THE NATURAL DENSITY OF SOME SETS OF SQUARE-FREE NUMBERS

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Abstract

Let P and T be disjoint sets of prime numbers with T finite. A simple formula is given for the natural density of the set of square-free numbers which are divisible by all of the primes in T and by none of the primes in P. If P is the set of primes congruent to r modulo m (where m and r are relatively prime numbers), then this natural density is shown to be 0 and if P is the set of Mersenne primes (and $T = \emptyset$), then it is appproximately .3834.

1. Main results

Gegenbauer proved in 1885 that the natural density of the set of square-free integers, i.e., the proportion of natural numbers which are square-free, is $6/\pi^2$ [3, Theorem 333; reference on page 272]. In 2008 J. A. Scott conjectured [8] and in 2010 G. J. O. Jameson proved [5] that the natural density of the set of odd square-free numbers is $4/\pi^2$ (so the proportion of natural numbers which are square-free and even is $2/\pi^2$). Jameson's argument was adapted from one used to compute the natural density of the set of all square-free numbers. In this note we use the classical result for all square-free numbers to reprove Jameson's result and indeed to generalize it.

Theorem 1. *Let P and T be disjoint sets of prime numbers with T finite. Then the proportion of all numbers which are square-free and divisible by all of the primes in T and by none of the primes in P is*

$$\frac{6}{\pi^2}\prod_{p\in T}\frac{1}{1+p}\prod_{p\in P}\frac{p}{1+p}.$$

As in the above theorem, throughout this paper P and T will be disjoint sets of prime numbers with T finite. The letter p will always denote a prime number. The term *numbers* will always refer to positive integers. Empty products, such as occurs in the first product above when T is empty, are understood to equal 1. If P is infinite, we will argue in Section 4 that the second product above is well-defined.

Examples. 1. Setting $P = \{2\}$ and T equal to the empty set $\emptyset$ in the theorem we see that the natural density of the set of odd square-free numbers is $\frac{6}{\pi^2}\frac{2}{2+1} = \frac{4}{\pi^2}$; taking $T = \{2\}$ and $P = \emptyset$ we see that the natural density of the set of even square-free numbers is $\frac{6}{\pi^2}\frac{1}{2+1} = \frac{2}{\pi^2}$. Thus one third of the square-free numbers are even and two thirds are odd. (These are Jameson's results of course.)

2. Taking $P = \{101\}$ and $T = \emptyset$ in the theorem we see that the set of square-free numbers not divisible by 101 has natural density $\frac{6}{\pi^2}\frac{101}{102}$. Thus slightly over 99% of square-free numbers are not divisible by 101.

3. Set $T = \{2, 5\}$ and $P = \{3, 7\}$ in the theorem. Then the theorem says that the natural density of the set of square-free numbers divisible by 10 but not by 3 or 7 is $\frac{6}{\pi^2}\frac{1}{3}\frac{1}{6}\frac{3}{4}\frac{7}{8} = \frac{7}{32\pi^2}$, so the proportion of square-free numbers which are divisible by 10 but not by 3 or 7 is 7/192.

Our interest in the case that P is infinite arose in part from a question posed by Ed Bertram: what is the natural density of the set of square-free numbers none of which is divisible by a prime congruent to 1 modulo 4? The answer is zero; more generally we have the following.

Theorem 2. *Let r and m be relatively prime numbers. Then the natural density of the set of square-free numbers divisible by no prime congruent to r modulo m is zero.*

This theorem is a corollary of the previous theorem since, as we shall see in Section 5, for any r and m as above, $\prod_{p\equiv r \pmod{m}} p/(1+p) = 0$.

By way of contrast with Theorem 2 we will prove in Section 6 a lemma giving a simple condition on an infinite set of primes P which will guarantee that the set of square-free numbers not divisible by any element of P has positive natural density. An immediate corollary is that the set of square-free numbers not divisible by any Mersenne prime has positive natural density; we will also show how to closely approximate the natural density of this set.

2. A Basic Lemma

Notation. For any finite set S of primes we set $d_S = \prod_{p\in S} p$.

For any real number x and set B of numbers, we let $B[x]$ denote the number of elements t of B with $t \leq x$. Recall that if $\lim_{x\to\infty} B[x]/x$ exists, then it is by definition the *natural density* of B [6, Definition 11.1]. In case it exists we will denote the natural density of B by B^*. We will also let $|B|$ denote the cardinality of B.

Let $\mathcal{A}$ denote the set of square-free numbers. Then we let $\mathcal{A}(T, P)$ denote the set of elements of $\mathcal{A}$ which are divisble by all elements of T and by no element of

P. (Thus $\mathcal{A} = \mathcal{A}(\emptyset, \emptyset)$ and $\mathcal{A}^* = 6/\pi^2$.) The set of square-free numbers analyzed in Theorem 1 is $A(T, P)$.

The above notation is used in the next lemma, which shows how the calculation of the natural density of the sets $A(T, P)$ reduces to the calculation of the natural density of sets of the form $A(\emptyset, S)$ and, when P is finite, also reduces to the calculation of the natural density of sets of the form $\mathcal{A}(S, \emptyset)$.

Lemma 1. *For any finite set of primes S disjoint from T and from P and for any real number x, we have $\mathcal{A}(T, S \cup P)[x] = \mathcal{A}(T \cup S, P)[xd_S]$. Moreover, the set $\mathcal{A}(T \cup S, P)$ has a natural density if and only if $\mathcal{A}(T, P \cup S)$ has a natural density, and if these natural densities exist, then $\mathcal{A}(T, P \cup S)^* = d_S \mathcal{A}(T \cup S, P)^*$.*

This lemma generalizes Lemmas 1 and 2 of [2].

Proof. The first assertion is immediate from the fact that multiplication by d_S gives a bijection from the set of elements of $\mathcal{A}(T, S \cup P)$ less than or equal to x to the set of elements of $\mathcal{A}(T \cup S, P)$ less than or equal to xd_S. This implies that

$$\frac{\mathcal{A}(T, P \cup S)[x]}{x} = d_S \frac{\mathcal{A}(T \cup S, P)[xd_S]}{xd_S}.$$

The lemma follows by taking the limit as x (and hence xd_S) goes to infinity. □

Remark 1. We might note that if we assume that for all T the sets $A(T, \emptyset)$ have natural densities, then it is easy to compute these natural densities. After all, for any T the set $\mathcal{A}$ is the disjoint union over all subsets S of T of the sets $\mathcal{A}(T \setminus S, S)$, so by Lemma 1

$$6/\pi^2 = \mathcal{A}^* = \sum_{S \subseteq T} \mathcal{A}(T \setminus S, S)^* = \sum_{S \subseteq T} d_S \mathcal{A}(T, \emptyset)^* = \mathcal{A}(T, \emptyset)^* \sum_{S \subseteq T} d_S.$$

But

$$\sum_{S \subseteq T} d_S = \sum_{d | d_T} d = \prod_{p \in T} (1 + p)$$

[6, Theorem 4.5], so indeed

$$A(T, \emptyset)^* = \frac{6}{\pi^2} \prod_{p \in P} \frac{1}{1 + p}.$$

We could now use Lemma 1 to obtain the formula of Theorem 1 in the case that P is finite.

3. Proof of Theorem 1 when P is Finite

The theorem in this case was proved in [2]. For the convenience of the reader we sketch the proof using the notation of this paper.

Lemma 2. *Let p be a prime number not in T. If the set $\mathcal{A}(T,\emptyset)$ has natural density D, then the set $\mathcal{A}(\{p\}\cup T,\emptyset)$ has natural density $\frac{1}{p+1}D$.*

Proof. For any real number x we set $E(x)=\mathcal{A}(\{p\}\cup T,\emptyset)[x]$. Let $\epsilon>0$.

Note that $\mathcal{A}(T,\emptyset)$ is the disjoint union of $\mathcal{A}(\{p\}\cup T,\emptyset)$ and $\mathcal{A}(T,\{p\})$. Hence by Lemma 1 (applied to $\mathcal{A}(T,\{p\})$) for any real number x,

$$A(T,\emptyset)[x/p]=E(x/p)+E(x)$$

and so by the choice of D there exists a number M such that if $x>M$ then

$$\left|\frac{E(x)}{x/p}+\frac{E(x/p)}{x/p}-D\right|<\epsilon/3.$$

We next pick an even number k such that $\frac{1}{p^k}<\frac{\epsilon}{3}$. Then

$$\left|E(x/p^k)\right|\le x/p^k<\frac{\epsilon}{3}x \tag{1}$$

and also (using the usual formula for summing a geometric series)

$$\left|-Dx\sum_{i=1}^{k}(-\frac{1}{p})^i-Dx\frac{1}{p+1}\right|=Dx\left|\frac{(-\frac{1}{p})-(-\frac{1}{p})^{k+1}}{1-(-\frac{1}{p})}+\frac{1}{p+1}\right| \tag{2}$$

$$=Dx\left|\frac{-1+\frac{1}{p^k}}{p+1}+\frac{1}{p+1}\right|<\frac{1}{p^k}Dx<\frac{\epsilon}{3}x.$$

Now suppose that $x>p^kM$. Then for all $0\le i\le k$ we have $x/p^i>M$ and hence (applying the choice of M above),

$$\left|(-1)^iE(\frac{x}{p^i})+(-1)^iE(\frac{x}{p^{i+1}})-(-1)^iD\frac{x}{p^{i+1}}\right|<\frac{\epsilon}{3}\frac{x}{p^{i+1}}. \tag{3}$$

Using the triangle inequality to combine the inequalities (1) and (2) and all the inequalities (3) for $0\le i<k$ and dividing through by x, we can conclude that

$$\left|\frac{E(x)}{x}-\frac{1}{p+1}D\right|<\frac{\epsilon}{3}\left(\sum_{i=1}^{k}\frac{1}{p^i}\right)+\frac{\epsilon}{3}+\frac{\epsilon}{3}<\epsilon.$$

Hence the natural density of $\mathcal{A}(\{p\}\cup T,\emptyset)$ is $\frac{1}{p+1}D$. □

Theorem 1 now follows in the case that P is empty from the above lemma by induction on the number of elements of T. That it is true when P is finite but not necessarily empty follows from Lemma 1: in the statement of that lemma replace P by $\emptyset$ and S by P; then we see that the natural density of $\mathcal{A}(T,P)$ is indeed

$$d_P\mathcal{A}(T\cup P,\emptyset)^* = \frac{6}{\pi^2}\prod_{p\in P} p \prod_{p\in T\cup P}\frac{1}{1+p} = \frac{6}{\pi^2}\prod_{p\in T}\frac{1}{1+p}\prod_{p\in P}\frac{p}{1+p}.$$

4. Proof of Theorem 1 when P is Infinite

In this and the next section we will use the simple fact that for all $x>1$ we have

$$\frac{1}{x} > \log(x+1)-\log x = \log\frac{x+1}{x} > \frac{1}{1+x} > \frac{1}{2x}. \tag{4}$$

We first show that the expression $\prod_{p\in P}\frac{p}{1+p}$ is well-defined when P is infinite. Let $p_1,p_2,p_3,\cdots$ be the strictly increasing sequence of elements of P. Since all the quotients $p_i/(1+p_i)$ are less than 1, the partial products of the infinite product $\prod_i p_i/(1+p_i)$ form a strictly decreasing sequence bounded below by 0; thus the infinite product $\prod_i p_i/(1+p_i)$ converges, say to α. Its limit is also unchanged by any rearrangement of its factors; this is easy to check if α is zero. Otherwise the infinite sum

$$\sum_i \log\frac{p_i}{1+p_i} = \sum_i \log(p_i)-\log(1+p_i)$$

converges absolutely (to $\log(\alpha)$) and therefore its value is unchanged under rearrangements; hence the corresponding fact is also true of the infinite product $\prod_i p_i/(1+p_i)$. Thus in all cases the expression $\prod_{p\in P}\frac{p}{1+p}$ is well-defined.

We now prove the theorem in the case that T is empty.

First suppose that $\alpha\neq 0$. Then $\sum_{p\in P}1/p<\infty$. After all, we have

$$-\log\alpha = \sum_{p\in P}\log(1+p)-\log p > \frac{1}{2}\sum_{p\in P}\frac{1}{p}.$$

Now observe that $\mathcal{A}\setminus\mathcal{A}(\emptyset,P)$ is the disjoint union

$$\mathcal{A}\setminus\mathcal{A}(\emptyset,P) = \cup_{k\geq 1}\mathcal{A}(\{p_k\},\{p_1,\cdots,p_{k-1}\})$$

since for all $b\in\mathcal{A}\setminus\mathcal{A}(\emptyset,P)$ there exists a least $k\geq 1$ with p_k dividing b, so that $b\in\mathcal{A}(\{p_k\},\{p_1,\cdots,p_{k-1}\})$.

For all n and k we have

$$\frac{\mathcal{A}(\{p_k\},\{p_1,\cdots,p_{k-1}\})[n]}{n} \leq \frac{|\{j:1\leq j\leq n \text{ and } p_k|j\}|}{n} \leq \frac{1}{p_k}.$$

Hence by Tannery's theorem (see [9, p. 292] or [4, p. 199]) the natural density of $\mathcal{A} \setminus \mathcal{A}(\emptyset, P)$ is therefore

$$\lim_{n\to\infty} \frac{(\mathcal{A} \setminus \mathcal{A}(\emptyset, P))[n]}{n} = \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{\mathcal{A}(\{p_k\}, \{p_1, \cdots, p_{k-1}\})[n]}{n}$$

$$= \sum_{k=1}^{\infty} \lim_{n\to\infty} \frac{\mathcal{A}(\{p_k\}, \{p_1, \cdots, p_{k-1}\})[n]}{n} = \sum_{k=1}^{\infty} \frac{6}{\pi^2} \frac{1}{1+p_k} \prod_{i<k} \frac{p_i}{1+p_i}$$

by the proof in the previous section of the theorem in the case that P is finite. Writing $1/(1+p_k) = 1 - p_k/(1+p_k)$ we can see that the natural density of $\mathcal{A}\setminus\mathcal{A}(\emptyset, P)$ is therefore a limit of telescoping sums

$$\frac{6}{\pi^2} \lim_{L\to\infty} \sum_{k=1}^{L} \left(\prod_{i<k} \frac{p_i}{1+p_i} - \prod_{i<k+1} \frac{p_i}{1+p_i} \right)$$

$$= \frac{6}{\pi^2} \lim_{L\to\infty} \left(1 - \prod_{i\leq L} \frac{p_i}{1+p_i} \right) = \frac{6}{\pi^2} (1-\alpha)$$

and thus

$$\mathcal{A}(\emptyset, P)^* = \frac{6}{\pi^2} - \frac{6}{\pi^2}(1-\alpha) = \frac{6}{\pi^2} \prod_{p\in P} \frac{p}{1+p},$$

which proves the theorem in the case that $\alpha \neq 0$ and $T = \emptyset$.

We next consider the case that $\alpha = \prod_{p\in P} p/(1+p) = 0$. Suppose that $\epsilon > 0$. By hypothesis there exists a number M with $\frac{6}{\pi^2} \prod_{i\leq M} \frac{p_i}{1+p_i} < \epsilon/2$. Then by our proof of the theorem in the case that P is finite there exists a number L such that if $n > L$ then

$$\frac{\mathcal{A}(\emptyset, \{p_1, p_2, \cdots, p_M\})[n]}{n} < \frac{\epsilon}{2} + \frac{6}{\pi^2} \prod_{i\leq M} \frac{p_i}{1+p_i}.$$

Thus if $n > L$ we have

$$0 \leq \frac{\mathcal{A}(\emptyset, P)[n]}{n} \leq \frac{\mathcal{A}(\emptyset, \{p_1, p_2, \cdots, p_M\})[n]}{n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore,

$$\lim_{n\to\infty} \frac{\mathcal{A}(\emptyset, P)[n]}{n} = 0 = \frac{6}{\pi^2} \prod_{p\in P} \frac{p}{1+p},$$

which proves the theorem if $\alpha = 0$ and $T = \emptyset$.

This completes the proof of the theorem in the case that $T = \emptyset$. The general case where T is arbitrary then follows from Lemma 1, applied with T and S replaced respectively by $\emptyset$ and T: $\mathcal{A}(T, P)^*$ equals

$$\mathcal{A}(\emptyset, P \cup T)^*/d_T = \frac{6}{\pi^2} \prod_{p \in T} \frac{1}{p} \prod_{p \in T} \frac{p}{1+p} \prod_{p \in P} \frac{p}{1+p} = \frac{6}{\pi^2} \prod_{p \in T} \frac{1}{1+p} \prod_{p \in P} \frac{p}{1+p},$$

which completes the proof of Theorem 1.

5. Proof of Theorem 2

Let us set $P = \{p : p \equiv r \pmod{m}\}$. It then suffices by Theorem 1 to prove that $\prod_{p \in P} p/(1+p) = 0$. For any real number $x > 3$ we have

$$\sum_{x > p \in P} 1/p - \frac{\log \log x}{\phi(m)} = O(1)$$

(see [1, Exercise 6, page 156]). Hence $\sum_{p \in P} 1/p = \infty$. Theorem 2 now follows from the next lemma, which applies to *any* infinite set P of primes.

Lemma 3. *Let P be an infinite set of primes. Then $\sum_{p \in P} 1/p = \infty$ if and only if $\prod_{p \in P} p/(1+p) = 0$. Moreover, if $\sum_{p \in P} 1/p \leq S < \infty$ for a real number S, then $\prod_{p \in P} p/(1+p) \geq e^{-S}$.*

Proof. The inequalities of display (4) imply that

$$\frac{1}{2} \sum_{p \in P} \frac{1}{p} \leq -\log \prod_{p \in P} \frac{p}{1+p} = \sum_{p \in P} \log \frac{p+1}{p} \leq \sum_{p \in P} \frac{1}{p}.$$

Our conclusions follow easily. □

6. Mersenne Primes

Lemma 4. *Let d be a number. Let $P = \{p_1, p_2, \cdots\}$ be an infinte set of primes such that for all $i \geq 1$ we have $p_i \geq 2^i - d$. Then the set of square-free numbers not divisible by any element of P has positive natural density.*

Proof. The infinite sum $\sum_{p \in P} 1/p$ converges since it is less than or equal to the infinite sum $\sum_{i < \infty} 1/(2^i - d)$, which itself converges by the limit comparison test. Thus by Lemma 3, $\prod_{p \in P} p/(1+p) > 0$; the lemma now follows from Theorem 1. □

An immediate corollary of this lemma (or just of Theorem 1 if the relevant set of primes turns out to be finite) is that the natural density of the set of square-free numbers not divisible by any Fermat prime is positive [3, Section 2.5]. Similarly, the natural density of the set of square-free numbers not divisible by any Mersenne prime is positive. The next theorem allows us to approximate this natural density closely.

Let q_i denote the i-th prime number and let P denote the set of Mersenne primes. For any number M let

$$P_M = \{p \in P : p = 2^{q_i} - 1 \text{ for some } i \leq M\}$$

(so, for example, $P_5 = \{3, 7, 31, 127\}$). Let $A_M = \prod_{p \in P_M} p/(1+p)$ (so, for example, $A_5 = \frac{3}{4}\frac{7}{8}\frac{31}{32}\frac{127}{128} \approx .63078$). If the number of Mersenne primes is finite, then there exists some M with $P = P_M$ and so $A_M = \prod_{p \in P} \frac{p}{1+p}$. In general, we have the following.

Theorem 3. *Whether P is finite or infinite, we have*

$$A_M \geq \prod_{p \in P} \frac{p}{1+p} \geq A_M \exp\left(-\frac{1}{2^M}\right).$$

Proof. For any number i we have $q_i \geq i+1$ so

$$\sum_{p \in P \setminus P_M} \frac{1}{p} \leq \sum_{i>M} \frac{1}{2^{q_i} - 1} \leq \sum_{i>M} \frac{1}{2^i} = \frac{1}{2^M},$$

so by Lemma 3

$$A_M \geq \prod_{p \in P} \frac{p}{1+p} = \prod_{p \in P_M} \frac{p}{1+p} \prod_{p \in P \setminus P_M} \frac{p}{1+p} \geq A_M \exp\left(-\frac{1}{2^M}\right).$$

□

Thus, for example, taking $M = 5$ we have

$$.631 \geq A_5 \geq \prod_{p \in P} \frac{p}{1+p} \geq \exp\left(-\frac{1}{2^5}\right) A_5 \geq .611$$

so the natural density D of the set of square-free numbers divisible by no Mersenne prime satisfies

$$.384 \geq .631 \frac{6}{\pi^2} \geq D \geq .611 \frac{6}{\pi^2} \geq .371.$$

If we repeat this calculation with $M = 17$ (so that now

$$P_M = \{3, 7, 31, 127, 2^{13} - 1, 2^{17} - 1, 2^{19} - 1, 2^{31} - 1\})$$

[3, Section 2.5], then we find that

$$.38342 > D > .38341.$$

Remark 2. An easy induction shows that for $i > 1$, $q_i - 2i$ is an increasing function of i; hence if $i > M > 1$ then $q_i \geq 2i + q_M - 2M$. Using this bound on q_i the argument in the proof of the above theorem can be modified to show that

$$\sum_{p \in P \setminus P_M} \frac{1}{p} \leq 3(2^{q_M} - 1)$$

so that by Lemma 3

$$D \geq \frac{6}{\pi^2} A_M \exp\left(-\frac{1}{3(2^{q_M} - 1)}\right) \geq \frac{6}{\pi^2} A_M \exp\left(-\frac{1}{3(M^{M \log 2} - 1)}\right)$$

where we have used Rosser's Theorem [7] to deduce the last inequality. Thus, for example, if $M = 5$ (so $q_M = 11$) we can compute that

$$D \geq \frac{6}{\pi^2} A_5 \exp\left(-\frac{1}{3(2^{11} - 1)}\right) > .383403.$$

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