u vertices selected from V_1 and $r - u$ vertices selected from V_2 . Since r is odd, exactly one of u and $r - u$ must be odd. Without loss of generality, let u be odd and $r - u$ be even. Since $K_{s,t}$ is complete bipartite, each of the u vertices selected from V_1 must share an edge with each of the $r - u$ vertices selected from V_2 . This means that the subgraph induced by these vertices must have $u \cdot (r - u)$ edges. As $r - u$ is even, $u \cdot (r - u)$ must also be even. □

Lemma 1. *Let r and u be positive integers such that r is even and $r > u$.*

1. *If $r \equiv 0 \pmod{4}$, then $\binom{u}{2} + \binom{r-u}{2} \equiv u \pmod{2}$.*
2. *If $r \equiv 2 \pmod{4}$, then $\binom{u}{2} + \binom{r-u}{2} \equiv (u + 1) \pmod{2}$.*

Proof. We run through cases, based on the value of r modulo 4.

Case 1: Suppose that $r \equiv 0 \pmod{4}$. When u is even (i.e. $u \equiv 0, 2 \pmod{4}$), then $\binom{u}{2}$ and $\binom{r-u}{2}$ have the same parity, implying that $\binom{u}{2} + \binom{r-u}{2}$ is even. When u is odd (i.e., $u \equiv 1, 3 \pmod{4}$), then $\binom{u}{2}$ and $\binom{r-u}{2}$ have different parities, implying that $\binom{u}{2} + \binom{r-u}{2}$ is odd.

Case 2: Suppose that $r \equiv 2 \pmod{4}$. When u is even (i.e., $u \equiv 0, 2 \pmod{4}$), then $\binom{u}{2}$ and $\binom{r-u}{2}$ have different parities, implying that $\binom{u}{2} + \binom{r-u}{2}$ is odd. When u is odd (i.e., $u \equiv 1, 3 \pmod{4}$), then $\binom{u}{2}$ and $\binom{r-u}{2}$ have the same parity, implying that $\binom{u}{2} + \binom{r-u}{2}$ is even. □

Theorem 7. *Let G be the disjoint union of two complete subgraphs of orders s and t with $r < s + t$. Then the following statements hold.*

1. *If r is even, then G is r -neutral.*
2. *If $r \equiv 1 \pmod{4}$, then G is r -void.*
3. *If $r \equiv 3 \pmod{4}$, then G is r -complete.*

Proof. First, consider the case where r is even. We select $u < r$ vertices from the K_s and $r - u$ vertices from the K_t . The subgraph induced by these r vertices will contain $\binom{u}{2} + \binom{r-u}{2}$ edges. Next, select $u + 1$ vertices from the K_s and $r - u - 1$ vertices from the K_t . The subgraph induced by these r vertices will contain $\binom{u+1}{2} + \binom{r-u-1}{2}$ edges. By Lemma 1, these two selections will yield differing numbers of edges modulo 2. Therefore G must be r -neutral. Now consider the case where r is odd. By Theorem 6, G is r -void, so by Theorem 2, if $r \equiv 1 \pmod{4}$, G is r -void and if $r \equiv 3 \pmod{4}$, G is r -complete. \square

We conclude this section with a result concerning the original lifting map $\Psi_n^{(2,3)}$.

Theorem 8. *The lifting $\Psi_n^{(2,3)}$ is a 2^{n-1} -to-one mapping.*

Proof. The fact that every $g \in \text{Im}(\Psi_n^{(2,3)})$ has the same number k of preimages follows from the linearity of $\Psi_n^{(2,3)}$. In order to determine the value of k , we select an element in the image whose preimages we can easily count. Specifically, consider the element in $\mathcal{H}_n^{(3)}$ that maps all hyperedges to color 0. The induced subhypergraph in color 0 is isomorphic to $K_n^{(3)}$, and by Theorem 4 of [1], the elements in $\mathcal{H}_n^{(2)}$ that map to this element are those in which color 0 is given to subgraphs that are complete of order n or are the disjoint union of two complete subgraphs, whose orders add to n . When n is even, the possibilities are

$$K_n, K_1 \dot{\cup} K_{n-1}, K_2 \dot{\cup} K_{n-2}, \dots, K_{n/2-1} \dot{\cup} K_{n/2+1}, K_{n/2} \dot{\cup} K_{n/2},$$

with these cases occurring

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n/2-1}, \frac{1}{2} \binom{n}{n/2}$$

times, respectively. When n is odd, the possibilities are

$$K_n, K_1 \dot{\cup} K_{n-1}, K_2 \dot{\cup} K_{n-2}, \dots, K_{(n-1)/2} \dot{\cup} K_{(n+1)/2},$$

with these cases occurring

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{(n-1)/2}$$

times, respectively. Applying the identity

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n,$$

along with the property $\binom{n}{k} = \binom{n}{n-k}$ to each of these cases, we obtain the statement of the theorem. \square

4. Some Applications of the Lifting Map to Ramsey Theory

At this point, a natural question to consider is whether or not this interpretation of the lifting map as a linear transformation allows us to generalize liftings to more than two colors, without having to restrict our attention to Gallai colorings (as was done in [2]). For example, one could define the lifting map $\Psi_{3,n}^{(2,3)}$ by identifying three colors with the elements in the finite field $\mathbb{F}_3 = \{0, 1, 2\}$ and assigning a hyperedge xyz the color given by adding the colors of xy , yz , and xz in \mathbb{F}_3 . The top image in Figure 2 shows how one particular 3-coloring of K_4 lifts under this generalization. Note that the green hyperedge bcd arises from a K_3 in the preimage that contains a single blue edge and two green edges. Under the usual lifting map, we would expect this set of vertices to lift to a blue hyperedge. Since this does not agree with the structure inherent in the original 2-color lifting, we make the following observation.

Remark 1. For finite fields having order greater than 2, the generalization of the lifting map as a linear transformation does not lift cliques in the predictable way that the 2-color lifting map does. In particular, Theorem 4 of [1] does not apply.

Rather than use the linear transformation generalization of the lifting map, we offer a different t -color generalization, which we denote by φ_t^* , that is similar to the lifting described in [2]. Specifically, φ_t^* takes a t -colored complete graph and maps it to a t -colored complete 3-uniform hypergraph by sending all rainbow K_3 -subgraphs to a hyperedge using the first color, and all other K_3 -subgraphs map to a hyperedge in the usual color described by the original lifting φ . The second image in Figure 2 demonstrates how φ_3^* lifts a particular 3-coloring of K_4 . The benefit of this generalization is that the first color lifts exactly how one would expect. For the other colors, the resulting images are subhypergraphs of the images of the usual 2-color lifting. This means that if the usual lifting wouldn't produce a particular subhypergraph in a given color, neither will this generalization. This allows us to make use of φ_t^* to construct lower bounds for certain multicolor hypergraph Ramsey numbers.

Recall that if H_1, H_2, \dots, H_t are r -uniform hypergraphs, then the *Ramsey number* $R(H_1, H_2, \dots, H_t; r)$ is the least positive integer p such that every t -coloring of the hyperedges of $K_p^{(r)}$ results in a subhypergraph isomorphic to H_i spanned by

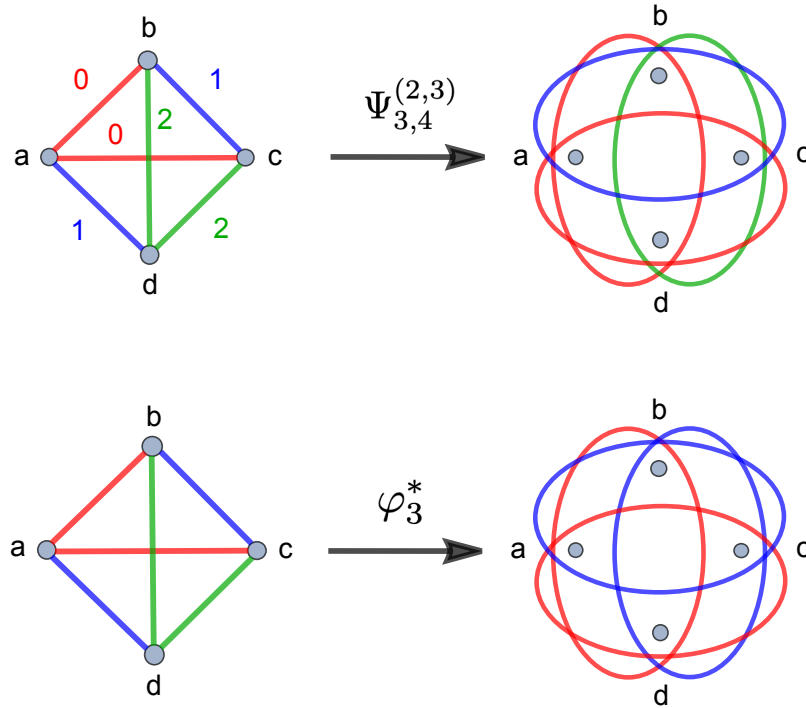


Figure 2: Two different generalizations of the lifting map to 3-colorings of K_4 . The top image shows how $\Psi_{3,4}^{(2,3)}$ lifts a particular 3-coloring of K_4 , while the bottom image shows how φ_3^* lifts the same coloring.

hyperedges in color i , for some $1 \leq i \leq t$. The following theorem generalizes Theorem 9 of [1] to t -colored Ramsey numbers.

Theorem 9. *Let $s_i \geq 3$ for all $1 \leq i \leq t$. Then*

$$R(K_{2s_1-1}^{(3)} - e, K_{2s_2-1}^{(3)} - e, \dots, K_{2s_t-1}^{(3)} - e; 3) \geq R(K_{s_1}, K_{s_2}, \dots, K_{s_t}; 2).$$

Proof. Let $p = R(K_{s_1}, K_{s_2}, \dots, K_{s_t}; 2)$ and consider a t -coloring of K_{p-1} that lacks a monochromatic copy of K_{s_i} in color i , for all $1 \leq i \leq t$. Then the image of this complete graph under φ_t^* sends the subgraph spanned by edges in each color to a subhypergraph of its image under φ as if only two colors were used. By Theorems 4 and 7 of [1], the resulting coloring avoids a copy of $K_{2s_i-1}^{(3)} - e$ in color i , for all $1 \leq i \leq t$, completing the proof of the theorem. \square

The following theorem generalizes Theorem 10 of [1].

Theorem 10. *Let $q \geq 3$ and $s_i \geq 3$, for all $1 \leq i \leq t$. Then*

$$R(K_{2s_1-1}^{(3)} - e, K_{2s_2-1}^{(3)} - e, \dots, K_{2s_t-1}^{(3)} - e, K_5^{(3)}, K_{q+1}^{(3)} - e; 3) > q(R(K_{s_1}, K_{s_2}, \dots, K_{s_t}; 2) - 1).$$

Proof. If $p = R(K_{s_1}, K_{s_2}, \dots, K_{s_t}; 2)$, then by Theorem 9, we can construct a t -colored $K_{p-1}^{(3)}$ that avoids a copy of $K_{2s_i-1}^{(3)} - e$ in color i , for all $1 \leq i \leq t$. Consider the disjoint union of q copies of this t -colored $K_{p-1}^{(3)}$, and label the copies V_1, V_2, \dots, V_q . Color the hyperedges that have one vertex in some V_i and the other two vertices in some V_j ($i \neq j$) using a $(t + 1)^{th}$ color and color the hyperedges that have all vertices coming from distinct V_i using a $(t + 2)^{th}$ color. It is easily confirmed that the largest complete hypergraph in the $(t + 2)^{th}$ color uses at most one vertex from any given V_i and adding in another vertex to such a hypergraph is lacking more than one hyperedge in the $(t + 2)^{th}$ color (hence, avoiding a $K_{q+1}^{(3)} - e$). The largest complete hypergraph in the $(t + 1)^{th}$ color uses at most two vertices from at most two distinct V_i . It follows that no $K_5^{(3)}$ exists in the $(t + 1)^{th}$ color and no $K_{q+1}^{(3)} - e$ exists in the $(t + 2)^{th}$ color. \square

$$\begin{aligned} R_3(K_3) = 17 &\implies R(K_5^{(3)} - e, K_5^{(3)} - e, K_5^{(3)} - e, K_5^{(3)}, K_{q+1}^{(3)} - e; 3) > 16q, \\ R_3(K_4) \geq 128 &\implies R(K_7^{(3)} - e, K_7^{(3)} - e, K_7^{(3)} - e, K_5^{(3)}, K_{q+1}^{(3)} - e; 3) > 127q, \\ R_3(K_5) \geq 417 &\implies R(K_9^{(3)} - e, K_9^{(3)} - e, K_9^{(3)} - e, K_5^{(3)}, K_{q+1}^{(3)} - e; 3) > 416q, \\ R_3(K_6) \geq 1070 &\implies R(K_{11}^{(3)} - e, K_{11}^{(3)} - e, K_{11}^{(3)} - e, K_5^{(3)}, K_{q+1}^{(3)} - e; 3) > 1069q, \\ R_3(K_7) \geq 3214 &\implies R(K_{13}^{(3)} - e, K_{13}^{(3)} - e, K_{13}^{(3)} - e, K_5^{(3)}, K_{q+1}^{(3)} - e; 3) > 3213q, \\ R_3(K_8) \geq 6132 &\implies R(K_{15}^{(3)} - e, K_{15}^{(3)} - e, K_{15}^{(3)} - e, K_5^{(3)}, K_{q+1}^{(3)} - e; 3) > 6133q, \\ R_3(K_9) \geq 14081 &\implies R(K_{17}^{(3)} - e, K_{17}^{(3)} - e, K_{17}^{(3)} - e, K_5^{(3)}, K_{q+1}^{(3)} - e; 3) > 14080q. \end{aligned}$$

Figure 3: Lower bounds implied by Theorem 10 and the 3-color diagonal Ramsey number lower bounds given in Table X of [5].

Using the 3-color lower bounds given in Table X of Section 6.1 of Radziszowski’s dynamic survey [5], Theorem 10 implies the lower bounds listed in Figure 3. Here, we use the usual notation $R_3(G)$ to denote the 3-color graph Ramsey number $R(G, G, G; 2)$. Many similar inequalities follow from other known multicolor Ramsey number lower bounds, which can be found in Sections 6.1 and 6.2 of [5]. While these are unlikely to be strong lower bounds in general, they give rare instances of explicit multicolor Ramsey number lower bounds for complete hypergraphs missing a single hyperedge. These theorems also fill a gap left by the “Stepping up” Lemma

of Erdős and Hajnal (see [4]), which used lower bounds for r -uniform hypergraph Ramsey numbers to construct lower bounds for $(r+1)$ -uniform hypergraph Ramsey numbers, but starting with $r = 3$.

In conclusion, we leave open the problem of studying the linear transformation generalization of the lifting map over general finite fields. At this time, it is unclear how such generalizations will lift complete subgraphs when three or more colors are used.

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