A FORMULA FOR THE NUMBER OF OVERPARTITIONS OF n IN TERMS OF THE NUMBER OF REPRESENTATIONS OF n AS A SUM OF r SQUARES

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Abstract

Let $\overline{p}(n)$ denote the number of overpartitions of n, and $c_r(n)$ be the number of representations of n as a sum of r squares where representations with different orders and different signs are counted as distinct. Then we prove that

$$\overline{p}(n) = \sum_{r=0}^{n} (-1)^{n+r} \binom{n+1}{r+1} c_r(n).$$

1. Main Result

An *overpartition* of n is a non-increasing sequence of natural numbers whose sum is n in which the first occurrence of a number may be overlined. Let $\overline{p}(n)$ denote the number of overpartitions of an integer n. For convenience, we let $\overline{p}(0) = 1$. For example, $\overline{p}(3) = 8$ because there are 8 possible overpartitions of 3:

$$3,\ \overline{3}, 2+1,\ \overline{2}+1,\ 2+\overline{1},\ \overline{2}+\overline{1},\ 1+1+1,\ \overline{1}+1+1.$$

Definition 1. [3, formula 7.324] Let $\theta(q)$ be the following infinite product:

$$\theta(q) := \prod_{j=1}^{\infty} \frac{1-q^j}{1+q^j} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2},$$

where $|q| < 1$.

From [1], we have that the generating function of $\overline{p}(n)$ is

$$\frac{1}{\theta(q)} = \sum_{n=0}^{\infty} \overline{p}(n) q^n.$$

Definition 2. For any positive integer r define $c_r(n)$ by $\theta(q)^r = \sum_{n=0}^{\infty} c_r(n)\,(-1)^n\,q^n$ where $c_r(n)$ is the number of representations of n as a sum of r squares where representations with different orders and different signs are counted as distinct. For convenience, we let $c_r(0) = 1$.

Our aim is to derive the following identity.

Theorem 1. *For all non-negative integers n we have*

$$\overline{p}(n) = \sum_{r=0}^{n} (-1)^{n+r} \binom{n+1}{r+1} c_r(n). \tag{1}$$

We require the following two lemmas for our proof.

Lemma 1. *For all positive integers n we have*

$$\overline{p}(n) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k\, k!\, B_{n,k} \left(\tilde{\theta}'(0), \tilde{\theta}''(0), \ldots, \tilde{\theta}^{(n-k+1)}(0)\right) \tag{2}$$

where $\tilde{\theta}(q) = \theta(q) - 1$ *and* $B_{n,k} \equiv \mathrm{B}_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ *are the* partial Bell polynomials *defined by [2, p. 134]*

$$\mathrm{B}_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{\substack{1 \le i \le n, \ell_i \in \mathbb{N} \\ \sum_{i=1}^{n} i\ell_i = n \\ \sum_{i=1}^{n} \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i}.$$

Proof. Let $f(q) = 1/(1+q)$. Using Faà di Bruno's formula [2, p. 137] we have

$$f\left(\tilde{\theta}(q)\right) = \sum_{n=0}^{\infty} h_n \frac{q^n}{n!} \tag{3}$$

where $h_0 = 1$ and

$$h_n = \sum_{k=1}^{n} f^k(0)\, B_{n,k} \left(\tilde{\theta}'(0), \tilde{\theta}''(0), \ldots, \tilde{\theta}^{(n-k+1)}(0)\right) \qquad (n \ge 1).$$

Since

$$\frac{d^n}{dq^n} f\left(\tilde{\theta}(q)\right) = \frac{d^n}{dq^n} \frac{1}{\theta(q)} = \frac{d^n}{dq^n} \sum_{n=0}^{\infty} \overline{p}(n) q^n\,,$$

we have

$$\begin{aligned} n!\,\overline{p}(n) &= \sum_{k=1}^{n} f^k(0)\, B_{n,k} \left(\tilde{\theta}'(0), \tilde{\theta}''(0), \ldots, \tilde{\theta}^{(n-k+1)}(0)\right) \\ &= \sum_{k=1}^{n} (-1)^k\, k!\, B_{n,k} \left(\tilde{\theta}'(0), \tilde{\theta}''(0), \ldots, \tilde{\theta}^{(n-k+1)}(0)\right). \end{aligned}$$

$\square$

Lemma 2. *We have, for positive integers n, k,*

$$B_{n,k}\left(\tilde{\theta}'(0), \tilde{\theta}''(0), \ldots, \tilde{\theta}^{(n-k+1)}(0)\right) = (-1)^n \frac{n!}{k!} \sum_{r=1}^{k} (-1)^{k-r} \binom{k}{r} c_r(n). \tag{4}$$

Proof. We start with the generating function for the partial Bell polynomials [2, Equation (3a') on p. 133] as follows to conclude (4):

$$\begin{aligned}
\sum_{n=k}^{\infty} B_{n,k}\left(\tilde{\theta}'(0), \tilde{\theta}''(0), \ldots, \tilde{\theta}^{(n-k+1)}(0)\right) \frac{q^n}{n!} &= \frac{1}{k!}\left(\sum_{j=1}^{\infty} \tilde{\theta}^{(j)}(0) \frac{q^j}{j!}\right)^k \\
&= \frac{1}{k!}(\theta(q) - 1)^k \\
&= \frac{1}{k!} \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} \theta(q)^r \\
&= \frac{1}{k!} \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} \sum_{n=0}^{\infty} (-1)^n c_r(n) q^n.
\end{aligned}$$

□

Proof of Theorem 1. Combining (2) and (4) we have

$$\begin{aligned}
\overline{p}(n) = (-1)^n \sum_{k=0}^{n} \sum_{r=0}^{k} (-1)^r \binom{k}{r} c_r(n) &= (-1)^n \sum_{r=0}^{n} (-1)^r c_r(n) \sum_{k=r}^{n} \binom{k}{r} \\
&= (-1)^n \sum_{r=0}^{n} (-1)^r \binom{n+1}{r+1} c_r(n).
\end{aligned}$$

Now we can conclude our main result. □

Remark 1. Similar methods have been employed by the author in [4] and [5].

References

[1] S. Corteel and J. Lovejoy, Overpartitions, *Trans. Amer. Math. Soc.* **356** (2004), 1623–1635.

[2] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, D. Reidel Publishing Co., Dordrecht, 1974.

[3] N. J. Fine, *Basic Hypergeometric Series and Applications*, American Mathematical Soc., 1988.

[4] S. K. Jha, A formula for the number of partitions of n in terms of the partial Bell polynomials, *Ramanujan J.*, 2021, to appear; published online at `https://doi.org/10.1007/s11139-020-00380-w`.

[5] S. K. Jha, An identity for the sum of inverses of odd divisors of n in terms of the number of representations of n as a sum of squares, *Rocky Mountain J. Math.* **51** (2), 581–583.