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WEIGHTED PARTITIONS AND GENERALIZED R-LAH NUMBERS

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Abstract

Using weighted ordered partitions, we provide a new combinatorial interpretation for the two-parameter polynomial generalization of the r-Lah numbers. Moreover, by the inclusion-exclusion principle, we give combinatorial proofs for an explicit formula and some combinatorial properties. Finally, we provide an expression involving symmetric functions.

1. Introduction

The r-Lah numbers, denoted $\left\lfloor{n \atop k}\right\rfloor_r$, are coefficients in the expression of the shifted rising and falling factorials of x, see [1],

$$(x+2r)^{\overline{n}} = \sum_{k=0}^{n} \left\lfloor{n \atop k}\right\rfloor_r x^{\underline{k}}, \tag{1}$$

with $x^{\overline{n}} = x(x+1)(x+2)\cdots(x+n-1)$ and $x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1)$.

Let $[n] := \{1,2,\ldots,n\}$ and $[k,n] := \{k,k+1,\ldots,n\}$. The r-Lah numbers are interpreted combinatorially as the number of partitions of $[n+r]$ into $k+r$ non-empty lists (ordered blocks) such that r elements are in distinct lists. They satisfy the following recurrence relation:

$$\left\lfloor{n+1 \atop k}\right\rfloor_r = \left\lfloor{n \atop k-1}\right\rfloor_r + (n+k+2r)\left\lfloor{n \atop k}\right\rfloor_r, \tag{2}$$

with initial values $\genfrac{\lfloor}{\rfloor}{0pt}{}{n}{k}_r = \delta_{k,r}$ for $n = r$ and $\genfrac{\lfloor}{\rfloor}{0pt}{}{n}{k}_r = 0$ for $n < k$ or $k < r$.

Many properties of the r-Lah numbers can be found in [1, 3, 5, 8]. Recently, Shattuck [9] considered a two-parameter polynomial generalization for the r-Lah numbers defined, for $1 \le k \le n+1$, by the recurrence relation:

$$\genfrac{\lfloor}{\rfloor}{0pt}{}{n+1}{k}_r^{\alpha,\beta} = \genfrac{\lfloor}{\rfloor}{0pt}{}{n}{k-1}_r^{\alpha,\beta} + (\alpha n + \beta k + (\alpha+\beta)r)\genfrac{\lfloor}{\rfloor}{0pt}{}{n}{k}_r^{\alpha,\beta}, \tag{3}$$

with initial values $\genfrac{\lfloor}{\rfloor}{0pt}{}{0}{k}_r^{\alpha,\beta} = \delta_{k,0}$ and $\genfrac{\lfloor}{\rfloor}{0pt}{}{n}{0}_r^{\alpha,\beta} = \prod_{j=0}^{n-1}(\alpha(j+r)+\beta r)$.

The coefficients $\genfrac{\lfloor}{\rfloor}{0pt}{}{n}{k}_r^{\alpha,\beta}$ correspond to the connection coefficients in the following expansion:

$$(x + (\alpha+\beta)r|\alpha)^{\overline{n}} = \sum_{k=0}^{n} \genfrac{\lfloor}{\rfloor}{0pt}{}{n}{k}_r^{\alpha,\beta} (x|\beta)^{\underline{k}}, \tag{4}$$

where $(x|\theta)^{\overline{n}}$ and $(x|\theta)^{\underline{n}}$ are the generalized n-th raising and falling factorials of an indeterminate x with increment θ, respectively, defined for $n = 0$ by $(x|\theta)^{\overline{0}} = (x|\theta)^{\underline{0}} = 1$ and for $n \ge 1$,

$$(x|\theta)^{\overline{n}} = x(x+\theta)(x+2\theta)\cdots(x+(n-1)\theta)$$

and

$$(x|\theta)^{\underline{n}} = x(x-\theta)(x-2\theta)\cdots(x-(n-1)\theta).$$

Note that, when $\alpha = \beta = 1$ we obtain the r-Lah numbers and for $r = 0$, the numbers $\genfrac{\lfloor}{\rfloor}{0pt}{}{n}{k}_0^{\alpha,\beta}$ which are reduced to the numbers $\genfrac{\lfloor}{\rfloor}{0pt}{}{n}{k}^{\alpha,\beta}$ studied in [2]. Also, for $(\alpha,\beta,r) = (\alpha,0,r)$, $(0,\alpha,r)$ and (α,α,r) we obtain the translated r-Whitney numbers of three kinds, respectively, see [4]. The coefficient $\genfrac{\lfloor}{\rfloor}{0pt}{}{n}{k}_r^{\alpha,\beta}$ corresponds to $S(n,k;-\alpha,\beta,r(\alpha+\beta))$ where $S(n,k;\alpha,\beta,r)$ is the generalized Stirling number introduced by Hsu and Shiue in [7]. Corcino et al. provide in [6] a combinatorial interpretation of $\beta^k k! S(n,k;\alpha,\beta,r)$ using a probabilistic approach. Shattuck [10] gives a combinatorial interpretation for $S(n,k;\alpha,\beta,r)$ as a distribution of some statistics on set partition.

Our aim is to give a simple combinatorial interpretation for $\genfrac{\lfloor}{\rfloor}{0pt}{}{n}{k}_r^{\alpha,\beta}$ using weighted partitions, which allows us to provide proofs of some combinatorial relations given in [9]. Furthermore, we propose a new combinatorial proof for an explicit formula by means of the inclusion-exclusion principle. Finally, we establish an expression involving symmetric functions.

2. Combinatorial Interpretation

Let $\Omega_r(n,k)$ be the set of all possible ways to distribute the set $[n+r]$ into $k+r$ lists, one element at a time, such that

(i) r distinguished elements have to be in distinct lists,
(ii) the first element being placed in the list must be assigned weight 1,
(iii) the list head has weight β,
(iv) the remaining elements in the list must have weight α.

We use the notation " $\cdot/\cdot/\cdots/\cdot$ " to represent a distribution of elements into lists. Given a distribution $\varepsilon \in \Omega_r(n,k)$, we define the weight of ε, denoted by $w(\varepsilon)$, to be the product of the weights of its elements. The total weight of $\Omega_r(n,k)$ is given by the sum of weights of all distributions.

As an example, letting $n=2$, $k=1$ and $r=2$, we distribute the set $\{1,2,3,4\}$ into 3 non-empty lists such that the elements 1 and 2 are in distinct lists. Firstly, we put the elements $\{1,2\}$ in two distinct lists and add another element in the third list, for example $1/2/3$: the weight of this partition is 1. Now, we add the fourth element in our given list. If we choose to add it to the first list, there are two possible ways: either as a list head, corresponding to $41/2/3$ with weight $\omega(41/2/3)=\beta$, or after the list head element corresponding to $14/2/3$ and $\omega(14/2/3)=\alpha$ and so on. All partitions of the set $\Omega_2(2,1)$ are given in the following figure.

$$\cdot/\cdot/\cdot \longrightarrow 1^{(1)}/2^{(1)}/\cdot \begin{cases} 1^{(1)}/2^{(1)}/3^{(1)} \\ 1^{(1)}/2^{(1)}/4^{(1)} \end{cases}$$

$$1^{(1)}/2^{(1)}/3^{(1)} \longrightarrow \begin{cases} 1^{(1)}4^{(\alpha)}/2^{(1)}/3^{(1)} = \alpha \\ 4^{(\beta)}1^{(1)}/2^{(1)}/3^{(1)} = \beta \\ 1^{(1)}/4^{(\beta)}2^{(1)}/3^{(1)} = \beta \\ 1^{(1)}/2^{(1)}4^{(\alpha)}/3^{(1)} = \alpha \\ 1^{(1)}/2^{(1)}/3^{(1)}4^{(\alpha)} = \alpha \\ 1^{(1)}/2^{(1)}/4^{(\beta)}3^{(1)} = \beta \end{cases}$$

$$1^{(1)}/2^{(1)}/4^{(1)} \longrightarrow \begin{cases} 1^{(1)}/2^{(1)}/3^{(1)}4^{(\alpha)} = \alpha \\ 1^{(1)}/2^{(1)}/4^{(\beta)}3^{(1)} = \beta \\ 1^{(1)}3^{(\alpha)}/2^{(1)}/4^{(1)} = \alpha \\ 3^{(\beta)}1^{(1)}/2^{(1)}/4^{(1)} = \beta \\ 1^{(1)}/3^{(\beta)}2^{(1)}/4^{(1)} = \beta \\ 1^{(1)}/2^{(1)}3^{(\alpha)}/4^{(1)} = \alpha \end{cases}$$

Figure 1: The partitions of the set $\{1,2,3,4\}$ in three weighted lists.

Thus, the total weight of $\Omega_2(2,1)$ is $5\alpha+5\beta$.

Definition 1. For any non-negative integers $n \geq k \geq r \geq 0$, we define:

$$\mathcal{G}_r(n,k;\alpha,\beta) = \sum_{\varepsilon \in \Omega_r(n,k)} \omega(\varepsilon). \tag{5}$$

Now, we provide the recurrence relation of the coefficient $\mathcal{G}_r(n,k;\alpha,\beta)$. We first verify the initial conditions. For $n=0$, it is clear that $\mathcal{G}_r(0,k;\alpha,\beta)=\delta_{0,k}$. For

$k = 0$, we compute the sum of weights of distributions from the set $[n + r]$ into r ordered lists. First, we place r elements into r distinct lists, each having weight 1. To distribute the remaining n elements, the first element has weight $\beta r + \alpha r$ coming from the r choices as list head with weight β or after the inserted element with weight α, the second one has weight $\beta r + \alpha(r + 1)$ and so on until the last element which has weight $\beta r + \alpha(r + n - 1)$. Hence, the total weight of this distribution is $\mathcal{G}_r(n, 0; \alpha, \beta) = \prod_{j=0}^{n-1}(\beta r + \alpha(r + j))$.

Now, we compute $\mathcal{G}_r(n + 1, k; \alpha, \beta)$ the sum of the weights of all distributions of the set $[n + r + 1]$ into $k + r$ ordered, non-empty lists such that r elements must be in distinct lists, according to the situation of the last element "$n + r + 1$".

- If the element "$n + r + 1$" is in a singleton list (with weight 1), the remaining elements of $[n + r]$ have to be distributed into $k + r - 1$ ordered lists with the weight $\mathcal{G}_r(n, k - 1; \alpha, \beta)$.
- If "$n + r + 1$" is not in a singleton list, then it belongs to one of the $k + r$ lists with some other elements. Total weight of distributing set $[n + r]$ into $r + k$ ordered and non-empty lists is $\mathcal{G}_r(n, k; \alpha, \beta)$, and there are $n + r$ choices to insert the element "$n + r + 1$" after any of the elements $[n + r]$ with weight α, and $k + r$ choices to insert it as list head with weight β. Hence, the weight is $(\alpha\,(n + r) + \beta(k + r))\,\mathcal{G}_r(n, k; \alpha, \beta)$.

Considering the two previous cases, we obtain the following recurrence relation:

$$\mathcal{G}_r(n + 1, k; \alpha, \beta) = \mathcal{G}_r(n, k - 1; \alpha, \beta) + (\alpha\,(n + r) + \beta(k + r))\,\mathcal{G}_r(n, k; \alpha, \beta). \tag{6}$$

Comparing initial values and recurrence relation of $\mathcal{G}_r(n, k; \alpha, \beta)$ and the generalized r-Lah numbers (3), we conclude that $\mathcal{G}_r(n, k; \alpha, \beta) = \left\lfloor {n \atop k} \right\rfloor_r^{\alpha,\beta}$, which provide us a new combinatorial interpretation $\left\lfloor {n \atop k} \right\rfloor_r^{\alpha,\beta}$.

3. Some Combinatorial Proofs

In this section, we give proofs of some properties given in [9], using the new combinatorial interpretation,

$$\left\lfloor {n \atop k} \right\rfloor_r^{\alpha,\beta} = \sum_{i=k}^{n} \left\lfloor {i-1 \atop k-1} \right\rfloor_r^{\alpha,\beta} \prod_{j=i}^{n-1}(\alpha j + \beta k + (\alpha + \beta) r), \tag{7}$$

$$\left\lfloor {n \atop k} \right\rfloor_{r+s}^{\alpha,\beta} = \sum_{i=k}^{n} \binom{n}{i} \left\lfloor {i \atop k} \right\rfloor_r^{\alpha,\beta} \prod_{j=0}^{n-i-1}(\alpha j + (\alpha + \beta) s), \tag{8}$$

$$\left\lfloor {n \atop k} \right\rfloor_r^{\alpha,\beta} = \sum_{i=0}^{k}(\alpha(n + r - i - 1) + \beta(k + r - i)) \left\lfloor {n-i-1 \atop k-i} \right\rfloor_r^{\alpha,\beta}. \tag{9}$$

Proof of (7). For a given $i \in [k, n]$, let us consider the elements of $[i + r - 1]$ which are not in the same list as the element "$n + r$". The sum of weights of distributions of elements $[i+r-1]$ into $k+r-1$ lists is $\left\lfloor {i-1 \atop k-1} \right\rfloor_r^{\alpha,\beta}$, and the weight of the distribution of the remaining elements $[i+r, n+r-1]$ into $k+r$ lists is $\prod_{j=i}^{n-1}(\alpha j+\beta k+(\alpha+\beta)r)$. Summing up yields the desired result. □

Proof of (8). The left hand-side counts the sum of weights of distributions of elements $[n + r + s]$ into $k + r + s$ lists such that $r + s$ elements are in distinct lists. Now, we show that the right-hand side counts the weight of the same distribution. We choose i $(k \leq i \leq n)$ elements from the set of n elements. There are $\binom{n}{i}$ ways to do it; the sum of weights of distributions of elements of $[i + r]$ into $k + r$ lists, such that r elements are in distinct lists, is given by $\left\lfloor {i \atop k} \right\rfloor_r^{\alpha,\beta}$. The weight obtained by distributing the remaining $n - i + s$ elements into s lists such that s elements are in distinct lists, is $\prod_{j=0}^{n-i-1}(\alpha j + (\alpha + \beta)s)$. Summing up yields the desired result. □

Proof of (9). Let i $(0 \leq i \leq k)$ be the number of lists that contain exactly one element, then the weight of such lists is 1. Now, it remains to count the weight obtained by distributing the elements of $[n+r-i]$ in $k+r-i$ lists such that r elements are in distinct lists. So, the weight of distributing the elements of $[n + r - i - 1]$ is $\left\lfloor {n-i-1 \atop k-i} \right\rfloor_r^{\alpha,\beta}$ and there are $n+2r-2i+k-1$ ways to add the element $n+r-i$ in the lists with a weight $\alpha(n + r - i - 1) + \beta(k + r - i)$. We conclude by summing. □

Shattuck [9] provided an explicit formula for the generalized r-Lah numbers using inductive reasoning. We give a combinatorial proof for this explicit expression by means of the inclusion-exclusion principle.

Theorem 1. *For any $n \geq k \geq r$, we have*

$$\left\lfloor {n \atop k} \right\rfloor_r^{\alpha,\beta} = \frac{1}{\beta^k (k+r)!} \sum_{j=0}^{k+r} (-1)^j \binom{k+r}{j} (k+r-j)^{\underline{r}} (\beta(k+r-j) + \alpha r|\alpha)^{\overline{n}}. \quad (10)$$

Proof. Let ϕ be the set of all possible ways to distribute the set $[n + r]$ into $k + r$ lists (labeled and not necessary non-empty), one element at a time, such that r elements are in distinct lists and satisfying

- the first element being placed in the list must have weight β,
- we assign weight β to the element inserted as list head,
- the remaining elements in the list must have weight α.

The total weight of the set ϕ is the sum of all weights of all the distributions.

Now, let Δ be the subset of elements of ϕ which have non-empty list. We want to count the total weight of the subset Δ.

For j $(1 \le j \le k+r)$, let A_j be the subset of $k+r$ labeled lists of ϕ such that the j-th list is empty. Then

$$\Delta = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_{k+r}},$$

where $\overline{A_j} = \phi \backslash A_j$. Applying the inclusion-exclusion principle, we get

$$|\Delta| = |\phi| - \sum_{j=1}^{k+r} (-1)^j \sum_{1 \le i_1 < i_2 < \cdots < i_j \le k+r} \left| A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j} \right|. \tag{11}$$

We compute the general term $\sum_{1 \le i_1 < i_2 < \cdots < i_j \le k+r} \left| A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j} \right|$. For a fixed j, there are $\binom{k+r}{j}$ ways to select j empty lists from $k+r$. Now, we distribute $n+r$ elements in the remaining $k+r-j$ lists, so we start by distributing the r elements in distinct lists which gives the weight value

$$\beta(k+r-j)\beta(k+r-j-1) \cdots \beta(k-j+1) = \beta^r (k+r-j)^{\underline{r}}.$$

We now distribute the remaining n elements: the first element has $k+2r-j$ choices to be placed with the weight $\beta(k+r-j)+\alpha r$; the second one has weight $\beta(k+r-j)+\alpha(r+1)$ coming from the $(k+r-j)$ choices as a list head with weight β or after the inserted element with weight α; and so on until the last element which has weight $\beta\,(k+r-j)+(n+r-1)\alpha$. So, the total weight of distributing the n elements is

$$(\beta(k+r-j)+\alpha r) \cdots (\beta\,(k+r-j)+(n+r-1)\alpha) = (\beta\,(k+r-j)+\alpha r|\alpha)^{\overline{n}}.$$

Thus,

$$\sum_{1 \le i_1 < \cdots < i_j \le k+r} \left| A_{i_1} \cap \cdots \cap A_{i_j} \right| = \binom{k+r}{j} \beta^r (k+r-j)^{\underline{r}} (\beta\,(k+r-j)+\alpha r|\alpha)^{\overline{n}},$$

and we get

$$|\Delta| = \sum_{j=0}^{k} (-1)^j \binom{k+r}{j} \beta^r (k+r-j)^{\underline{r}} (\beta\,(k+r-j)+\alpha r|\alpha)^{\overline{n}}.$$

We divide by $(k+r)!$ to avoid the repeated permutations and by β^{k+r} to give weight 1 to the $k+r$ first elements inserted in the lists. $\square$

4. An Expression Involving a Symmetric Function

Belbachir and Bousbaa [3] established an identity for the r-Lah numbers in terms of elementary symmetric functions. The following theorem generalizes this identity to our case.

Theorem 2. *The generalized r-Lah numbers satisfy*

$$\left\lfloor \begin{matrix} n+k \\ n \end{matrix} \right\rfloor_r^{\alpha,\beta} = \sum_{r \le i_1 \le \cdots \le i_k \le n+r} \prod_{j=1}^{k} ((\alpha+\beta) i_j + \alpha(j-1)). \tag{12}$$

Proof. The left-hand side $\left\lfloor \begin{smallmatrix} n+k \\ k \end{smallmatrix} \right\rfloor_r^{\alpha,\beta}$ counts the total weight of distributing a set $[n+r+k]$ into $n+r$ ordered non-empty lists. Now, we show that the right-hand side counts the same quantity.

First, we constitute $n+r$ lists with the elements of $[n+r]$ (each list contains one element with weight 1). It remains to count the weight of the remaining elements $[n+r+1, n+r+k]$.

To insert the element "$n+r+1$", we have two situations:

(A1) if it belongs to the r first lists containing the elements of $[r]$, then we have r possibilities to affect it with the weight $(\alpha+\beta)r$,

(A2) otherwise, it belongs to a list labeled i_1 $(r+1 \le i_1 \le n+r)$ and again, we consider all the possible situations of the element already in the list i_1 and we distinguish two other cases:

- **(i)** first, the initial element holds in the list i_1, we put the element "$n+r+1$" before the initial one with weight β or after it with weight α,
- **(ii)** secondly, we move the initial element to one of the first i_1-1 lists with weight $(\alpha+\beta)(i_1-1)$ and we put the element "$n+r+1$" in the list i_1 with weight 1. Note that, we move the elements only from right to left to avoid the double counting situations.

Thus, from (i) and (ii), the weight of the element $n+r+1$ is $(\alpha+\beta)i_1$.

Form (A1) and (A2), we sum over all the possible insertions of the element "$n+r+1$", we get the total weight of elements of $[n+r+1]$ as $\sum_{r \le i_1 \le n+r} (\alpha+\beta) i_1$.

Now, to insert the element "$n+r+2$", we consider the elements of the lists $1, \ldots, i_1$ as fixed ones due to the insertion of the previous element $n+r+1$ where we consider all the situations. We have two ones (same as before):

(B1) if we add the element $n+r+2$ to one of the lists $1, \ldots, i_1$, the weight is $((\alpha+\beta)i_1+\alpha)$,

(B2) else, it belongs to a list i_2 $(i_1+1 \le i_2 \le n)$, with weight $((\alpha+\beta)i_2+\alpha)$ (indeed, the weight of element $n+r+2$ is $(\beta+\alpha)$ if it is inserted before or after the initial element of the list i_2 or $(\alpha+\beta)(i_2-1)+\alpha$ if it is inserted in the list i_2 and the initial element of the list i_2 is moved to the previous lists).

Then from (B1) and (B2), the weight of the element $n+r+2$ is

$$((\alpha+\beta)i_1+\alpha)+\sum_{i_2=i_1+1}^{n+r}(\alpha+\beta)i_2+\alpha=\sum_{i_2=i_1}^{n+r}(\alpha+\beta)i_2+\alpha.$$

Altogether, the weight of the elements $n+r+1$ and $n+r+2$ is

$$\sum_{i_1=r}^{n+r}(\alpha+\beta)i_1\sum_{i_2=i_1}^{n+r}((\alpha+\beta)i_2+\alpha)=\sum_{1\leq i_1\leq i_2\leq n+r}((\alpha+\beta)i_1)((\alpha+\beta)i_2+\alpha).$$

We carry on by the same process for the remaining $k-2$ elements. So, for the last element "$n+r+k$", we consider the elements of the lists $1,\ldots,i_{k-1}$ as fixed ones, then the weight of the element $n+r+k$ is $(\alpha+\beta)(i_{k-1}-1)+\alpha(k-1)$ if it is inserted in these lists. Otherwise, the weight is $(\alpha+\beta)i_k+\alpha(k-1)$ if it is inserted in a list i_k $(i_{k-1}+1\leq i_k\leq n+r)$. This gives the total weight of distributions of elements of $[n+r+k]$ into $n+r$ ordered non-empty lists,

$$\sum_{r\leq i_1\leq i_2\leq\cdots\leq i_k\leq n+r}((\alpha+\beta)i_1)((\alpha+\beta)i_2+\alpha)\cdots((\alpha+\beta)i_k+\alpha(k-1)).$$

□

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