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ON REPRESENTATIONS OF THE RIEMANN ZETA FUNCTION, ITS SQUARE AND OF THE POSITIVE INTEGERS

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Abstract

We prove that the Riemann zeta function and its square can be represented as multiple harmonic type of series and that every positive integer can be represented as a sum of multiple harmonic numbers.

1. Introduction

The Riemann zeta function ζ is defined in the half-plane $\mathrm{Re}(s) > 1$ by the absolutely convergent series

$$\zeta(s) = \sum_{j=1}^{\infty} \frac{1}{j^s} \tag{1}$$

and in the entire complex plane $\mathbb{C}$ (except the singularity at $s = 1$) by an analytic continuation. The importance of the Riemann zeta function is that it is related to number theory and other fields of mathematics. It is well known that the Riemann zeta function has many integral and series representations in different regions of convergence. In this paper we prove that the Riemann zeta function and its square can be represented as multiple harmonic type of series and that every positive integer can be represented as a finite sum of multiple harmonic numbers. It is based on the author's dissertation [4] in which he studied the higher order linear elliptic differential operators with singular coefficients acting in the Hilbert space $L^2(\Omega)$ (where Ω is an (not necessarily bounded) open domain with smooth boundary in $\mathbb{R}^n$, $n \geq 1$) and observed some connections between the norm estimates of fractional powers of the (Friedrichs) self-adjoint extensions of the operators and the elementary number theory. The connections depend only on the powers of the (Friedrichs) self-adjoint extensions and the norms of them but not on the extensions themselves and lead to representations of the Riemann zeta function, its square and of the positive integers. The estimates can be formally formulated for positive ($T \geq 0$) or lower semi-bounded ($T \geq -c$ for some $c \geq 0$) self-adjoint linear operators in abstract Hilbert spaces. If T is a lower semi-bounded, then for any fixed $\lambda \geq c$, $T_\lambda = T + \lambda$ will be positive, so it is enough to consider positive operators.

Let T be a positive self-adjoint linear operator in a Hilbert space $(H, \|\cdot\|)$ with (dense) domain $D(T)$. If $l \geq 1$ is an integer, $\tau_1, \tau_2, \ldots, \tau_l \in [0,1]$, $\tau = \sum_{j=1}^{l} \tau_j$ and $f \in D(T^l)$, then

$$\|T^\tau f\| \leq \prod_{j=0}^{l} \|T^j f\|^{t_{j,l}}, \tag{2}$$

where the numbers (the powers) $t_{j,l}$ (that are in $[0,1]$ by Lemma 2) are defined recursively by:

$$t_{0,l} = \prod_{k=1}^{l} (1 - \tau_k), \quad l \geq 1, \tag{3}$$

$$t_{j,l} = t_{j,l-1}(1 - \tau_l) + t_{j-1,l-1}\tau_l, \quad 1 \leq j \leq l-1,\, l \geq 2, \tag{4}$$

$$t_{l,l} = \prod_{k=1}^{l} \tau_k, \quad l \geq 1. \tag{5}$$

The numbers in (4) are (by Lemma 1) sums of the form

$$t_{j,l} = \sum_{\pi} \tau_{\pi(1)}\tau_{\pi(2)} \cdots \tau_{\pi(j)}(1 - \tau_{\pi(j+1)})(1 - \tau_{\pi(j+2)}) \cdots (1 - \tau_{\pi(l)}), \tag{6}$$

where the summation is taken over all permutations π of $\{1, 2, \ldots, l\}$ such that

$$\pi(1) < \cdots < \pi(j) \quad \text{and} \quad \pi(j+1) < \cdots < \pi(l).$$

The number of such permutations is $\binom{l}{j}$ which is the same as the number of j-combinations of $\{1, 2, \ldots, l\}$, or equivalently the number of $(l-j)$-combinations. Thus there are $\binom{l}{j}$ terms of the form

$$\tau_{i_1}\tau_{i_2} \cdots \tau_{i_j}(1 - \tau_{i_{j+1}})(1 - \tau_{i_{j+2}}) \cdots (1 - \tau_{i_l}) \tag{7}$$

in the sum (6) and any such term can be thought to be obtained from the term

$$\tau_1\tau_2 \cdots \tau_j(1 - \tau_{j+1})(1 - \tau_{j+2}) \cdots (1 - \tau_l)$$

(which is in (6)) by a permutation of the variables $\tau_1, \tau_2, \ldots, \tau_l$ such that $\pi(k) = i_k$, $1 \leq k \leq l$. It is clear that there are also $\binom{l}{j}$ terms of the form (3) and (5) (when $j = 0$ and $j = l$) because $\binom{l}{0} = \binom{l}{l} = 1$ for every integer $l \geq 1$.

If $l = 1$, in which case $\tau = \tau_1 \in [0,1]$, $f \in D(T)$, we see from (3) and (5) that $t_{0,1} = 1 - \tau$, $t_{1,1} = \tau$ and get from (2) the interpolation inequality ([3], [6])

$$\|T^\tau f\| \leq \|Tf\|^\tau \|f\|^{1-\tau}.$$

Instead of considering $[0,1]$, we could consider any closed interval $[\alpha, \beta]$, $0 \leq \alpha \leq \beta < \infty$, and generalize the estimate (2) but, since our focus is on the numbers $t_{j,l}$, we do not pay no more attention on the operators and their estimates.

The aim of this paper is to give elementary proofs of representations of the Riemann zeta function, its square and of the positive integers. The proofs are mainly based on combinatorial argument. The first result states that the Riemann zeta function and its square can be represented as multiple type of harmonic series involving all positive composite numbers. It is as follows.

Theorem 1. *For $s > 1$, the Riemann zeta function and its square can be written such that*

1)

$$\zeta(s) = \frac{1}{m(s)}\Big(1 + \sum_{j \geq 2} \sum_{k_j > k_{j-1} > \cdots > k_2 \geq 2} \frac{1}{(k_j^s - 1)(k_{j-1}^s - 1) \cdots (k_2^s - 1)}\Big),$$

2)

$$\zeta^2(s) = \frac{1}{m(s)}\Big(1 + \sum_{j \geq 2} j \sum_{k_j > k_{j-1} > \cdots > k_2 \geq 2} \frac{1}{(k_j^s - 1)(k_{j-1}^s - 1) \cdots (k_2^s - 1)}\Big).$$

Here $m(s)$ is defined by

$$m(s) = \prod_{(h)} \Big(1 + \frac{1}{h^s - 1}\Big), \tag{8}$$

where h ranges over all positive composite numbers and $s > 1$.

As far as the author knows, the results obtained in this paper have not been published previously in any journal.

2. Preliminaries

In this section we give two lemmas concerning combinatorial properties of the numbers $t_{j,l}$ defined in the introduction. We begin with a following lemma.

Lemma 1. *The numbers $t_{j,l}$, $1 \leq j \leq l-1$, $l \geq 2$ in* (4) *are sums of the form* (6).

Proof. We give a proof by induction on l. For $l = 2$, $j = 1$ and

$$t_{1,2} = \tau_1(1 - \tau_2) + \tau_2(1 - \tau_1),$$

where $\pi_1(1) = 1$, $\pi_1(2) = 2$ (the identity permutation) and $\pi_2(1) = 2$, $\pi_2(2) = 1$ (the transposition), so the lemma holds for $l = 2$. Suppose that the lemma holds (for $l \geq 2$); we will prove that it also holds for $l + 1$. We have

$$\begin{aligned} t_{j,l+1} &= (1-\tau_{l+1})t_{j,l} + \tau_{l+1}t_{j-1,l} \\ &= (1-\tau_{l+1})\sum_{\pi}\tau_{\pi(1)}\cdots\tau_{\pi(j)}(1-\tau_{\pi(j+1)})\cdots(1-\tau_{\pi(l)}) \\ &\quad + \tau_{l+1}\sum_{\pi}\tau_{\pi(1)}\cdots\tau_{\pi(j-1)}(1-\tau_{\pi(j)})(1-\tau_{\pi(j+1)})\cdots(1-\tau_{\pi(l)}) \\ &= \sum_{\pi}\tau_{\pi(1)}\cdots\tau_{\pi(j)}(1-\tau_{\pi(j+1)})\cdots(1-\tau_{\pi(l)})(1-\tau_{l+1}) \\ &\quad + \sum_{\pi}\tau_{\pi(1)}\cdots\tau_{\pi(j-1)}\tau_{l+1}(1-\tau_{\pi(j)})(1-\tau_{\pi(j+1)})\cdots(1-\tau_{\pi(l)}), \end{aligned}$$

where $1 - \tau_{l+1}$ appears in every terms of the first sum (of $\binom{l}{j}$ terms) and τ_{l+1} appears in every terms of the second sum (of $\binom{l}{j-1}$ terms). Therefore there are $\binom{l+1}{j} = \binom{l}{j} + \binom{l}{j-1}$ permutations and terms of the form

$$\tau_{i_1}\tau_{i_2}\cdots\tau_{i_j}(1-\tau_{i_{j+1}})(1-\tau_{i_{j+2}})\cdots(1-\tau_{i_l})(1-\tau_{i_{l+1}})$$

in $t_{j,l+1}$. After renumbering (and keeping the notations), it can be written as

$$t_{j,l+1} = \sum_{\pi}\tau_{\pi(1)}\cdots\tau_{\pi(j)}(1-\tau_{\pi(j+1)})\cdots(1-\tau_{\pi(l)})(1-\tau_{\pi(l+1)}),$$

and therefore it is of the form (6). This proves Lemma 1. □

In the next lemma we prove three basic properties of the numbers $t_{j,l}$ that we will apply in Lemma 3.

Lemma 2. *For $l \geq 1$ and $0 \leq j \leq l$, the numbers $t_{j,l}$ in (3)-(5) satisfy the following properties:*

1)

$$0 \leq t_{j,l} \leq 1; \tag{9}$$

2)

$$\sum_{j=0}^{l} t_{j,l} = 1; \tag{10}$$

3)

$$\sum_{j=1}^{l} \tau_j = \sum_{j=0}^{l} j t_{j,l}. \tag{11}$$

Proof. 1) The property (9) follows immediately from the equality (10), or can be proved by a direct induction proof on l.

2) The approach is a direct induction proof on l. Since $t_{0,1} = 1-\tau_1$ and $t_{1,1} = \tau_1$, the equality (10) is true for $l = 1$. Suppose that it is true (for $l \geq 1$); we will prove that it is also true for $l+1$. We have

$$\begin{aligned}
\sum_{j=0}^{l+1} t_{j,l+1} &= t_{0,l+1} + t_{l+1,l+1} + \sum_{j=1}^{l} t_{j,l+1} \\
&= t_{0,l+1} + t_{l+1,l+1} + \sum_{j=1}^{l} \big((1-\tau_{l+1})t_{j,l} + \tau_{l+1}t_{j-1,l}\big) \qquad \text{by (4)} \\
&= t_{0,l+1} + t_{l+1,l+1} + (1-\tau_{l+1})\sum_{j=1}^{l} t_{j,l} + \tau_{l+1}\sum_{j=1}^{l} t_{j-1,l} \\
&= t_{0,l+1} + t_{l+1,l+1} + (1-\tau_{l+1})\sum_{j=0}^{l} t_{j,l} - (1-\tau_{l+1})t_{0,l} + \tau_{l+1}\sum_{j=0}^{l} t_{j,l} - \tau_{l+1}t_{l,l} \\
&= t_{0,l+1} + t_{l+1,l+1} + 1 - \tau_{l+1} - (1-\tau_{l+1})t_{0,l} + \tau_{l+1} - \tau_{l+1}t_{l,l} \\
&= 1
\end{aligned}$$

because $(1-\tau_{l+1})t_{0,l} = t_{0,l+1}$ and $\tau_{l+1}t_{l,l} = t_{l+1,l+1}$.

3) As in the previous case, we use induction on l. For $l = 1$, it equals $\tau_1 = 1 \cdot t_{1,1}$, so (11) is true for that value of l. Suppose that (11) is true (for $l \geq 1$). Then for $l+1$, we have

$$\begin{aligned}
\sum_{j=0}^{l+1} jt_{j,l+1} &= \sum_{j=1}^{l} jt_{j,l+1} + (l+1)t_{l+1,l+1} \\
&= \sum_{j=1}^{l} j\big(t_{j,l}(1-\tau_{l+1}) + t_{j-1,l}\tau_{l+1}\big) + (l+1)t_{l+1,l+1} \\
&= (1-\tau_{l+1})\sum_{j=0}^{l} jt_{j,l} + \tau_{l+1}\sum_{j=0}^{l-1} jt_{j,l-1} + (l+1)t_{l+1,l+1} \\
&= (1-\tau_{l+1})\sum_{j=0}^{l} jt_{j,l} + \tau_{l+1}\sum_{j=0}^{l}(j+1)t_{j,l} - \tau_{l+1}(l+1)t_{l,l} + (l+1)t_{l+1,l+1} \\
&= \sum_{j=1}^{l} \tau_j + \tau_{l+1} - \tau_{l+1}(l+1)t_{l,l} + (l+1)t_{l+1,l+1} \\
&= \sum_{j=1}^{l+1} \tau_j.
\end{aligned}$$

Thus Lemma 2 is proved. □

Remark 1. Since $\sum_{j=0}^{l}\binom{l}{j}=2^l$, there are altogether 2^l terms of the form (3), (5) and (7) in the sum $\sum_{j=0}^{l} t_{j,l}$.

Denote by $q_{j,l}$ the product of all terms in $t_{j,l}$, $0\le j\le l$, $l\ge 1$. For $j=0$ and $j=l$, $q_{0,l}=t_{0,l}$ and $q_{l,l}=t_{l,l}$. For $1\le j\le l-1$, $l\ge 2$,

$$q_{j,l}=\tau_1^{m_{j-1,l-1}}\cdots\tau_l^{m_{j-1,l-1}}(1-\tau_1)^{n_{j,l-1}}\cdots(1-\tau_l)^{n_{j,l-1}}, \tag{12}$$

where

$$m_{j-1,l-1}=\binom{l-1}{j-1},\quad n_{j,l-1}=\binom{l-1}{j},\quad m_{j-1,l-1}+n_{j,l-1}=\binom{l}{j}.$$

The equalities (12) can be proved by induction on l. If $l=2$, then $j=1$ and $t_{1,2}=\tau_1(1-\tau_2)+(1-\tau_1)\tau_2$. Therefore $q_{1,2}=\tau_1\tau_2(1-\tau_1)(1-\tau_2)$, so $m_{0,1}=1$ (for both τ_1 and τ_2), $n_{1,1}=1$ (for both $1-\tau_1$ and $1-\tau_2$) with $\binom{2}{1}=m_{0,1}+n_{1,1}$. Thus the assertion is true for $l=2$. Suppose that it is true (for $l\ge 2$). We prove that it is true for $l+1$. Since

$$t_{j,l+1}=(1-\tau_{l+1})t_{j,l}+\tau_{l+1}t_{j-1,l},$$

we have

$$\begin{aligned}
q_{j,l+1}&=(1-\tau_{l+1})^{\binom{l}{j}}q_{j,l}\tau_{l+1}^{\binom{l}{j-1}}q_{j-1,l}\\
&=(1-\tau_{l+1})^{\binom{l}{j}}\tau_1^{\binom{l-1}{j-1}}\cdots\tau_l^{\binom{l-1}{j-1}}(1-\tau_1)^{\binom{l-1}{j}}\cdots(1-\tau_l)^{\binom{l-1}{j}}\\
&\qquad\times\tau_{l+1}^{\binom{l}{j-1}}\tau_1^{\binom{l-1}{j-2}}\cdots\tau_l^{\binom{l-1}{j-2}}(1-\tau_1)^{\binom{l-1}{j-1}}\cdots(1-\tau_l)^{\binom{l-1}{j-1}}\\
&=\tau_1^{\binom{l}{j-1}}\cdots\tau_{l+1}^{\binom{l}{j-1}}(1-\tau_1)^{\binom{l}{j}}\cdots(1-\tau_{l+1})^{\binom{l}{j}}
\end{aligned}$$

such that

$$\binom{l+1}{j}=m_{j-1,l}+n_{j,l},\quad m_{j-1,l}=\binom{l}{j-1},\quad n_{j,l}=\binom{l}{j}.$$

If we define $m_{0,l}=0$ and $n_{0,l}=1$ for $q_{0,l}$ and $m_{l,l}=1$ and $n_{l,l}=0$ for $q_{l,l}$, then the numbers $m_{0,l}$ $m_{j-1,l-1}$, $m_{l,l}$, (res. $n_{0,l}$, $n_{j,l-1}$, $n_{l,l}$) tell us how many times $\tau_1,\ldots,\tau_l$ (res. $1-\tau_1,\ldots,1-\tau_l$) appear in $t_{0,l}$, $t_{j,l}$ and $t_{l,l}$ for $1\le j\le l-1$, $l\ge 2$. Since

$$\prod_{j=0}^{l}q_{j,l}=\tau_1^{2^{l-1}}\cdots\tau_l^{2^{l-1}}(1-\tau_1)^{2^{l-1}}\cdots(1-\tau_l)^{2^{l-1}},$$

it follows that each of the numbers $\tau_1,\ldots,\tau_l$ and $1-\tau_1,\ldots,1-\tau_l$ appears 2^{l-1} times in (2).

Remark 2. There are a number of assertions which can be obtained as special cases of (10) and (11). We mention two that are well-known.

(a) For any integer $l \geq 1$ and any $\alpha \in]0,1[$,

$$\sum_{j=0}^{l} \binom{l}{j} \alpha^j (1-\alpha)^{l-j} = 1.$$

This is the sum of all probabilities of a binomial random variable on getting j successes on l trials, each with probability α.

(b) For any integer $l \geq 1$ and any $\alpha \in]0,1[$,

$$l\alpha = \sum_{j=0}^{l} j \binom{l}{j} \alpha^j (1-\alpha)^{l-j}.$$

This is the expected value of a binomial random variable on l trials, each with probability α of success. For both of these assertions, we refer to [2].

The assertion (a) follows from (10) and the assertion (b) form (11) when all the numbers τ_j are equal: $\tau_j = \alpha \in]0,1[$ for all $1 \leq j \leq l$. Thus (10) and (11) are the generalizations of the assertions (a) and (b) when interpreted probabilistically.

3. The Proofs of Representations

We need the following lemma.

Lemma 3. *Let* $\tau_1 = 1$, $\tau_j = \frac{1}{j^s}$, $2 \leq j \leq l$, $l \geq 2$, $s > 0$. *Then we have*

$$\prod_{k=2}^{l} \big(1 + \frac{1}{k^s - 1}\big) = 1 + \sum_{j=2}^{l} \sum_{k_j=j}^{l} \sum_{k_{j-1}=j-1}^{k_j - 1} \cdots \sum_{k_2=2}^{k_3-1} \frac{1}{(k_j^s - 1)(k_{j-1}^s - 1)\cdots(k_2^s - 1)}. \quad (13)$$

Proof. Our starting point are (the recurrence relations) (3)-(5). In order to apply them, we rewrite them as follows:

$$t_{0,l} = \prod_{k=1}^{l} (1 - \tau_k), \quad (14)$$

$$t_{j,l} = \Big(\sum_{k_j=j}^{l} \sum_{k_{j-1}=j-1}^{k_j-1} \cdots \sum_{k_2=2}^{k_3-1} \sum_{k_1=1}^{k_2-1} \frac{\tau_{k_j}\tau_{k_{j-1}}\cdots\tau_{k_2}\tau_{k_1}}{(1-\tau_{k_j})\cdots(1-\tau_{k_1})} \Big) t_{0,l}, \ \ 1 \leq j \leq l, l \geq 1. \quad (15)$$

Written out explicitly, they are

$$t_{0,l} = \prod_{k=1}^{l}(1-\tau_k), \tag{16}$$

$$t_{1,l} = \Big(\sum_{k_1=1}^{l} \frac{\tau_{k_1}}{1-\tau_{k_1}}\Big)t_{0,l}, \tag{17}$$

$$t_{2,l} = \Big(\sum_{k_2=2}^{l}\sum_{k_1=1}^{k_2-1} \frac{\tau_{k_2}}{1-\tau_{k_2}}\frac{\tau_{k_1}}{1-\tau_{k_1}}\Big)t_{0,l}, \tag{18}$$

$$t_{3,l} = \Big(\sum_{k_3=3}^{l}\sum_{k_2=2}^{k_3-1}\sum_{k_1=1}^{k_2-1} \frac{\tau_{k_3}}{1-\tau_{k_3}}\frac{\tau_{k_2}}{1-\tau_{k_2}}\frac{\tau_{k_1}}{1-\tau_{k_1}}\Big)t_{0,l}, \tag{19}$$

$$\vdots$$

$$t_{l-1,l} = \Big(\sum_{k_{l-1}=l-1}^{l}\sum_{k_{l-2}=l-2}^{k_{l-1}-1}\cdots\sum_{k_1=1}^{k_2-1} \frac{\tau_{k_{l-1}}}{1-\tau_{k_{l-1}}}\frac{\tau_{k_{l-2}}}{1-\tau_{k_{l-2}}}\cdots\frac{\tau_{k_1}}{1-\tau_{k_1}}\Big)t_{0,l}, \tag{20}$$

$$t_{l,l} = \prod_{k=1}^{l}\tau_k. \tag{21}$$

Then (10) becomes

$$t_{0,l} + \Big(\sum_{j=1}^{l}\sum_{k_j=j}^{l}\sum_{k_{j-1}=j-1}^{k_j-1}\cdots\sum_{k_2=2}^{k_3-1}\sum_{k_1=1}^{k_2-1}\frac{\tau_{k_j}\tau_{k_{j-1}}\cdots\tau_{k_2}\tau_{k_1}}{(1-\tau_{k_j})\cdots(1-\tau_{k_1})}\Big)t_{0,l} = 1$$

or

$$\prod_{k=1}^{l}\big(1+\frac{\tau_k}{1-\tau_k}\big) = 1 + \sum_{j=1}^{l}\sum_{k_j=j}^{l}\sum_{k_{j-1}=j-1}^{k_j-1}\cdots\sum_{k_2=2}^{k_3-1}\sum_{k_1=1}^{k_2-1}\frac{\tau_{k_j}\tau_{k_{j-1}}\cdots\tau_{k_2}\tau_{k_1}}{(1-\tau_{k_j})\cdots(1-\tau_{k_1})} \tag{22}$$

since

$$\frac{1}{t_{0,l}} = \prod_{k=1}^{l}\big(1+\frac{\tau_k}{1-\tau_k}\big).$$

The expressions (16)-(21) hold for all $(\tau_1,\cdots,\tau_l) \in [0,1[^l$, also when $\tau_j = 1$ for some j (or even for all j) but then some of the terms $t_{j,l}$ (all except $t_{l,l} = 1$) will disappear. For example, if $\tau_j = 1$ for at least one j, then $t_{0,l} = 0$. On the contrary, (22) does not hold for any $\tau_j = 1$. Therefore we will not use it as such but modified.

Assume now that $\tau_1 = 1$ and $\tau_j \in]0,1[$, $2 \le j \le l$, $l \ge 2$. Then it follows from (3)-(5) (or from (14)-(21)) that

$$t_{0,l} = 0, \quad t_{1,l} = \prod_{k=2}^{l}(1-\tau_k) \tag{23}$$

and

$$t_{j,l}=\Big(\sum_{k_j=j}^{l}\sum_{k_{j-1}=j-1}^{k_j-1}\cdots\sum_{k_3=3}^{k_4-1}\sum_{k_2=2}^{k_3-1}\frac{\tau_{k_j}\tau_{k_{j-1}}\cdots\tau_{k_3}\tau_{k_2}}{(1-\tau_{k_j})\cdots(1-\tau_{k_2})}\Big)t_{1,l},\ 2\le j\le l. \tag{24}$$

In this case (10) takes the form

$$t_{1,l}+\Big(\sum_{j=2}^{l}\sum_{k_j=j}^{l}\sum_{k_{j-1}=j-1}^{k_j-1}\cdots\sum_{k_2=2}^{k_3-1}\frac{\tau_{k_j}\tau_{k_{j-1}}\cdots\tau_{k_3}\tau_{k_2}}{(1-\tau_{k_j})\cdots(1-\tau_{k_2})}\Big)t_{1,l}=1$$

or

$$\prod_{k=2}^{l}\big(1+\frac{\tau_k}{1-\tau_k}\big)=1+\sum_{j=2}^{l}\sum_{k_j=j}^{l}\sum_{k_{j-1}=j-1}^{k_j-1}\cdots\sum_{k_3=3}^{k_4-1}\sum_{k_2=2}^{k_3-1}\frac{\tau_{k_j}\tau_{k_{j-1}}\cdots\tau_{k_3}\tau_{k_2}}{(1-\tau_{k_j})\cdots(1-\tau_{k_2})}. \tag{25}$$

If we choose $\tau_j=\frac{1}{j^s}$, $2\le j\le l$, $l\ge 2$, $s>0$ in (25), we get

$$\prod_{k=2}^{l}\big(1+\frac{1}{k^s-1}\big)=1+\sum_{j=2}^{l}\sum_{k_j=j}^{l}\sum_{k_{j-1}=j-1}^{k_j-1}\cdots\sum_{k_2=2}^{k_3-1}\frac{1}{(k_j^s-1)(k_{j-1}^s-1)\cdots(k_2^s-1)},$$

which proves Lemma 3. □

Now we are ready to prove Theorem 1.

Proof. 1): If $s>1$ and $l\to\infty$, then the product on the left side of (13) converges since the series

$$\sum_{k=2}^{\infty}\frac{1}{k^s-1}$$

converges for $s>1$. Then the expression of nested sums on the right side of (13) also converges. Since every integer $k\ge 2$ is either a prime or a composite, we can write

$$\prod_{k=2}^{\infty}\big(1+\frac{1}{k^s-1}\big)=\prod_{(p)}\big(1+\frac{1}{p^s-1}\big)\prod_{(h)}\big(1+\frac{1}{h^s-1}\big)=\zeta(s)m(s), \tag{26}$$

where p ranges over all positive prime numbers ($p\ge 2$) and h ranges over all positive composite numbers. Here

$$\zeta(s)=\prod_{(p)}\big(1+\frac{1}{p^s-1}\big) \tag{27}$$

is the Euler product formula of the Riemann zeta function ([5, (16.6)]) for $s > 1$ (or $\mathrm{Re}(s) > 1$) and m is given by (8) for $s > 1$ (or $\mathrm{Re}(s) > 1$). Hence we have the representation of the Riemann zeta function:

$$\zeta(s) \;=\; \frac{1}{m(s)}\Big(1 \;+\; \sum_{j=2}^{\infty}\sum_{k_j=j}^{\infty}\sum_{k_{j-1}=j-1}^{k_j-1}\cdots\sum_{k_2=2}^{k_3-1}\frac{1}{(k_j^s-1)(k_{j-1}^s-1)\cdots(k_2^s-1)}\Big),$$

for $s > 1$, therefore for $\mathrm{Re}(s) > 1$.

2): Consider next the property (11) and apply it when $\tau_1 = 1$ and $\tau_j \in\,]0,1[$, $2 \le j \le l$, $l \ge 2$. Then, by (23) and (24),

$$\sum_{j=1}^{l}\tau_j = t_{1,l} + \sum_{j=2}^{l} j t_{j,l}$$

or

$$\sum_{j=1}^{l}\tau_j = \prod_{k=2}^{l}(1-\tau_k) + \Big(\sum_{j=2}^{l} j \sum_{k_j=j}^{l}\sum_{k_{j-1}=j-1}^{k_j-1}\cdots\sum_{k_2=2}^{k_3-1}\frac{\tau_{k_j}\tau_{k_{j-1}}\cdots\tau_{k_3}\tau_{k_2}}{(1-\tau_{k_j})\cdots(1-\tau_{k_2})}\Big)\prod_{k=2}^{l}(1-\tau_k).$$

This is equal to

$$\Big(\sum_{j=1}^{l}\tau_j\Big)\prod_{k=2}^{l}\big(1+\frac{\tau_k}{1-\tau_k}\big) = 1 + \sum_{j=2}^{l} j \sum_{k_j=j}^{l}\sum_{k_{j-1}=j-1}^{k_k-1}\cdots\sum_{k_2=2}^{k_3-1}\frac{\tau_{k_j}\tau_{k_{j-1}}\cdots\tau_{k_3}\tau_{k_2}}{(1-\tau_{k_j})\cdots(1-\tau_{k_2})}.$$

Putting $\tau_j = \frac{1}{j^s}$, $1 \le j \le l$, $l \ge 2$, $s > 0$ into this, gives

$$\Big(\sum_{j=1}^{l}\frac{1}{j^s}\Big)\prod_{k=2}^{l}\big(1+\frac{1}{k^s-1}\big) = 1 + \sum_{k=2}^{l} j \sum_{k_j=j}^{l}\sum_{k_{j-1}=j-1}^{k_j-1}\cdots\sum_{k_2=2}^{k_3-1}\frac{1}{(k_j^s-1)(k_{j-1}^s-1)\cdots(k_2^s-1)}.$$

If $s > 1$, then the left side converges and (by (1) and (26)) have the limit $\zeta^2(s)m(s)$ as $l \to +\infty$. Hence so does the right side and have the same limit. Thus

$$\zeta^2(s) = \frac{1}{m(s)}\Big(1 + \sum_{j=2}^{\infty} j \sum_{k_j=j}^{\infty}\sum_{k_{j-1}=j-1}^{k_j-1}\cdots\sum_{k_2=2}^{k_3-1}\frac{1}{(k_j^s-1)(k_{j-1}^s-1)\cdots(k_2^s-1)}\Big),$$

for $s > 1$, therefore for $\mathrm{Re}(s) > 1$. This is the representation of the square of the Riemann zeta function, and completes the proof of Theorem 1. □

Remark 3. For $s > 1$ (or $\mathrm{Re}(s) > 1$), the Riemann zeta function can also be represented as a multiple harmonic type of series involving all positive prime numbers,

$$\zeta(s) = 1 + \sum_{j \geq 2} \sum_{p_j > p_{j-1} > \cdots > p_2 \geq 2} \frac{1}{(p_j^s - 1)(p_{j-1}^s - 1) \cdots (p_2^s - 1)},$$

where p_j is the jth positive prime number. This follows from (25) by taking $\tau_j = \frac{1}{p_j^s}$, $s > 1$, and letting $l \to +\infty$ because then the left side of it converges to the Riemann zeta function.

As a consequence of Lemma 3, we obtain our second result.

Theorem 2. *Every positive integer $l - 1$ ($l \geq 2$) can be written such that*

$$l - 1 = \sum_{j=2}^{l} \sum_{l-1 \geq k_{j-1} > k_{j-2} > \cdots > k_2 > k_1 \geq 1} \frac{1}{k_{j-1} k_{j-2} \cdots k_2 k_1}. \tag{28}$$

Proof. We set $s = 1$ in (13). On the left side, we have $\prod_{k=2}^{l} \left(1 + \frac{1}{k-1}\right) = l$. Renumbering $k_2 - 1 = i_1, k_3 - 1 = i_2, \ldots, k_j - 1 = i_{j-1}$ on the right side then gives

$$l - 1 = \sum_{j=2}^{l} \sum_{i_{j-1}=j-1}^{l-1} \sum_{i_{j-2}=j-2}^{i_{j-1}-1} \cdots \sum_{i_2=2}^{i_3-1} \sum_{i_1=1}^{i_2-1} \frac{1}{i_{j-1} i_{j-2} \cdots i_2 i_1}. \tag{29}$$

Thus every positive integer can be represented as a finite sum of multiple harmonic numbers. This proves Theorem 2. □

Remark 4. It is well known that for $n \geq 1$, the harmonic numbers $H_n(1)$ are not integers except in the case $n = 1$. In 2012, Chen and Tang [1, Theorem 1] generalized this result and proved that the multiple harmonic sums (numbers)

$$H_n(\{1\}^j) = \sum_{n \geq k_j > k_{j-1} > \cdots > k_2 > k_1 \geq 1} \frac{1}{k_j k_{j-1} \cdots k_2 k_1}, \quad 1 \leq j \leq n,$$

are not integers except in the cases $n = 1$, $j = 1$ and $n = 3$, $j = 2$.

Remark 5. Theorem 2 can also be proved by induction on l. For it, we write the right side of the representation in (28) explicitly (using the form of (29)),

$$\begin{aligned}
&\sum_{j=2}^{l} \sum_{k_{j-1}=j-1}^{l-1} \sum_{k_{j-2}=j-2}^{k_{j-1}-1} \cdots \sum_{k_2=2}^{k_3-1} \sum_{k_1=1}^{k_2-1} \frac{1}{k_{j-1}k_{j-2}\cdots k_2k_1} \\
&\quad = \sum_{k_1=1}^{l-1} \frac{1}{k_1} + \sum_{k_2=2}^{l-1} \frac{1}{k_2} \sum_{k_1=1}^{k_2-1} \frac{1}{k_1} + \sum_{k_3=3}^{l-1} \frac{1}{k_3} \sum_{k_2=2}^{k_3-1} \frac{1}{k_2} \sum_{k_1=1}^{k_2-1} \frac{1}{k_1} + \cdots \\
&\qquad + \sum_{k_{l-2}=l-2}^{l-1} \frac{1}{k_{l-2}} \sum_{k_{l-3}=l-3}^{k_{l-2}-1} \frac{1}{k_{l-3}} \cdots \sum_{k_2=2}^{k_3-1} \frac{1}{k_2} \sum_{k_1=1}^{k_2-1} \frac{1}{k_1} \\
&\qquad + \sum_{k_{l-1}=l-1}^{l-1} \frac{1}{k_{l-1}} \sum_{k_{l-2}=l-2}^{k_{l-1}-1} \frac{1}{k_{l-2}} \cdots \sum_{k_2=2}^{k_3-1} \frac{1}{k_2} \sum_{k_1=1}^{k_2-1} \frac{1}{k_1}. \qquad (30)
\end{aligned}$$

If $l = 2$, we see from (28) and (30) that $2 - 1 = \sum_{k_1=1}^{2-1} \frac{1}{k_1} = 1$, so it is true for that value of l. Suppose now that (28) is true (for $l \geq 2$) and prove that it is also true for $l + 1$. Replacing l on the right side of (28) by $l + 1$ and using (30), we get

$$\begin{aligned}
&\sum_{j=2}^{l+1} \sum_{k_{j-1}=j-1}^{l} \frac{1}{k_{j-1}} \sum_{k_{j-2}=j-2}^{k_{j-1}-1} \frac{1}{k_{j-2}} \cdots \sum_{k_2=2}^{k_3-1} \frac{1}{k_2} \sum_{k_1=1}^{k_2-1} \frac{1}{k_1} \\
&= \sum_{k_1=1}^{l} \frac{1}{k_1} + \sum_{k_2=2}^{l} \frac{1}{k_2} \sum_{k_1=1}^{k_2-1} \frac{1}{k_1} + \sum_{k_3=3}^{l} \frac{1}{k_3} \sum_{k_2=2}^{k_3-1} \frac{1}{k_2} \sum_{k_1=1}^{k_2-1} \frac{1}{k_1} + \cdots \\
&\quad + \sum_{k_{l-1}=l-1}^{l} \frac{1}{k_{l-1}} \sum_{k_{l-2}=l-2}^{k_{l-1}-1} \frac{1}{k_{l-2}} \cdots \sum_{k_2=2}^{k_3-1} \frac{1}{k_2} \sum_{k_1=1}^{k_2-1} \frac{1}{k_1} \\
&\quad + \sum_{k_l=l}^{l} \frac{1}{k_l} \sum_{k_{l-1}=l-1}^{k_l-1} \frac{1}{k_{l-1}} \sum_{k_{l-2}=l-2}^{k_{l-1}-1} \frac{1}{k_{l-2}} \cdots \sum_{k_2=2}^{k_3-1} \frac{1}{k_2} \sum_{k_1=1}^{k_2-1} \frac{1}{k_1} \\
&= \sum_{k_1=1}^{l-1} \frac{1}{k_1} + \sum_{k_2=2}^{l-1} \frac{1}{k_2} \sum_{k_1=1}^{k_2-1} \frac{1}{k_1} + \sum_{k_3=3}^{l-1} \frac{1}{k_3} \sum_{k_2=2}^{k_3-1} \frac{1}{k_2} \sum_{k_1=1}^{k_2-1} \frac{1}{k_1} + \cdots \\
&\quad + \sum_{k_{l-1}=l-1}^{l-1} \frac{1}{k_{l-1}} \sum_{k_{l-2}=l-2}^{k_{l-1}-1} \frac{1}{k_{l-2}} \cdots \sum_{k_2=2}^{k_3-1} \frac{1}{k_2} \sum_{k_1=1}^{k_2-1} \frac{1}{k_1} \\
&\quad + \frac{1}{l} + \frac{1}{l} \sum_{k_1=1}^{l-1} \frac{1}{k_1} + \frac{1}{l} \sum_{k_2=2}^{l} \frac{1}{k_2} \sum_{k_1=1}^{k_2-1} \frac{1}{k_1} + \frac{1}{l} \sum_{k_3=3}^{l-1} \frac{1}{k_3} \sum_{k_2=2}^{k_3-1} \frac{1}{k_2} \sum_{k_1=1}^{k_2-1} \frac{1}{k_1} + \cdots \\
&\quad + \frac{1}{l} \sum_{k_{l-1}=l-1}^{l-1} \frac{1}{k_{l-1}} \sum_{k_{l-2}=l-2}^{k_{l-1}-1} \frac{1}{k_{l-2}} \cdots \sum_{k_2=2}^{k_3-1} \frac{1}{k_2} \sum_{k_1=1}^{k_2-1} \frac{1}{k_1}
\end{aligned}$$

$$= \sum_{j=2}^{l} \sum_{k_{j-1}=j-1}^{l-1} \frac{1}{k_{j-1}} \sum_{k_{j-2}=j-2}^{k_{j-1}-1} \frac{1}{k_{j-2}} \cdots \sum_{k_2=2}^{k_3-1} \frac{1}{k_2} \sum_{k_1=1}^{k_2-1} \frac{1}{k_1}$$
$$+ \frac{1}{l}\Big(1 + \sum_{j=2}^{l} \sum_{k_{j-1}=j-1}^{l-1} \frac{1}{k_{j-1}} \sum_{k_{j-2}=j-2}^{k_{j-1}-1} \frac{1}{k_{j-2}} \cdots \sum_{k_2=2}^{k_3-1} \frac{1}{k_2} \sum_{k_1=1}^{k_2-1} \frac{1}{k_1}\Big)$$
$$= l - 1 + \frac{1}{l} l = l. \qquad \text{by (28)}$$

This proves Theorem 2.

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