THE $n$-COLOR PARTITION FUNCTION AND SOME COUNTING THEOREMS

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Abstract
Recently, Merca and Schmidt found some decompositions for the partition function $p(n)$ in terms of the classical Möbius function as well as Euler’s totient. In this paper, we define a counting function $T_{r,k}^n(m)$ on the set of $n$-color partitions of $m$ for given positive integers $k, r$ and relate the function with the $n$-color partition function and other well-known arithmetic functions like the Möbius function, Liouville function, etc. and their divisor sums. Furthermore, we use a counting method of Erdős to obtain some counting theorems for $n$-color partitions that are analogous to those found by Andrews and Deutsch for the partition function.

1. Introduction

A partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ of a positive integer $m$ is a finite sequence of non-increasing positive integers $\lambda_i$, called parts, such that $m = \sum_{i=1}^{k} \lambda_i$. The partition function $p(m)$ is the number of partitions of $m$. For example, the partitions of 4 are (4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1), and hence, $p(4) = 5$.

An $n$-color partition (also called a partition with “$n$ copies of $n$”) of a positive integer $m$ is a partition in which a part of size $n$ can appear in $n$ different colors denoted by subscripts in $n_1, n_2, \ldots, n_n$ and the parts satisfy the order:

$$1_1 < 2_1 < 3_1 < 2_2 < 3_2 < 3_3 < 4_1 < 4_2 < 4_3 < 4_4 < \cdots$$

Let $PL(m)$ denote the number of $n$-color partitions of $m$. For example, $PL(4) = 13$, since there are 13 $n$-color partitions of 4, namely, (4), (4_1), (4_2), (4_3), (4_4), (3_1, 1_1), (3_1, 2), (3_2, 1), (3_3), (2_1, 2), (2_1, 1_1, 1), (1_1, 1, 1, 1, 1), and hence, $PL(4) = 13$. 


(3, 1), (3, 1), (2, 2), (2, 2), (2, 1, 1), (2, 1, 1) and (1, 1, 1, 1, 1). The generating function of \( PL(m) \) is given by
\[
\sum_{m=0}^{\infty} PL(m) q^m = \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^m}. 
\]
(1)

In [4, p. 1421], MacMahon observed that the right-hand side of Equation (1) also

\[ 
\text{generates the number of plane partitions of } m \text{ (Also see [9, Corollary 7.20.3]), where a plane partition } \pi \text{ of } m \text{ is an array of non-negative integers,}
\]

\[
\begin{array}{c}
m_{11} \quad m_{12} \quad m_{13} \\
m_{21} \quad m_{22} \quad m_{23} \\
\vdots \quad \vdots \quad \vdots 
\end{array}
\]

\( \sum m_{ij} = m \) and the rows and columns are in decreasing order, that is, \( m_{ij} \geq m_{i(j+1)}, m_{ij} \geq m_{i(j+1)} \), for all \( i, j \geq 1 \). For example, the plane partitions of 4 are

\[
\begin{array}{cccccccc}
4, & 31, & 3, & 22 & 2, & 211, & 21, & 2, \\
1 & & 2 & & & & & 1 \\
111, & 111, & 11, & 11, & 1, \\
1 & 1 & 11 & 1 & 1 \\
1 & & & 1 & 1 \\
\end{array}
\]

For further reading on \( n \)-color partitions and plane partitions we refer to [1, 4, 7, 8].

Recently, Merca and Schmidt in [5, 6] found some decompositions for the partition function \( p(m) \) in terms of the classical elementary functions, namely, the Möbius function and Euler’s totient. In this paper, we find some connections between the \( n \)-color partition function (equivalently, the plane partition function) \( PL(m) \) and elementary arithmetic functions and their divisor sums.

We define an associated function \( T_k^r(n) \) in two separate scenarios:

1. For \( r \leq k \), \( T_k^r(n) = \frac{1}{k} \times (\text{the number of } k^{\text{es}} \text{ in the } n^{\text{-color partitions of }} n \text{ with the smallest part at least } r) \).

2. For \( r > k \), \( T_k^r(n) \) equals the number of the \( n^{\text{-color partitions of }} n - k \text{ with the smallest part at least } r \) except the possibility of the part \( k < r \) being present in only one color.
We consider the following three examples. First we consider $T_2^3(5)$. We note that the number of 3’s in the $n$-color partitions of 5 with the smallest part at least 2 is 6, which is evident from the relevant partitions $3_1 + 2_1, 3_2 + 2_1, 3_3 + 2_1, 3_1 + 2_2, 3_2 + 2_1$ and $3_3 + 2_2$. Therefore, $T_2^3(5) = \frac{1}{3} \times 6 = 2$.

Next, we consider $T_3^2(5)$. Here $n - k = 5 - 2 = 3$. The $n$-color partitions of 3 with the smallest part at least 3 except the possibility of the part 2 being present in only one color are $3_1$, $3_2$ and $3_3$. Hence, $T_3^2(5) = 3$.

Finally, we consider $T_3^2(7)$. In this case, $n - k = 7 - 2 = 5$ and the $n$-color partitions of 5 with the smallest part at least 3, except the possibility of the part 2 being present in only one color, are $5_1, 5_2, 5_3, 5_4, 5_5, 3_1 + 2, 3_2 + 2, 3_3 + 2$. Thus, $T_3^2(7) = 8$.

The generating function of $T_r^k(m)$ is given in the following lemma.

**Lemma 1.** We have

$$\sum_{m=k}^{\infty} T_k^r(m)q^m = \frac{q^k}{1 - q^k} \prod_{m=r}^{\infty} \frac{1}{(1 - q^m)^m}.$$ 

We have the following main theorem that connects $PL(m), T_r^k(m),$ and elementary arithmetic functions.

**Theorem 1.** Let $A(m)$ be an arithmetic function for $m \geq 1$ and $B(m)$ be its divisor sum, that is,

$$B(m) = \sum_{d|m} A(d).$$

Also, define the functions $\ell_r(m)$ for $m \geq 0$ and $r \geq 1$ recursively as

$$\ell_1(m) = 1,$$

and

$$\ell_r(m) = \sum_{k=0}^{\lfloor m/r \rfloor} \binom{r + k - 1}{k} \ell_{r-1}(m - kr), \text{ for } r \geq 2. \quad (2)$$

Additionally, we set

$$\ell_0(m) = \begin{cases} 1, & \text{if } m = 0, \\ 0, & \text{for } m \geq 1. \end{cases}$$

Then for $m \geq 1$ and $1 \leq r \leq m$, we have

$$\sum_{k=1}^{m} PL(m - k)B(k) = \sum_{k=1}^{m} \sum_{j=0}^{m-k} A(k)T_r^k(m - j)\ell_{r-1}(j). \quad (3)$$
Recently, Andrews and Deutsch in [2] used a counting technique of Erdős to derive certain theorems that involves counting parts of the integer partition. The following theorem is one such result.

**Theorem 2.** Given $k \geq 1$, let $S_k(n)$ be the number of appearances of $k$ in the partitions of $n$. Also, in each partition of $n$, we count the number of times a part divisible by $k$ appears uniquely; then sum these numbers over all the partitions of $n$. Let this number be $S_{|k|}(n)$. Then,

$$S_{|k|}(n) = S_{2k}(n + k).$$

In this paper, we generalize the above theorem to the case of counting the number of times a part congruent to $s \pmod{k}$ appears uniquely for some $s$ satisfying $0 \leq s < k$, then summing these numbers over all the integer partitions of $n$. Furthermore, we apply the same techniques to give counting theorems for $n$-color partitions involving the counting function $T_{1k}(n)$, which is a special case of $T_{rk}(n)$ defined earlier.

The rest of the paper is organized as follows. In Sections 2 and 3, we prove Lemma 1 and Theorem 1, respectively. In Section 4, we present some corollaries and a detailed worked out example. In Section 5, we present an interesting identity involving PL$(n)$ and Euler’s totient $\phi$ that is analogous to a recent result of Merca and Schmidt in [5]. In the final section, we present a generalization of Theorem 2 and some counting theorems for $n$-color partitions involving $T_{1k}(n)$.

### 2. Proof of Lemma 1

**Proof.** If $G(q)$ denotes the generating function of the number of $n$-color partitions of $m$ with the least part being at least $r$, then

$$G(q) = \prod_{m=r}^{\infty} \frac{1}{(1 - q^m)^m}.$$

Marking the part $k \geq r$ with a counter $u$, let

$$G(q, u) = \frac{1}{(1 - q^r)(1 - q^{k-1})^{k-1}(1 - uq^k)(1 - q^{k+1})^{k+1} \cdots}.$$  

Note that $G(q, 1) = G(q)$. Each term of $G(q, u)$ is of the form $\ell \times u^j \times q^m$ where $j$ is the number of part $k$ present in the $n$-color partitions of $m$ and $\ell$ is the number of such $n$-color partitions where $j$ number of part $k$ are present. If we take derivative with respect to $u$ then the term becomes $\ell \times j \times u^{j-1} \times q^m$ and the terms without $u$ vanishes. Next, taking $u = 1$ helps to sum up the $q^m$ terms for each $m$, and we get the required generating function.
Hence, taking the symbolic derivative at \( u = 1 \), we obtain the generating function of \( k \times T_k^r(m) \) for \( r \leq k \) as

\[
\frac{dG(u)}{du}\bigg|_{u=1} = \frac{1}{(1-q)^r} \cdots \frac{1}{(1-q^{k-1})^{k-1}} \frac{kq^k}{(1-q^k)^{k+1}} \frac{1}{(1-q^{k+1})^{k+1}} \cdots
\]

\[
= \frac{kq^k}{1-q^k} \prod_{m=r}^{\infty} \frac{1}{(1-q^m)^m}.
\]

In case of \( r > k \), we consider \( h(m) \) to be the number of the \( n \)-color partitions of \( m \) with the least part being \( r \) except the possibility of the part \( k < r \) being present in only one color. Then

\[
\sum_{m=0}^{\infty} h(m)q^m = \frac{1}{1-q^k} \prod_{m=r}^{\infty} \frac{1}{(1-q^m)^m},
\]

from which, we have

\[
\sum_{m=0}^{\infty} h(m)q^{m+k} = \frac{q^k}{1-q^k} \prod_{m=r}^{\infty} \frac{1}{(1-q^m)^m},
\]

which can be rewritten, after adjusting the index of the sum on the left-hand side, as

\[
\sum_{m=k}^{\infty} h(m-k)q^m = \frac{q^k}{1-q^k} \prod_{m=r}^{\infty} \frac{1}{(1-q^m)^m}.
\]

Of course, the above gives the required generating function of \( T_k^r(m) \) for \( r > k \). \( \Box \)

3. Proof of Theorem 1

Observe that

\[
\left( \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^m} \right) \left( \sum_{k=1}^{\infty} \frac{A(k)q^k}{1-q^k} \right) = \left( \sum_{m=0}^{\infty} PL(m)q^m \right) \left( \sum_{k=1}^{\infty} B(k)q^k \right)
\]

\[
= \sum_{m=1}^{\infty} \left( \sum_{k=1}^{m} PL(m-k)B(k) \right) q^m. \quad (4)
\]
Again,
\[
\left( \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^m} \right) \left( \sum_{k=1}^{\infty} \frac{A(k)q^k}{1-q^k} \right) = 
\left( \prod_{m=1}^{r-1} \frac{1}{(1-q^m)^m} \right) \left( \prod_{m=r}^{\infty} \frac{1}{(1-q^m)^m} \sum_{k=1}^{\infty} \frac{A(k)q^k}{1-q^k} \right)
\]
\[= \left( \sum_{j=0}^{\infty} \ell_{r-1}(j)q^j \right) \left( \sum_{k=1}^{\infty} \sum_{m=k}^{\infty} A(k)T^r_k(m)q^m \right)
\]
\[= \left( \sum_{j=0}^{\infty} \ell_{r-1}(j)q^j \right) \left( \sum_{m=1}^{\infty} \sum_{k=1}^{m} A(k)T^r_k(m-j)q^m \right)
\]
\[= \sum_{m=1}^{\infty} \left( \sum_{j=0}^{m-1} \sum_{k=1}^{m-j} A(k)T^r_k(m-j)\ell_{r-1}(j) \right) q^m
\]
\[= \sum_{m=1}^{\infty} \left( \sum_{k=1}^{m} \sum_{j=0}^{m-k} A(k)T^r_k(m-j)\ell_{r-1}(j) \right) q^m. \tag{5}\]

Comparing Equation (4) and Equation (5) we arrive at the desired result.

Now we work out the definition of \( \ell_r(m) \). As in the proof, for \( r \geq 1 \),
\[
\sum_{m=0}^{\infty} \ell_r(m)q^m = \prod_{m=1}^{r} \frac{1}{(1-q^m)^m},
\]
from which, for \( r \geq 2 \), we see that
\[
\sum_{m=0}^{\infty} \ell_r(m)q^m = \left( \sum_{m=0}^{\infty} \ell_{r-1}(m)q^m \right) \left( \frac{1}{(1-q^r)^r} \right)
\]
\[= \left( \sum_{m=0}^{\infty} \ell_{r-1}(m)q^m \right) \left[ \sum_{k=0}^{\infty} \binom{r+k-1}{k} q^{kr} \right]
\]
\[= \sum_{m=0}^{\infty} \left( \sum_{k=0}^{\lfloor m/r \rfloor} \binom{r+k-1}{k} \ell_{r-1}(m-kr) \right) q^m. \tag{6}\]

Furthermore,
\[
\sum_{m=0}^{\infty} \ell_1(m)q^m = \frac{1}{1-q} = \sum_{m=0}^{\infty} q^m.
\tag{7}\]

From Equation (6) and Equation (7), we arrive at the definition of \( \ell_r(m) \) for \( r \geq 1 \).
It remains to show that our definition for $\ell_0(m)$ is consistent with the result. That is, we need to prove that
\[
\sum_{k=1}^{m} PL(m-k)B(k) = \sum_{k=1}^{m} A(k)T^1_k(m).
\]
(8)

We have
\[
\left( \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^m} \right) \left( \sum_{k=1}^{\infty} \frac{A(k)q^k}{1-q^k} \right) = \sum_{k=1}^{\infty} A(k) \sum_{m=k}^{\infty} T^1_k(m)q^m
\]
\[
= \sum_{m=1}^{\infty} \left( \sum_{k=1}^{m} A(k)T^1_k(m) \right) q^m. \]
(9)

Comparing Equation (4) and Equation (9), we arrive at Equation (8).

4. Corollaries and an Example

In this section, we substitute $A(m)$ and $B(m)$ in Theorem 1 with some well known pairs of arithmetic functions to arrive at some interesting corollaries.

**Corollary 1.** We have
\[
\sum_{k=1}^{m} PL(m-k) = T^1_1(m).
\]

**Proof.** Taking $A(m) = \lfloor \frac{1}{m} \rfloor$, $B(m) = \sum_{d|m} A(d) = 1$ and $r = 1$ in Equation (3) we easily arrive at the corollary.

**Corollary 2.** For $\mu$ being the Möbius function, and $m \geq r \geq 1$, we have
\[
PL(m-1) = \sum_{k=1}^{m} \sum_{j=0}^{m-k} \mu(k)T^r_k(m-j)\ell_{r-1}(j).
\]

**Proof.** Take $A(m) = \mu(m)$. Hence
\[
B(m) = \sum_{d|m} \mu(d) = \lfloor \frac{1}{m} \rfloor.
\]
Putting these in Equation (3) we arrive at the required result.
Corollary 3. If $\tau(m)$ is the number of positive divisors of $m$ for $m \geq 1$, and $m \geq r \geq 1$, then

$$\sum_{k=1}^{m} \text{PL}(m-k)\tau(k) = \sum_{k=1}^{m} \sum_{j=0}^{m-k} T_{r}^{k}(m-j)\ell_{r-1}(j).$$

Proof. Follows readily by substituting $A(m) = 1$ and $B(m) = \sum_{d|m} A(d) = \sum_{d|m} 1 = \tau(m)$ in Equation (3).

Corollary 4. For $\lambda(m)$ being the Liouville function, and $m \geq r \geq 1$, we have

$$\sum_{k=1}^{\lfloor\sqrt{m}\rfloor} \text{PL}(m-k^2) = \sum_{k=1}^{m} \sum_{j=0}^{m-k} \lambda(k)T_{r}^{k}(m-j)\ell_{r-1}(j).$$

Proof. Let $A(m) = \lambda(m)$. It is well known that

$$B(m) = \sum_{d|m} A(d) = \sum_{d|m} \lambda(d) = \begin{cases} 1, & \text{if } m \text{ is a square;} \\ 0, & \text{otherwise.} \end{cases}$$

Substituting the above in Equation (3) we readily arrive at the corollary.

Corollary 5. For $\alpha \geq 1$, let $\sigma_{\alpha}(m) = \sum_{d|m} d^\alpha$. Then, for $m \geq r \geq 1$,

$$\sum_{k=1}^{m} \text{PL}(m-k)\sigma_{\alpha}(k) = \sum_{k=1}^{m} \sum_{j=0}^{m-k} k^\alpha T_{r}^{k}(m-j)\ell_{r-1}(j). \quad (10)$$

Proof. Take $A(m) = m^{\alpha}$ so that

$$B(m) = \sum_{d|m} A(d) = \sum_{d|m} d^\alpha = \sigma_{\alpha}(m).$$

We substitute the above in Equation (3) to arrive at the proffered identity.

Corollary 6. For $m \geq r \geq 1$,

$$\sum_{k=1}^{m} \text{PL}(m-k) \log k = \sum_{1 \leq k \leq m, \ \text{prime}, \ p \prime, c \geq 1} \sum_{j=0}^{m-k} T_{r}^{k}(m-j)\ell_{r-1}(j) \log p. \quad (11)$$
**Proof.** Take $A(m) = \Lambda(m)$, the Von Mangoldt function. Then

$$B(m) = \sum_{d|m} \Lambda(m) = \log m.$$ 

The result now follows by substituting these in Equation (3). \qed

**Example.** We work out the case $m = 11$ and $r = 3$ in Equation (11). First of all, we generate the required values for $\ell_2(j)$. Using the definition Equation (2), we have

$$\ell_2(m) = \sum_{k=0}^{\lfloor m/2 \rfloor} \left( \binom{2+k-1}{k} \ell_1(m-2k) = \sum_{k=0}^{\lfloor m/2 \rfloor} (k+1) \right) = 1 + 2 + \ldots + (\lfloor m/2 \rfloor + 1).$$

Using the above, we have the values as shown in the following table.

<table>
<thead>
<tr>
<th>$j$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell_2(j)$</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>6</td>
<td>6</td>
<td>10</td>
<td>10</td>
<td>15</td>
<td>15</td>
</tr>
</tbody>
</table>

Setting $m = 11$, the left-hand side of Equation (11) becomes

$$\sum_{k=1}^{11} \text{PL}(11-k) \log k.$$ 

The corresponding coefficients of the $\log k$ terms are given in the following table.

<table>
<thead>
<tr>
<th>$\log k$</th>
<th>$\log 2$</th>
<th>$\log 3$</th>
<th>$\log 5$</th>
<th>$\log 7$</th>
<th>$\log 11$</th>
</tr>
</thead>
<tbody>
<tr>
<td>corresponding coefficients</td>
<td>497</td>
<td>190</td>
<td>49</td>
<td>13</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table 1**

Setting $m = 11$ and $r = 3$, the right-hand side of Equation (11) becomes

$$\sum_{1<k<11, \ k=p^c, \ p \ prime, \ c \geq 1}^{11-k} \sum_{j=0}^{11-k} T_k^3(11-j) \ell_2(j) \log p.$$ 

The coefficients of the $\log p$ terms are given in the following table, which matches with the values in Table 1 calculated for the left-hand side of Equation (11) for $m = 11$. 
As a demonstration, we now explicitly calculate the coefficient of \( \log 3 \), that is 190, from the combinatorial procedure. To this end, for the partitions of 11 with the smallest part at least 3, the following table helps to calculate \( T_3^3(11) \).

<table>
<thead>
<tr>
<th>Integer partition</th>
<th>Number of corresponding ( n )-color partitions</th>
<th>Total number of parts 3 present</th>
</tr>
</thead>
<tbody>
<tr>
<td>((8,3))</td>
<td>24</td>
<td>24</td>
</tr>
<tr>
<td>((5,3,3))</td>
<td>30</td>
<td>60</td>
</tr>
<tr>
<td>((4,4,3))</td>
<td>30</td>
<td>30</td>
</tr>
</tbody>
</table>

Thus, \( T_3^3(11) = \frac{1}{3}(24 + 60 + 30) = 38 \).

Next, for the partitions of 10 with the smallest part at least 3, we have the following table.

<table>
<thead>
<tr>
<th>Integer partition</th>
<th>Number of corresponding ( n )-color partitions</th>
<th>Total number of parts 3 present</th>
</tr>
</thead>
<tbody>
<tr>
<td>((7,3))</td>
<td>21</td>
<td>21</td>
</tr>
<tr>
<td>((4,3,3))</td>
<td>24</td>
<td>48</td>
</tr>
</tbody>
</table>

Therefore, \( T_3^3(10) = \frac{1}{3}(21 + 48) = 23 \).

For the partitions of 9 with the smallest part at least 3, the following table helps to calculate \( T_3^3(9) \).

<table>
<thead>
<tr>
<th>Integer partition</th>
<th>Number of corresponding ( n )-color partitions</th>
<th>Total number of parts 3 present</th>
</tr>
</thead>
<tbody>
<tr>
<td>((6,3))</td>
<td>18</td>
<td>18</td>
</tr>
<tr>
<td>((3,3,3))</td>
<td>10</td>
<td>30</td>
</tr>
</tbody>
</table>

Hence, \( T_3^3(9) = \frac{1}{3}(18 + 30) = 16 \).

For the partitions of 8 with the smallest part at least 3, we have the following
table that helps to calculate $T_3^3(8)$.

<table>
<thead>
<tr>
<th>Integer partition</th>
<th>Number of corresponding $n$-color partitions</th>
<th>Total number of parts 3 present</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5,3)</td>
<td>15</td>
<td>15</td>
</tr>
</tbody>
</table>

Thus, $T_3^3(8) = \frac{1}{3} \times 15 = 5$.

In a similar way, we calculate $T_3^3(7) = 4$, $T_3^3(6) = 4$, $T_3^3(3) = 1$ and $T_3^3(9) = 1$.

Now, using the table of $\ell_2(j)$ and the above values, we arrive at

$$\sum_{j=0}^{8} T_3^3(11-j)\ell_2(j) + \sum_{j=0}^{2} T_9^3(11-j)\ell_2(j) = 187 + 3 = 190,$$

which coincides with the coefficient 190 of log 3 in Table 2.

5. A Special Identity Involving Euler’s Totient $\phi$

We recall from [3, Theorem 309] that

$$\sum_{m=0}^{\infty} \frac{\phi(m)q^m}{1-q^m} = \frac{q}{(1-q)^2}. \quad (12)$$

Due to the existence of such a closed form, we can pursue a different approach for the case of $\phi$ function, as done in [5].

**Theorem 3.** For $m \geq 0$,

$$\text{PL}(m+2) - \text{PL}(m) = \frac{1}{2} \sum_{k=3}^{m+5} \phi(k)T_k^3(m+5). \quad (13)$$

**Proof.** Notice that

$$(1-q)(1-q^2)^2 \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^m} \sum_{k=1}^{\infty} \phi(k)q^k$$

$$= (q + q^2 - 3q^3 - q^4 + 2q^5) \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^m} + \sum_{m=3}^{\infty} \sum_{k=3}^{m} \phi(k)T_k^3(m)q^m. \quad (14)$$

Again, using the closed form Equation (12), we have

$$(1-q)(1-q^2)^2 \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^m} \sum_{k=1}^{\infty} \phi(k)q^k$$

$$= (q + q^2 - q^3 - q^4) \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^m}. \quad (15)$$
From Equation (14) and Equation (15), we find that
\[
\sum_{m=3}^{\infty} \left( \sum_{k=3}^{m} \phi(k)T^3_k(m) \right) q^m = 2(q^3 - q^5) \left( \sum_{m=0}^{\infty} \text{PL}(m)q^m \right) \\
= 2 \left( \sum_{m=3}^{\infty} \text{PL}(m-3) - \sum_{m=5}^{\infty} \text{PL}(m-5) \right) q^m.
\]

Equating the coefficients of $q^{m+5}$, for $m \geq 0$, from both sides of the above, we readily arrive at Equation (13) to finish the proof.

**Example.** We verify Theorem 3 for the case $m = 6$.

The left-hand side of (13) is $\text{PL}(8) - \text{PL}(6) = 160 - 48 = 112$.

On the other hand, the right-hand side of Equation (13) can be worked out as
\[
\frac{1}{2} \sum_{k=3}^{11} \phi(k)T^3_k(11) \\
= \frac{1}{2} \left( \phi(3)T^3_3(11) + \phi(4)T^3_4(11) + \phi(5)T^3_5(11) + \phi(6)T^3_6(11) + \phi(7)T^3_7(11) + \phi(8)T^3_8(11) + \phi(11)T^3_{11}(11) \right) \\
= \frac{1}{2} (2 \times 38 + 2 \times 22 + 4 \times 12 + 2 \times 5 + 6 \times 4 + 4 \times 3 + 10 \times 1) \\
= 112.
\]

Thus the result is verified for $n = 6$.

6. A Generalization of Theorem 2 and Some Counting Theorems for $n$-color Partitions

We state a generalization of Theorem 2 as follows.

**Theorem 4.** Given $k \geq 1$, in each partition of $n$ we count the number of times a part congruent to $s \pmod{k}$ appears uniquely for some $s$ satisfying $0 \leq s < k$, then sum these numbers over all the partitions of $n$. Let us call this $S_{s(k)}(n)$. Then
\[
S_{s(k)}(n) = S_{2k}(n + k - s) + S_{2k}(n - s) - S_{2k}(n - 2s).
\]

**Proof.** Approaching as in [2], assuming $n \geq 1$, $k \geq 1$, the generating function of $S_{s(k)}(n)$ is given by
\[
\sum_{n \geq 1} S_{2k}(n)q^n = \sum_{j=1}^{\infty} \frac{q^{kj+s}}{\prod_{n \neq kj+s}(1 - q^n)}.
\]
from which, we have

\[
\sum_{n \geq 1} S_{2k}(n)q^n = \frac{1}{\prod_n (1 - q^n)} \sum_{j=1}^{\infty} q^{kj+s}(1 - q^{kj+s})
\]

\[
= \frac{1}{\prod_n (1 - q^n)} \left( q^s \frac{q^k}{1 - q^k} - \frac{q^{2k}}{1 - q^{2k}} \right) - q^{2k} \frac{1}{\prod_n (1 - q^n)} \left( q^{-(k-s)} + q^s - q^{2s} \right)
\]

\[
= \left( \sum_{n \geq 1} S_{2k}(n)q^n \right) \left( q^{-(k-s)} + q^s - q^{2s} \right).
\]

Comparing the coefficients of \( q^n \) from both sides, we arrive at the desired result. \( \square \)

The following theorem presents the case for \( n \)-color partitions in the spirit of Theorem 2. This shows how functions of the form \( T^r_k(n) \) can be useful in such counting theorems.

**Theorem 5.** In each \( n \)-color partition of \( n \), we count the number of times a part divisible by \( k \) appears uniquely, then sum these numbers over all the \( n \)-color partitions of \( n \). Let us multiply this sum by \( \frac{1}{k} \) and call it \( T_{\mid k}^1(n) \). Then

\[
T_{\mid k}^1(n) = \sum_{j \geq 0} \left( T_{2k}^1(n - (2j - 1)k) + T_{2k}^1(n - 2jk) + T_{2k}^1(n - (2j + 1)k) \right).
\]

**Proof.** The generating function of \( T_{\mid k}^1(n) \) is given by

\[
k \sum_n T_{\mid k}^1(n)q^n = \sum_{j=1}^{\infty} \frac{kjq^{kj}}{(1 - q^{kj})^{k-j-1} \prod_{n \neq kj} (1 - q^n)^n}.
\]

Hence,

\[
\sum_n T_{\mid k}^1(n)q^n = \frac{1}{\prod_n (1 - q^n)^n} \left( \sum_{j=1}^{\infty} jq^{kj} - \sum_{j=1}^{\infty} jq^{2kj} \right)
\]

\[
= \frac{1}{\prod_n (1 - q^n)^n} \left( \frac{q^k}{(1 - q^k)^2} - \frac{q^{2k}}{(1 - q^{2k})^2} \right)
\]

\[
= \frac{1}{\prod_n (1 - q^n)^n} \frac{q^{2k}}{(1 - q^{2k})^2} \left( q^{-k} + 1 + q^k \right)
\]

\[
= \left( \sum_n T_{2k}^1(n)q^n \right) \left( \sum_{j=1}^{\infty} (q^{(2j-1)k} + q^{2jk} + q^{(2j+1)k}) \right).
\]
Comparing the coefficients of $q^n$ from both sides of Equation (16), we obtain the desired result.

In fact, we can also generalize this theorem to any part congruent to $s$ (mod $k$) as follows.

**Theorem 6.** In each $n$-color partition of $n$ we count the number of times a part congruent to $s$ (mod $k$) appears uniquely for some $s$ satisfying $0 \leq s < k$, then sum these numbers over all the $n$-color partitions of $n$. Let us call this $T_{s(k)}(n)$. Then,

$$T_{s(k)}(n) = (k + s)(T_{2k}^1(n + k - s) - T_{2k}^1(n - 2s)) + sT_{2k}^1(n - s) + 2k \sum_{l \geq 1} T_{2k}^1(n + k - s - kl) - k \sum_{l \geq 1} T_{2k}^1(n - 2s - 2kl).$$

**Proof.** The generating function of $T_{s(k)}(n)$ is given by

$$\sum_{n \geq 0} T_{s(k)}(n)q^n = \sum_{j \geq 1} \frac{(kj + s)q^{kj+s}}{(1 - q^{kj+s})^{kj+s-1} \prod_{n \neq kj+s}(1 - q^n)^n}.$$

Now,

$$\sum_{n \geq 0} T_{s(k)}(n)q^n = \sum_{j \geq 1} \frac{(kj + s)q^{kj+s}}{(1 - q^{kj+s})^{kj+s-1} \prod_{n \neq kj+s}(1 - q^n)^n}$$

$$= \sum_{j \geq 1} \frac{(kj + s)q^{kj+s}}{\prod_{n \geq 1}(1 - q^n)^n}$$

$$= \frac{1}{\prod_{n \geq 1}(1 - q^n)^n} \left(kq^s \sum_{j \geq 1} jq^{kj} + sq^s \sum_{j \geq 1} q^{kj} - kq^{2s} \sum_{j \geq 1} j q^{2kj} - sq^{2s} \sum_{j \geq 1} q^{2kj}\right)$$

$$= \frac{1}{\prod_{n \geq 1}(1 - q^n)^n} \left(kq^s \frac{q^k}{(1 - q^k)^2} + sq^s \frac{q^k}{1 - q^k} - kq^{2s} \frac{q^{2k}}{(1 - q^{2k})^2} - sq^{2s} \frac{q^{2k}}{1 - q^{2k}}\right)$$

$$= \frac{q^{2k}}{1 - q^{2k}} \frac{1}{\prod_{n \geq 1}(1 - q^n)^n} \left(kq^{-(k-s)} \frac{1 + q^k}{1 - q^k} + sq^{-(k-s)} (1 + q^k) - kq^{2s} \frac{1}{1 - q^{2k}} - sq^{2s}\right).$$
\[
\sum_{n \geq 0} T_{2k}(n) q^n = \left( \sum_{n \geq 0} T_{2k}(n) q^n \right) \left( kq^{-(k-s)} \left( 1 + 2 \sum_{l \geq 1} q^{kl} \right) + sq^{-(k-s)}(1 + q^k) \right) - kq^{2s} \left( l + \sum_{l \geq 1} q^{2kl} \right) \]
\[
= \left( \sum_{n \geq 0} T_{2k}(n) q^n \right) \left( kq^{-(k-s)} + sq^{-(k-s) + sq^2} - sq^2 + 2kq^{-(k-s)} \sum_{l \geq 1} q^{kl} \right) - kq^{2s} \sum_{l \geq 1} q^{2kl} \]
\[
= (k + s) \sum_n T_{2k}^1(n) q^{n-k-s} + s \sum_n T_{2k}^1(n) q^{n+s} - (k + s) \sum_n T_{2k}^1(n) q^{n+2s} + 2kq^{-(k-s)} \sum_n \left( \sum_{l \geq 1} T_{2k}^1(n - kl) \right) q^n - kq^{2s} \sum_n \left( \sum_{l \geq 1} T_{2k}^1(n - 2kl) \right) q^n \]
\[
= (k + s) \sum_n T_{2k}^1(n + k - s) q^n + s \sum_n T_{2k}^1(n - s) q^n - (k + s) \sum_n T_{2k}^1(n - 2s) q^n + 2k \sum_n \left( \sum_{l \geq 1} T_{2k}^1(n + k - s - kl) \right) q^n - k \sum_n \left( \sum_{l \geq 1} T_{2k}^1(n - 2s - 2kl) \right) q^n. \]

Comparing the coefficient of \( q^n \) from both sides, we arrive at the desired result. \( \square \)

It is to be noted that Theorem 5 can be obtained from Theorem 6 by putting \( s = 0 \), taking \( T_{|k|}(n) = \frac{1}{k} T_{0(k)}(n) \), and then rearranging the terms.

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