THE n-COLOR PARTITION FUNCTION AND SOME COUNTING THEOREMS

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Abstract

Recently, Merca and Schmidt found some decompositions for the partition function $p(n)$ in terms of the classical Möbius function as well as Euler's totient. In this paper, we define a counting function $T_k^r(m)$ on the set of n-color partitions of m for given positive integers k, r and relate the function with the n-color partition function and other well-known arithmetic functions like the Möbius function, Liouville function, etc. and their divisor sums. Furthermore, we use a counting method of Erdös to obtain some counting theorems for n-color partitions that are analogous to those found by Andrews and Deutsch for the partition function.

1. Introduction

A *partition* $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ of a positive integer m is a finite sequence of non-increasing positive integers λ_i, called parts, such that $m = \sum_{i=1}^k \lambda_i$. The partition function $p(m)$ is the number of partitions of m. For example, the partitions of 4 are $(4),\ (3,1),\ (2,2),\ (2,1,1),\ \ (1,1,1,1)$, and hence, $p(4) = 5$.

An n-color partition (also called a partition with "n copies of n") of a positive integer m is a partition in which a part of size n can appear in n different colors denoted by subscripts in $n_1, n_2, \ldots, n_n$ and the parts satisfy the order:

$$1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < 4_1 < 4_2 < 4_3 < 4_4 < \cdots .$$

Let $\mathrm{PL}(m)$ denote the number of n-color partitions of m. For example, $\mathrm{PL}(4) = 13$, since there are 13 n-color partitions of 4, namely, (4_1), (4_2), (4_3), (4_4), $(3_1, 1_1)$,

$(3_2, 1_1)$, $(3_3, 1_1)$, $(2_1, 2_1)$, $(2_2, 2_1)$, $(2_2, 2_2)$, $(2_1, 1_1, 1_1)$, $(2_2, 1_1, 1_1)$ and $(1_1, 1_1, 1_1, 1_1)$. The generating function of $\mathrm{PL}(m)$ is given by

$$\sum_{m=0}^{\infty} \mathrm{PL}(m) q^m = \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^m}. \tag{1}$$

In [4, p. 1421], MacMahon observed that the right-hand side of Equation (1) also generates the number of plane partitions of m (Also see [9, Corollary 7.20.3]), where a plane partition π of m is an array of non-negative integers,

$$\begin{matrix} m_{11} & m_{12} & m_{13} & \cdots \\ m_{21} & m_{22} & m_{23} & \cdots \\ \vdots & \vdots & \vdots & \end{matrix}$$

such that $\sum m_{ij} = m$ and the rows and columns are in decreasing order, that is, $m_{ij} \geq m_{(i+1)j}$, $m_{ij} \geq m_{i(j+1)}$, for all $i, j \geq 1$. For example, the plane partitions of 4 are

$$4, \qquad 31, \qquad \begin{matrix} 3, \\ 1 \end{matrix} \qquad 22 \qquad \begin{matrix} 2, \\ 2 \end{matrix} \qquad 211, \qquad \begin{matrix} 21, \\ 1 \end{matrix} \qquad \begin{matrix} 2, \\ 1 \\ 1 \end{matrix}$$

$$111, \qquad \begin{matrix} 111, \\ 1 \end{matrix} \qquad \begin{matrix} 11, \\ 1 \\ 1 \end{matrix} \qquad \begin{matrix} 11, \\ 11 \end{matrix} \qquad \begin{matrix} 1. \\ 1 \\ 1 \\ 1 \end{matrix}$$

For further reading on n-color partitions and plane partitions we refer to [1, 4, 7, 8].

Recently, Merca and Schmidt in [5, 6] found some decompositions for the partition function $p(m)$ in terms of the classical elementary functions, namely, the Möbius function and Euler's totient. In this paper, we find some connections between the n-color partition function (equivalently, the plane partition function) PL(m) and elementary arithmetic functions and their divisor sums.

We define an associated function $T_k^r(n)$ in two separate scenarios:

1. For $r \leq k$, $T_k^r(n) = \frac{1}{k} \times$ (the number of k's in the n-color partitions of n with the smallest part at least r).

2. For $r > k$, $T_k^r(n)$ equals the number of the n-color partitions of $n - k$ with the smallest part at least r except the possibility of the part $k < r$ being present in only one color.

We consider the following three examples. First we consider $T_3^2(5)$. We note that the number of 3's in the n-color partitions of 5 with the smallest part at least 2 is 6, which is evident from the relevant partitions $3_1+2_1, 3_2+2_2, 3_3+2_1, 3_1+2_2, 3_2+2_1$ and 3_3+2_2. Therefore, $T_3^2(5)=\dfrac{1}{3}\times 6=2$.

Next, we consider $T_2^3(5)$. Here $n-k=5-2=3$. The n-color partitions of 3 with the smallest part at least 3 except the possibility of the part 2 being present in only one color are $3_1, 3_2$ and 3_3. Hence, $T_2^3(5)=3$.

Finally, we consider $T_2^3(7)$. In this case, $n-k=7-2=5$ and the n-color partitions of 5 with the smallest part at least 3, except the possibility of the part 2 being present in only one color, are $5_1, 5_2, 5_3, 5_4, 5_5, 3_1+2, 3_2+2$, 3_3+2. Thus, $T_2^3(7)=8$.

The generating function of $T_k^r(m)$ is given in the following lemma.

Lemma 1. *We have*

$$\sum_{m=k}^{\infty} T_k^r(m)q^m = \frac{q^k}{1-q^k}\prod_{m=r}^{\infty}\frac{1}{(1-q^m)^m}.$$

We have the following main theorem that connects $\mathrm{PL}(m)$, $T_k^r(m)$, and elementary arithmetic functions.

Theorem 1. *Let $A(m)$ be an arithmetic function for $m \geqslant 1$ and $B(m)$ be its divisor sum, that is,*

$$B(m)=\sum_{d|m} A(d).$$

Also, define the functions $\ell_r(m)$ for $m \geqslant 0$ and $r \geqslant 1$ recursively as

$$\ell_1(m)=1,$$

and

$$\ell_r(m)=\sum_{k=0}^{\lfloor m/r\rfloor}\binom{r+k-1}{k}\ell_{r-1}(m-kr), \text{ for } r \geqslant 2. \tag{2}$$

Additionally, we set

$$\ell_0(m)=\begin{cases}1, & \text{if } m=0,\\ 0, & \text{for } m \geqslant 1.\end{cases}$$

Then for $m \geqslant 1$ and $1 \leqslant r \leqslant m$, we have

$$\sum_{k=1}^{m}\mathrm{PL}(m-k)B(k)=\sum_{k=1}^{m}\sum_{j=0}^{m-k}A(k)T_k^r(m-j)\ell_{r-1}(j). \tag{3}$$

Recently, Andrews and Deutsch in [2] used a counting technique of Erdős to derive certain theorems that involves counting parts of the integer partition. The following theorem is one such result.

Theorem 2. *Given $k \geq 1$, Let $S_k(n)$ be the number of appearances of k in the partitions of n. Also, in each partition of n, we count the number of times a part divisible by k appears uniquely; then sum these numbers over all the partitions of n. Let this number be $S_{|k}(n)$. Then,*

$$S_{|k}(n) = S_{2k}(n+k).$$

In this paper, we generalize the above theorem to the case of counting the number of times a part congruent to $s(\text{mod } k)$ appears uniquely for some s satisfying $0 \leqslant s < k$, then summing these numbers over all the integer partitions of n. Furthermore, we apply the same techniques to give counting theorems for n-color partitions involving the counting function $T_k^1(n)$, which is a special case of $T_k^r(n)$ defined earlier.

The rest of the paper is organized as follows. In Sections 2 and 3, we prove Lemma 1 and Theorem 1, respectively. In Section 4, we present some corollaries and a detailed worked out example. In Section 5, we present an interesting identity involving $\mathrm{PL}(n)$ and Euler's totient ϕ that is analogous to a recent result of Merca and Schmidt in [5]. In the final section, we present a generalization of Theorem 2 and some counting theorems for n-color partitions involving $T_k^1(n)$.

2. Proof of Lemma 1

Proof. If $G(q)$ denotes the generating function of the number of n-color partitions of m with the least part being at least r, then

$$G(q) = \prod_{m=r}^{\infty} \frac{1}{(1-q^m)^m}.$$

Marking the part $k \geq r$ with a counter u, let

$$G(q,u) = \frac{1}{(1-q^r)^r \cdots (1-q^{k-1})^{k-1}(1-uq^k)^k(1-q^{k+1})^{k+1}\cdots}.$$

Note that $G(q,1) = G(q)$. Each term of $G(q,u)$ is of the form $\ell \times u^j \times q^m$ where j is the number of part k present in the n-color partitions of m and ℓ is the number of such n-color partitions where j number of part k are present. If we take derivative with respect to u then the term becomes $\ell \times j \times u^{j-1} \times q^m$ and the terms without u vanishes. Next, taking $u = 1$ helps to sum up the q^m terms for each m, and we get the required generating function.

Hence, taking the symbolic derivative at $u = 1$, we obtain the generating function of $k \times T_k^r(m)$ for $r \leq k$ as

$$\begin{aligned}\left.\frac{dG(u)}{du}\right|_{u=1} &= \frac{1}{(1-q^r)^r} \cdots \frac{1}{(1-q^{k-1})^{k-1}} \frac{kq^k}{(1-q^k)^{k+1}} \frac{1}{(1-q^{k+1})^{k+1}} \cdots \\ &= \frac{kq^k}{1-q^k} \prod_{m=r}^{\infty} \frac{1}{(1-q^m)^m}.\end{aligned}$$

In case of $r > k$, we consider $h(m)$ to be the number of the n-color partitions of m with the least part being r except the possibility of the part $k < r$ being present in only one color. Then

$$\sum_{m=0}^{\infty} h(m)q^m = \frac{1}{1-q^k} \prod_{m=r}^{\infty} \frac{1}{(1-q^m)^m},$$

from which, we have

$$\sum_{m=0}^{\infty} h(m)q^{m+k} = \frac{q^k}{1-q^k} \prod_{m=r}^{\infty} \frac{1}{(1-q^m)^m},$$

which can be rewritten, after adjusting the index of the sum on the left-hand side, as

$$\sum_{m=k}^{\infty} h(m-k)q^m = \frac{q^k}{1-q^k} \prod_{m=r}^{\infty} \frac{1}{(1-q^m)^m}.$$

Of course, the above gives the required generating function of $T_k^r(m)$ for $r > k$. $\square$

3. Proof of Theorem 1

Observe that

$$\begin{aligned}\left(\prod_{m=1}^{\infty} \frac{1}{(1-q^m)^m}\right)\left(\sum_{k=1}^{\infty} \frac{A(k)q^k}{1-q^k}\right) &= \left(\sum_{m=0}^{\infty} \mathrm{PL}(m)q^m\right)\left(\sum_{k=1}^{\infty} B(k)q^k\right) \\ &= \sum_{m=1}^{\infty}\left(\sum_{k=1}^{m} \mathrm{PL}(m-k)B(k)\right)q^m. \qquad (4)\end{aligned}$$

Again,

$$
\begin{aligned}
&\left(\prod_{m=1}^{\infty}\frac{1}{(1-q^m)^m}\right)\left(\sum_{k=1}^{\infty}\frac{A(k)q^k}{1-q^k}\right)\\
&\qquad\qquad = \left(\prod_{m=1}^{r-1}\frac{1}{(1-q^m)^m}\right)\left(\prod_{m=r}^{\infty}\frac{1}{(1-q^m)^m}\sum_{k=1}^{\infty}\frac{A(k)q^k}{1-q^k}\right)\\
&\qquad\qquad = \left(\sum_{j=0}^{\infty}\ell_{r-1}(j)q^j\right)\left(\sum_{k=1}^{\infty}A(k)\sum_{m=k}^{\infty}T_k^r(m)q^m\right)\\
&\qquad\qquad = \left(\sum_{j=0}^{\infty}\ell_{r-1}(j)q^j\right)\left(\sum_{m=1}^{\infty}\sum_{k=1}^{m}A(k)T_k^r(m)q^m\right)\\
&\qquad\qquad = \sum_{m=1}^{\infty}\left(\sum_{j=0}^{m-1}\sum_{k=1}^{m-j}A(k)T_k^r(m-j)\ell_{r-1}(j)\right)q^m\\
&\qquad\qquad = \sum_{m=1}^{\infty}\left(\sum_{k=1}^{m}\sum_{j=0}^{m-k}A(k)T_k^r(m-j)\ell_{r-1}(j)\right)q^m. \qquad (5)
\end{aligned}
$$

Comparing Equation (4) and Equation (5) we arrive at the desired result.

Now we work out the definition of $\ell_r(m)$. As in the proof, for $r \geqslant 1$,

$$\sum_{m=0}^{\infty}\ell_r(m)q^m = \prod_{m=1}^{r}\frac{1}{(1-q^m)^m},$$

from which, for $r \geqslant 2$, we see that

$$
\begin{aligned}
\sum_{m=0}^{\infty}\ell_r(m)q^m &= \left(\sum_{m=0}^{\infty}\ell_{r-1}(m)q^m\right)\left(\frac{1}{(1-q^r)^r}\right)\\
&= \left(\sum_{m=0}^{\infty}\ell_{r-1}(m)q^m\right)\left[\sum_{k=0}^{\infty}\binom{r+k-1}{k}q^{kr}\right]\\
&= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{\lfloor m/r\rfloor}\binom{r+k-1}{k}\ell_{r-1}(m-kr)\right)q^m. \qquad (6)
\end{aligned}
$$

Furthermore,

$$\sum_{m=0}^{\infty}\ell_1(m)q^m = \frac{1}{1-q} = \sum_{m=0}^{\infty}q^m. \qquad (7)$$

From Equation (6) and Equation (7), we arrive at the definition of $\ell_r(m)$ for $r \geqslant 1$.

It remains to show that our definition for $\ell_0(m)$ is consistent with the result. That is, we need to prove that

$$\sum_{k=1}^{m} \mathrm{PL}(m-k)B(k) = \sum_{k=1}^{m} A(k)T_k^1(m). \tag{8}$$

We have

$$\begin{aligned}
&\left(\prod_{m=1}^{\infty} \frac{1}{(1-q^m)^m}\right)\left(\sum_{k=1}^{\infty} \frac{A(k)q^k}{1-q^k}\right) \\
&= \sum_{k=1}^{\infty} A(k) \sum_{m=k}^{\infty} T_k^1(m)q^m \\
&= \sum_{m=1}^{\infty} \left(\sum_{k=1}^{m} A(k)T_k^1(m)\right) q^n. \qquad (9)
\end{aligned}$$

Comparing Equation (4) and Equation (9), we arrive at Equation (8).

4. Corollaries and an Example

In this section, we substitute $A(m)$ and $B(m)$ in Theorem 1 with some well known pairs of arithmetic functions to arrive at some interesting corollaries.

Corollary 1. *We have*

$$\sum_{k=1}^{m} \mathrm{PL}(m-k) = T_1^1(m).$$

Proof. Taking $A(m) = \left\lfloor \frac{1}{m} \right\rfloor$, $B(m) = \sum_{d|m} A(d) = 1$ and $r = 1$ in Equation (3) we easily arrive at the corollary. □

Corollary 2. *For μ being the Möbius function, and $m \geq r \geq 1$, we have*

$$\mathrm{PL}(m-1) = \sum_{k=1}^{m} \sum_{j=0}^{m-k} \mu(k)T_k^r(m-j)\ell_{r-1}(j).$$

Proof. Take $A(m) = \mu(m)$. Hence

$$B(m) = \sum_{d|m} \mu(d) = \left\lfloor \frac{1}{m} \right\rfloor.$$

Putting these in Equation (3) we arrive at the required result. □

Corollary 3. *If $\tau(m)$ is the number of positive divisors of m for $m \geqslant 1$, and $m \geq r \geq 1$, then*

$$\sum_{k=1}^{m} \mathrm{PL}(m-k)\tau(k) = \sum_{k=1}^{m}\sum_{j=0}^{m-k} T_k^r(m-j)\ell_{r-1}(j).$$

Proof. Follows readily by substituting

$$A(m) = 1 \quad \text{and} \quad B(m) = \sum_{d|m} A(d) = \sum_{d|m} 1 = \tau(m)$$

in Equation (3). □

Corollary 4. *For $\lambda(m)$ being the Liouville function, and $m \geq r \geq 1$, we have*

$$\sum_{k=1}^{\lfloor\sqrt{m}\rfloor} \mathrm{PL}(m-k^2) = \sum_{k=1}^{m}\sum_{j=0}^{m-k} \lambda(k) T_k^r(m-j)\ell_{r-1}(j).$$

Proof. Let $A(m) = \lambda(m)$. It is well known that

$$B(m) = \sum_{d|m} A(d) = \sum_{d|m} \lambda(d) = \begin{cases} 1, & \text{if } m \text{ is a square;} \\ 0, & \text{otherwise.} \end{cases}.$$

Substituting the above in Equation (3) we readily arrive at the corollary. □

Corollary 5. *For $\alpha \geqslant 1$, let $\sigma_\alpha(m) = \sum_{d|m} d^\alpha$. Then, for $m \geq r \geq 1$,*

$$\sum_{k=1}^{m} \mathrm{PL}(m-k)\sigma_\alpha(k) = \sum_{k=1}^{m}\sum_{j=0}^{m-k} k^\alpha T_k^r(m-j)\ell_{r-1}(j). \tag{10}$$

Proof. Take $A(m) = m^\alpha$ so that

$$B(m) = \sum_{d|m} A(d) = \sum_{d|m} d^\alpha = \sigma_\alpha(m).$$

We substitute the above in Equation (3) to arrive at the proffered identity. □

Corollary 6. *For $m \geq r \geq 1$,*

$$\sum_{k=1}^{m} \mathrm{PL}(m-k)\log k = \sum_{\substack{1<k\leqslant m,\\ k=p^c, p\ prime, c\geqslant 1}} \sum_{j=0}^{m-k} T_k^r(m-j)\ell_{r-1}(j)\log p. \tag{11}$$

Proof. Take $A(m) = \Lambda(m)$, the Von Mangoldt function. Then

$$B(m) = \sum_{d|m} \Lambda(m) = \log m.$$

The result now follows by substituting these in Equation (3). $\square$

Example. We work out the case $m = 11$ and $r = 3$ in Equation (11). First of all, we generate the required values for $\ell_2(j)$. Using the definition Equation (2), we have

$$\begin{aligned}\ell_2(m) = \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{2+k-1}{k} \ell_1(m-2k) &= \sum_{k=0}^{\lfloor m/2 \rfloor} (k+1) \\ &= 1 + 2 + ... + (\lfloor m/2 \rfloor + 1).\end{aligned}$$

Using the above, we have the values as shown in the following table.

j	0	1	2	3	4	5	6	7	8	9
$\ell_2(j)$	1	1	3	3	6	6	10	10	15	15

Setting $m = 11$, the left-hand side of Equation (11) becomes

$$\sum_{k=1}^{11} \mathrm{PL}(11-k) \log k.$$

The corresponding coefficients of the $\log k$ terms are given in the following table.

$\log k$	$\log 2$	$\log 3$	$\log 5$	$\log 7$	$\log 11$
corresponding coefficients	497	190	49	13	1

Table 1

Setting $m = 11$ and $r = 3$, the right-hand side of Equation (11) becomes

$$\sum_{\substack{1<k\leqslant 11, \\ k=p^c,\ p \text{ prime},\ c \geqslant 1}} \sum_{j=0}^{11-k} T_k^3(11-j)\ell_2(j) \log p.$$

The coefficients of the $\log p$ terms are given in the following table, which matches with the values in Table 1 calculated for the left-hand side of Equation (11) for $m = 11$.

$\log p$	corresponding coefficients
$\log 2$	$\sum_{j=0}^{9} T_2^3(11-j)\ell_2(j) + \sum_{j=0}^{7} T_4^3(11-j)\ell_2(j)$ $+\sum_{j=0}^{3} T_8^3(11-j)\ell_2(j) = 497$
$\log 3$	$\sum_{j=0}^{8} T_3^3(11-j)\ell_2(j) + \sum_{j=0}^{2} T_9^3(11-j)\ell_2(j) = 190$
$\log 5$	$\sum_{j=0}^{6} T_5^3(11-j)\ell_2(j) = 49$
$\log 7$	$\sum_{j=0}^{4} T_7^3(11-j)\ell_2(j) = 13$
$\log 11$	1

Table 2

As a demonstration, we now explicitly calculate the coefficient of $\log 3$, that is 190, from the combinatorial procedure. To this end, for the partitions of 11 with the smallest part at least 3, the following table helps to calculate $T_3^3(11)$.

Integer partition	Number of corresponding n-color partitions	Total number of parts 3 present
(8,3)	24	24
(5,3,3)	30	60
(4,4,3)	30	30

Thus, $T_3^3(11) = \dfrac{1}{3}(24+60+30) = 38.$

Next, for the partitions of 10 with the smallest part at least 3, we have the following table.

Integer partition	Number of corresponding n-color partitions	Total number of parts 3 present
(7,3)	21	21
(4,3,3)	24	48

Therefore, $T_3^3(10) = \dfrac{1}{3}(21+48) = 23.$

For the partitions of 9 with the smallest part at least 3, the following table helps to calculate $T_3^3(9)$.

Integer partition	Number of corresponding n-color partitions	Total number of parts 3 present
(6,3)	18	18
(3,3,3)	10	30

Hence, $T_3^3(9) = \dfrac{1}{3}(18+30) = 16.$

For the partitions of 8 with the smallest part at least 3, we have the following table that helps to calculate $T_3^3(8)$.

Integer partition	Number of corresponding n-color partitions	Total number of parts 3 present
(5,3)	15	15

Thus, $T_3^3(8) = \frac{1}{3} \times 15 = 5$.

In a similar way, we calculate $T_3^3(7) = 4, T_3^3(6) = 4, T_3^3(3) = 1$ and $T_3^3(9) = 1$.

Now, using the table of $\ell_2(j)$ and the above values, we arrive at

$$\sum_{j=0}^{8} T_3^3(11-j)\ell_2(j) + \sum_{j=0}^{2} T_9^3(11-j)\ell_2(j) = 187 + 3 = 190,$$

which coincides with the coefficient 190 of $\log 3$ in Table 2.

5. A Special Identity Involving Euler's Totientϕ

We recall from [3, Theorem 309] that

$$\sum_{m=0}^{\infty} \frac{\phi(m)q^m}{1-q^m} = \frac{q}{(1-q)^2}. \tag{12}$$

Due to the existence of such a closed form, we can pursue a different approach for the case of ϕ function, as done in [5].

Theorem 3. *For $m \geqslant 0$,*

$$\mathrm{PL}(m+2) - \mathrm{PL}(m) = \frac{1}{2} \sum_{k=3}^{m+5} \phi(k) T_k^3(m+5). \tag{13}$$

Proof. Notice that

$$\begin{aligned}(1-q)(1-q^2)^2 &\prod_{m=1}^{\infty} \frac{1}{(1-q^m)^m} \sum_{k=1}^{\infty} \frac{\phi(k)q^k}{1-q^k} \\ &= (q+q^2-3q^3-q^4+2q^5) \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^m} + \sum_{m=3}^{\infty}\sum_{k=3}^{m} \phi(k) T_k^3(m) q^m. \end{aligned} \tag{14}$$

Again, using the closed form Equation (12), we have

$$\begin{aligned}(1-q)(1-q^2)^2 &\prod_{m=1}^{\infty} \frac{1}{(1-q^m)^m} \sum_{k=1}^{\infty} \frac{\phi(k)q^k}{1-q^k} \\ &= (q+q^2-q^3-q^4) \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^m}. \end{aligned} \tag{15}$$

From Equation (14) and Equation (15), we find that

$$\sum_{m=3}^{\infty}\left(\sum_{k=3}^{m}\phi(k)T_k^3(m)\right)q^m = 2(q^3-q^5)\left(\sum_{m=0}^{\infty}\mathrm{PL}(m)q^m\right)$$
$$= 2\left(\sum_{m=3}^{\infty}\mathrm{PL}(m-3)-\sum_{m=5}^{\infty}\mathrm{PL}(m-5)\right)q^m.$$

Equating the coefficients of q^{m+5}, for $m \geq 0$, from both sides of the above, we readily arrive at Equation (13) to finish the proof. $\square$

Example. We verify Theorem 3 for the case $m = 6$.

The left-hand side of (13) is $\mathrm{PL}(8) - \mathrm{PL}(6) = 160 - 48 = 112$.

On the other hand, the right-hand side of Equation (13) can be worked out as

$$\begin{aligned}
&\frac{1}{2}\sum_{k=3}^{11}\phi(k)T_k^3(11)\\
&= \frac{1}{2}\big(\phi(3)T_3^3(11)+\phi(4)T_4^3(11)+\phi(5)T_5^3(11)+\phi(6)T_6^3(11)+\phi(7)T_7^3(11)\\
&\quad+\phi(8)T_8^3(11)+\phi(11)T_{11}^3(11)\big)\\
&= \frac{1}{2}(2\times 38+2\times 22+4\times 12+2\times 5+6\times 4+4\times 3+10\times 1)\\
&= 112.
\end{aligned}$$

Thus the result is verified for $n = 6$.

6. A Generalization of Theorem 2 and Some Counting Theorems for *n*-color Partitions

We state a generalization of Theorem 2 as follows.

Theorem 4. *Given $k \geqslant 1$, in each partition of n we count the number of times a part congruent to $s(mod\ k)$ appears uniquely for some s satisfying $0 \leqslant s < k$, then sum these numbers over all the partitions of n. Let us call this $S_{s(k)}(n)$. Then*

$$S_{s(k)}(n) = S_{2k}(n+k-s) + S_{2k}(n-s) - S_{2k}(n-2s).$$

Proof. Approaching as in [2], assuming $n \geqslant 1$, $k \geqslant 1$, the generating function of $S_{s(k)}(n)$ is given by

$$\sum_{n\geqslant 1} S_{2k}(n)q^n = \sum_{j=1}^{\infty}\frac{q^{kj+s}}{\prod_{n\neq kj+s}(1-q^n)},$$

from which, we have

$$\begin{aligned}\sum_{n\geqslant 1} S_{2k}(n)q^n &= \frac{1}{\prod_n(1-q^n)}\sum_{j=1}^{\infty} q^{kj+s}(1-q^{kj+s})\\ &= \frac{1}{\prod_n(1-q^n)}\left(q^s\frac{q^k}{1-q^k}-q^{2s}\frac{q^{2k}}{1-q^{2k}}\right)\\ &= \frac{q^{2k}}{1-q^{2k}}\frac{1}{\prod_n(1-q^n)}\left(q^{-(k-s)}+q^s-q^{2s}\right)\\ &= \left(\sum_{n\geqslant 1} S_{2k}(n)q^n\right)\left(q^{-(k-s)}+q^s-q^{2s}\right).\end{aligned}$$

Comparing the coefficients of q^n from both sides, we arrive at the desired result. □

The following theorem presents the case for n-color partitions in the spirit of Theorem 2. This shows how functions of the form $T_k^r(n)$ can be useful in such counting theorems.

Theorem 5. *In each n-color partition of n, we count the number of times a part divisible by k appears uniquely, then sum these numbers over all the n-color partitions of n. Let us multiply this sum by $\frac{1}{k}$ and call it $T_{|k}(n)$. Then*

$$T_{|k}(n) = \sum_{j\geqslant 0}\Big(T_{2k}^1(n-(2j-1)k)+T_{2k}^1(n-2jk)+T_{2k}^1(n-(2j+1)k)\Big).$$

Proof. The generating function of $T_{|k}(n)$ is given by

$$k\sum_n T_{|k}(n)q^n = \sum_{j=1}^{\infty}\frac{kjq^{kj}}{(1-q^{kj})^{kj-1}\prod_{n\neq kj}(1-q^n)^n}.$$

Hence,

$$\begin{aligned}\sum_n T_{|k}(n)q^n &= \frac{\sum_{j=1}^{\infty} jq^{kj}(1-q^{kj})}{\prod_n(1-q^n)^n}\\ &= \frac{1}{\prod_n(1-q^n)^n}\Big(\sum_{j=1}^{\infty} jq^{kj}-\sum_{j=1}^{\infty} jq^{2kj}\Big)\\ &= \frac{1}{\prod_n(1-q^n)^n}\left(\frac{q^k}{(1-q^k)^2}-\frac{q^{2k}}{(1-q^{2k})^2}\right)\end{aligned}$$

$$\begin{aligned}&= \frac{1}{\prod_n(1-q^n)^n}\frac{q^{2k}}{(1-q^{2k})^2}\Big(q^{-k}+1+q^k\Big)\\ &= \Big(\sum_n T_{2k}^1(n)q^n\Big)\Big(\sum_{j=1}^{\infty}(q^{(2j-1)k}+q^{2jk}+q^{(2j+1)k})\Big). \qquad (16)\end{aligned}$$

Comparing the coefficients of q^n from both sides of Equation (16), we obtain the desired result. $\square$

In fact, we can also generalize this theorem to any part congruent to $s(\text{mod } k)$ as follows.

Theorem 6. *In each n-color partition of n we count the number of times a part congruent to $s(\text{mod } k)$ appears uniquely for some s satisfying $0 \leqslant s < k$, then sum these numbers over all the n-color partitions of n. Let us call this $T_{s(k)}(n)$. Then,*

$$\begin{aligned}T_{s(k)}(n) =&(k+s)(T^1_{2k}(n+k-s) - T^1_{2k}(n-2s)) + sT^1_{2k}(n-s)\\ &+ 2k\sum_{l\geqslant 1} T^1_{2k}(n+k-s-kl) - k\sum_{l\geqslant 1} T^1_{2k}(n-2s-2kl).\end{aligned}$$

Proof. The generating function of $T_{s(k)}(n)$ is given by

$$\sum_{n\geqslant 0} T_{s(k)}(n)q^n = \sum_{j\geqslant 1} \frac{(kj+s)q^{kj+s}}{(1-q^{kj+s})^{kj+s-1}\prod_{n\neq kj+s}(1-q^n)^n}.$$

Now,

$$\begin{aligned}
&\sum_{n\geqslant 0} T_{s(k)}(n)q^n\\
&= \sum_{j\geqslant 1} \frac{(kj+s)q^{kj+s}}{(1-q^{kj+s})^{kj+s-1}\prod_{n\neq kj+s}(1-q^n)^n}\\
&= \sum_{j\geqslant 1} \frac{(kj+s)q^{kj+s}(1-q^{kj+s})}{\prod_{n\geqslant 1}(1-q^n)^n}\\
&= \frac{1}{\prod_{n\geqslant 1}(1-q^n)^n}\left(kq^s\sum_{j\geqslant 1} jq^{kj} + sq^s\sum_{j\geqslant 1} q^{kj} - kq^{2s}\sum_{j\geqslant 1} jq^{2kj} - sq^{2s}\sum_{j\geqslant 1} q^{2kj}\right)\\
&= \frac{1}{\prod_{n\geqslant 1}(1-q^n)^n}\left(kq^s\frac{q^k}{(1-q^k)^2} + sq^s\frac{q^k}{1-q^k} - kq^{2s}\frac{q^{2k}}{(1-q^{2k})^2} - sq^{2s}\frac{q^{2k}}{1-q^{2k}}\right)\\
&= \frac{q^{2k}}{1-q^{2k}}\frac{1}{\prod_{n\geqslant 1}(1-q^n)^n}\left(kq^{-(k-s)}\frac{1+q^k}{1-q^k} + sq^{-(k-s)}(1+q^k) - kq^{2s}\frac{1}{1-q^{2k}}\right.\\
&\quad\left. - sq^{2s}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{n\geqslant 0} T^1_{2k}(n)q^n\right)\left(kq^{-(k-s)}\left(1+2\sum_{l\geqslant 1} q^{kl}\right) + sq^{-(k-s)}(1+q^k)\right.\\
&\quad \left. - kq^{2s}\left(l+\sum_{l\geqslant 1} q^{2kl}\right)\right)\\
&= \left(\sum_{n\geqslant 0} T^1_{2k}(n)q^n\right)\left(kq^{-(k-s)} + sq^{-(k-s)+sq^s-kq^{2s}-sq^{2s}+2kq^{-(k-s)}}\sum_{l\geqslant 1} q^{kl}\right.\\
&\quad \left. - kq^{2s}\sum_{l\geqslant 1} q^{2kl}\right)\\
&= (k+s)\sum_n T^1_{2k}(n)q^{n-k+s} + s\sum_n T^1_{2k}(n)q^{n+s} - (k+s)\sum_n T^1_{2k}(n)q^{n+2s}\\
&\quad + 2kq^{-(k-s)}\sum_n\left(\sum_{l\geqslant 1} T^1_{2k}(n-kl)\right)q^n - kq^{2s}\sum_n\left(\sum_{l\geqslant 1} T^1_{2k}(n-2kl)\right)q^n\\
&= (k+s)\sum_n T^1_{2k}(n+k-s)q^n + s\sum_n T^1_{2k}(n-s)q^n - (k+s)\sum_n T^1_{2k}(n-2s)q^n\\
&\quad + 2k\sum_n\left(\sum_{l\geqslant 1} T^1_{2k}(n+k-s-kl)\right)q^n - k\sum_n\left(\sum_{l\geqslant 1} T^1_{2k}(n-2s-2kl)\right)q^n.
\end{aligned}
$$

Comparing the coefficient of q^n from both sides, we arrive at the desired result. $\square$

It is to be noted that Theorem 5 can be obtained from Theorem 6 by putting $s=0$, taking $T_{|k}(n) = \frac{1}{k}T_{0(k)}(n)$, and then rearranging the terms.

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