#A86 INTEGERS 21 (2021)

ON SOME p-ADIC PROPERTIES AND SUPERCONGRUENCES OF DELANNOY AND SCHRÖDER NUMBERS

Tamás Lengyel
Department of Mathematics, Occidental College, USA
lengyel@oxy.edu

Received: 2/27/21, Revised: 6/7/21, Accepted: 8/13/21, Published: 8/27/21

Abstract

The Delannoy number $d(n)$ is defined as the number of paths from $(0,0)$ to (n,n) with steps (1,0), (1,1), and (0,1), which is equal to the number of paths from $(0,0)$ to $(2n,0)$ using only steps $(1,1)$, $(2,0)$ and $(1,-1)$. The Schröder number $s(n)$ counts only those paths that never go below the x-axis. We discuss some p-adic properties of the sequences $\{d(p^n)\}_{n\to\infty}$, and $\{d(ap^n+b)\}_{n\to\infty}$ with $a\in\mathbb{N}$, $(a,p)=1$, $b\in\mathbb{Z}$, and prime p. We also present similar p-adic properties of the Schröder numbers. We provide several supercongruences for these numbers and their differences. Some conjectures are also proposed.

1. Introduction

The central Delannoy number $d(n)$ is defined as the number of paths from $(0,0)$ to (n,n) in an $n\times n$ grid using only steps north, northeast and east (i.e., steps (1,0), (1,1), and (0,1)). With $n\geq 0$ the first few values are: 1, 3, 13, 63, 321, 1683, 8989, cf. A001850, [8]. It is also the number of paths from $(0,0)$ to $(2n,0)$ using only steps $(1,1)$, $(2,0)$ and $(1,-1)$. The corresponding paths are called Delannoy paths. It is well-known that

$$d(n)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}=\sum_{k=0}^{n}\binom{n}{k}^2 2^k, n\geq 0. \tag{1}$$

We prefer the second definition (cf. [11, Example and Note 3]) in (1) in most cases below, however, we also use the first definition in some proofs. We note that the generating function of these numbers is

$$\sum_{n=0}^{\infty}d(n)x^n=\frac{1}{\sqrt{1-6x+x^2}}=1+3x+13x^2+63x^3+321x^4+....,$$

although we do not directly use it in this paper.

The Schröder number $s(n)$ counts only the Delannoy paths from $(0,0)$ to $(2n,0)$ that never go below the x-axis.

It turns out that $d(n) = P_n(3)$ and $s(n) = (-P_{n-1}(3) + 6P_n(3) - P_{n+1}(3))/2$ with $P_n(x)$ being a Legendre polynomial $P_n(x) = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\left(\frac{x-1}{2}\right)^k$. The first definition in (1) implies that $d(p-1) \equiv 1 \pmod{p}$. Via the Legendre polynomial correspondence several supercongruences are derived in [3] and [4] for the sums $w(n,A,B) = \sum_{k=0}^{n}\binom{n}{k}^A\binom{n+k}{k}^B\epsilon^k$ where $A, B \in \mathbb{N}$ and $\epsilon = \pm 1$ with $\mathbb{N}$ being the set of non-negative integers. Although [3, Theorem 4] does not provide applicable results for $d(n)$, the methods can be extended to yield supercongruences for $d\big((mp^n-1)/2\big)$ with $m, n \in \mathbb{N}$, m odd, and $p \equiv 1 \pmod{4}$. In this case p can be written as $p = a^2 + b^2$, with $a, b \in \mathbb{Z}$. Let $a \equiv 1 \pmod{4}$ and i be a p-adic integer such that $i^2 = -1$. We fix the sign of bi such that $a \equiv bi \pmod{p}$. For instance, the congruences

$$d\left(\frac{p-1}{2}\right) \equiv (-1)^{(p-1)/4}2a \pmod{p}$$

and

$$d\left(\frac{mp^n-1}{2}\right) \equiv (-1)^{(p-1)/4}(a+bi)\, d\left(\frac{mp^{n-1}-1}{2}\right) \pmod{p^{2n}}$$

hold true according to [4, Theorem 1 and Example 2]. For example, for $p = 5$ we get that

$$d\left(\frac{m5^n-1}{2}\right) \equiv -\left(1 - 2\cdot, \ldots 223032431212_5\right) d\left(\frac{m5^{n-1}-1}{2}\right) \pmod{5^{2n}},$$

and the exponent $2n$ seems to be the best possible. The technical difficulties involved in the proofs suggest that supercongruences for $d(ap^n+b)$ might be difficult to come by, even less to be proven. In this paper, we explore some of these possibilities. We also included several conjectures.

We note that for primes $p > 3$, Liu [7] considered modulo p-power congruential properties of certain sums involving the first $p-1$ Delannoy and Schröder numbers. The results are unrelated to ours.

Note that the second definition in (1) implies that

$$d(n) = [x^n](1+2x)^n(1+x)^n = [x^n](1+3x+2x^2)^n. \tag{2}$$

For instance, $d(2) = 13$ is the coefficient of the quadratic term in $1 + 6x + 13x^2 + 12x^3 + 4x^4$. However, the appealing relation (2) involving generalized central trinomial coefficients does not seem to help in proving p-adic orders and supercongruences except if $p = 3$ (cf. Theorems 10–12).

We prove or conjecture results on the p-adic convergence of the sequences $\{d(ap^n + b)\}_{n\to\infty}$ and $\{s(ap^n + b)\}_{n\to\infty}$ with $a \in \mathbb{N}$, $(a,p) = 1$, $b \in \mathbb{Z}$, and prime p. (Here the notation (a,p) is used to denote the GCD of a and p.)

The existence of p-adic limits is established or conjectured in Theorem 9, identity (13), and Conjectures 4 and 7. In some cases the exact limit is determined, e.g., identities (3) and (9), and Theorems 10 and 17 provide the limit 0, and Theorems 3 and 15 come with other limit values.

In the proofs we apply various techniques relying on divisibilities properties of binomial coefficients (cf. Theorems 1 and 2), generating functions (cf. identities (11) and (29) used in Theorems 12 and Theorem 17, respectively), and infinite incongruent disjoint covering systems (in Theorem 13).

Regarding the Delannoy numbers, Section 2 contains the main results, Theorems 3, 5, 8–14 and Corollary 1, while Section 3 is devoted to the proofs. Theorems 3 and 4 deal with the case of $p = 2$, Theorems 5–12 cover the case with $p = 3$, and Theorems 13 and 14 take care of the odd primes, in general. In the last section we derive three results, Theorems 15, 17–18 and Corollary 2, on the Schröder numbers.

We will repeatedly use the so called power-lifting lemma from mathematical folklore.

Lemma 1. *If $2 \nmid n$, p is an odd prime, $p \nmid A$, $p \nmid B$, and $p \mid A \pm B$, then $\nu_p(A^n \pm B^n) = \nu_p(A \pm B) + \nu_p(n)$.*

We also need an improved version of Lucas' theorem on binomial coefficients taken modulo p^{q+1} where q is the p-adic order of the binomial coefficient.

Theorem 1 (Anton, 1869, Stickelberger, 1890, Hensel, 1902, cf. [5])**.** *Let $N = n_0+n_1p+\cdots+n_dp^d$, $M = m_0+m_1p+\cdots+m_dp^d$ and $R = N-M = r_0+r_1p+\cdots+r_dp^d$ with $0 \le n_i, m_i, r_i \le p-1$ for each i, be the base p representations of N, M, and $R = N - M$, respectively. Then with $q = \nu_p\big(\binom{N}{M}\big)$,*

$$(-1)^q \frac{1}{p^q}\binom{N}{M} \equiv \left(\frac{n_0!}{m_0!r_0!}\right)\left(\frac{n_1!}{m_1!r_1!}\right)\cdots\left(\frac{n_d!}{m_d!r_d!}\right) \pmod{p}.$$

We also use a generalization of the Jacobstahl–Kazandzidis congruences that involve the Bernoulli numbers B_n.

Theorem 2 ([1, Corollary 11.6.22])**.** *Let M and N such that $0 \le M \le N$ and p prime. We have*

$$\binom{pN}{pM} \equiv \begin{cases} \left(1 - \frac{B_{p-3}}{3}p^3NM(N-M)\right)\binom{N}{M} & \left(\text{mod } p^4NM(N-M)\binom{N}{M}\right), \text{ if } p \ge 5,\\ (1+45NM(N-M))\binom{N}{M} & \left(\text{mod } p^4NM(N-M)\binom{N}{M}\right), \text{ if } p = 3,\\ (-1)^{M(N-M)}P(N,M)\binom{N}{M} & \left(\text{mod } p^4NM(N-M)\binom{N}{M}\right), \text{ if } p = 2,\end{cases}$$

where $P(N,M) = 1+6NM(N-M)-4NM(N-M)(N^2-NM+M^2)+2(NM(N-M))^2$.

Remark 1 ([6, Remark 3.7])**.** It is well known that $\nu_p(B_n) \geq -1$ by the von Staudt–Clausen theorem.

2. Main Results

This section contains the main results.

Theorem 3. *For $n \geq 2$ and $a \geq 1$ odd, we have that*

$$\nu_2(d(a2^n) - 1) \geq 2n + 2.$$

It follows that the 2-adic limit $\lim_{n\to\infty} d(a2^n) = 1$.

Corollary 1. *For $p = 2$ and $n \geq 2$ even, we have that*

$$d(n) \equiv 1 \pmod{2^{2\nu_2(n)}}.$$

The proof of Theorem 3 guarantees the corollary while the exponent can be improved by an additional term of 2 if $4 \mid n$.

The next conjectures and theorem provide exact 2-adic orders. Apparently, the proofs of the conjectures might be difficult.

Conjecture 1. For $a \geq 1$ odd and $n \geq 2$, we have that

$$\nu_2(d(a2^n) - 1) = 3n$$

and

$$\nu_2(d(a2^n - 1) + 1) = 3n.$$

With respect to the second part we have the much stronger

Conjecture 2. For $m \geq 0$ we have that

$$\nu_2(d(4m + 3) + 1) = 3(\nu_2(m + 1) + 2).$$

Equivalently, we can also write that $\nu_2(d(m) + 1) = 3\nu_2(m + 1)$ if $m \equiv 3 \pmod 4, m \geq 3$. By setting $a2^n - 1 = 4m + 3, n \geq 2$, i.e., $m = a2^{n-2} - 1$, this conjecture claims that $\nu_2(d(a2^n - 1) + 1) = 3(\nu_2(a2^{n-2}) + 2) = 3n$.

We note that although one can directly guess the above conjecture, however, we used the IntegerSequences *Mathematica* package (cf. [9]) to guess the underlying structure and then we formulated the conjecture based on our finding.

It is easy to see the next lemma which covers the cases with $n = 1$.

Lemma 2. *For $m \geq 0$, we have*

$$\nu_2(d(2m+1)+1) = 2, \text{ if } 2 \mid m,$$

and

$$\nu_2(d(2m)+1) = 1.$$

In a similar spirit, we have another conjecture and lemma that take care of the first part of the original conjecture, Conjecture 1.

Conjecture 3. For $m \geq 0$ we have that

$$\nu_2(d(2m)-1) = 3\nu_2(2m), \text{ if } 2 \mid m.$$

Lemma 3. *For $m \geq 0$, we have*

$$\nu_2(d(2m)-1) = 2, \text{ if } 2 \nmid m,$$

and

$$\nu_2(d(2m+1)-1) = 1.$$

Theorem 4. *For $b \geq 1$ odd, we have that $\nu_2(d(a2^n+b)-1) = 1$ with $n \geq 1$, and $\nu_2(d(a2^n+3)+1) = 6$ with $n \geq 3$.*

Conjecture 4. For $a \geq 1$ odd and n sufficiently large, we have that

$$\nu_2(d(a2^{n+1}+b) - d(a2^n+b)) = \begin{cases} 3n, & \text{if } b = 0 \text{ or } -1, \\ n + 2\nu_2(b(b+1)), & \text{otherwise.} \end{cases}$$

The conjecture suggests that the 2-adic limit of the sequence $\{d(a2^n+b)\}_{n\to\infty}$ exists.

Now we are moving to odd primes.

Theorem 5. *For any odd prime $p \geq 5$ and $n \in \mathbb{N}$ we have that*

$$\nu_p(d(p^n)-3) \geq 1,$$

i.e., the least significant p-adic digit of $d(p^n)$ is 3: $d(p^n) \equiv 3 \pmod{p}$. For $p = 3$ and $n \geq 0$, we have the supercongruence

$$d(3^n) \equiv 3^{n+1} \pmod{3^{n+2}}; \tag{3}$$

and thus,

$$\nu_3(d(3^n)) = n+1. \tag{4}$$

For an alternative proof of (3) and (4) see Theorem 10.

Remark 2. Numerical experimentation suggests the 3-adic limit

$$\lim_{n\to\infty} d(3^n)/3^{n+1} = ...111202121222021_3.$$

Other interesting cases with $p = 3$ are covered in the next seven theorems and a conjecture.

Theorem 6. *For $n \geq 0$, we have the congruence*

$$d(3^n - 1) \equiv 1 \pmod 3; \tag{5}$$

and thus,

$$\nu_3(d(3^n - 1)) = 0. \tag{6}$$

Theorem 7. *For $n \geq 1$, we have the congruence*

$$d(3^n - 2) \equiv 3 \pmod 9; \tag{7}$$

and thus,

$$\nu_3(d(3^n - 2)) = 1. \tag{8}$$

Theorem 8. *For $n \geq 1$, we have the supercongruence*

$$d(3^n + 1) \equiv 2 \cdot 3^n \pmod{3^{n+1}}; \tag{9}$$

and thus,

$$\nu_3(d(3^n + 1)) = n. \tag{10}$$

The next theorem provides two supercongruences for differences of certain Delannoy numbers.

Theorem 9. *For $n \geq 1$, we have that*

$$d(3^{n+1}) - d(3^n) \equiv 2 \cdot 3^{n+1} \pmod{3^{n+2}}$$

and

$$\nu_3(d(3^{n+1}) - 3d(3^n)) \geq n + 3.$$

We also have that

$$d(3^{n+1} + 1) - d(3^n + 1) \equiv 3^n \pmod{3^{n+1}}$$

and

$$\nu_3(d(3^{n+1} + 1) - 3d(3^n + 1)) \geq n + 2.$$

In order to generalize some of the above results, Theorems 5 and 8, to $d(a3^n+b)$ in Theorems 10–12, we apply (2) and binomial expansion in the form

$$(3x+(1+2x^2))^{a3^n+b} = \sum_{k=0}^{a3^n+b} \binom{a3^n+b}{k}(3x)^k(1+2x^2)^{a3^n+b-k};$$

therefore,

$$\begin{aligned} d(a3^n+b) &= [x^{a3^n+b}](3x+(1+2x^2))^{a3^n+b} \\ &= \sum_{\substack{k=0 \\ a3^n+b-k \text{ even}}}^{a3^n+b} \binom{a3^n+b}{k} 3^k \binom{a3^n+b-k}{\frac{a3^n+b-k}{2}} 2^{\frac{a3^n+b-k}{2}}, \end{aligned} \tag{11}$$

Identity (11) is instrumental in the full characterization of $\nu_3(d(a3^n+b))$ with fixed integers a and b and sufficiently large n.

Remark 3. We note that identity (11) provides an alternative proof of Theorem 8 with $a=b=1$, $k=0$, and $n\geq 1$. In fact,

$$\binom{3^n+1}{\frac{3^n+1}{2}} 2^{\frac{3^n+1}{2}} \equiv 2\cdot 3^n \pmod{3^{n+1}}$$

since

$$\binom{3^n+1}{\frac{3^n+1}{2}} \equiv (-1)^n 3^n \pmod{3^{n+1}}$$

by Theorem 1, and $\frac{3^n+1}{2} \equiv n-1 \pmod 2$ (for $(4-1)^n+3 \equiv (-1)^n+3 \equiv 2n \pmod 4$). We also observe that

$$\nu_3\left(\binom{3^n+1}{k}3^k\right) \geq \begin{cases} n-\nu_3(k-1)+k, & \text{if } k\equiv 1 \pmod 3 \\ n+k, & \text{if } k\not\equiv 1 \pmod 3 \end{cases} \geq n+2$$

for $k\geq 2$.

In a similar fashion, we get that $d(3^n+2)\equiv 3^{n+1} \pmod{3^{n+2}}$, $\nu_3(d(3^n+2)) = n+1$ for $n\geq 1$ with $k=1$ and $d(3^n+3)\equiv 2\cdot 3^{n-1} \pmod{3^n}$, $\nu_3(d(3^n+3)) = n-1$ for $n\geq 2$ with $k=0$. Note that the last congruence can be also derived by the previous two congruences via a standard holonomic recurrence for the central Delannoy numbers, $d(n) = (3(2n-1)d(n-1)-(n-1)d(n-2))/n$.

Remark 4. Alternative proofs of (6) and (8) are provided by Theorem 12. Other alternative proofs of (3) and (4) as well as (10) follow by Theorems 10 and 11, respectively.

For the special case with $b=0$ we mention the following theorem.

Theorem 10. *For $a \geq 1$ odd with $(a,3)=1$, we have that*

$$d(a3^n) \equiv a\binom{a-1}{\frac{a-1}{2}}3^{n+1}2^{\frac{a-1}{2}}$$
$$\equiv \begin{cases} \binom{a-1}{\frac{a-1}{2}}3^{n+1} & \pmod{3^{n+2+\nu_3\left(\binom{a-1}{\frac{a-1}{2}}\right)}}, \quad \text{if } a \equiv 1, 11 \pmod{12}, \\ 2\binom{a-1}{\frac{a-1}{2}}3^{n+1} & \pmod{3^{n+2+\nu_3\left(\binom{a-1}{\frac{a-1}{2}}\right)}}, \quad \text{if } a \equiv 5, 7 \pmod{12}; \end{cases}$$

and therefore,

$$\nu_3(d(a3^n)) = n + 1 + \nu_3\left(\binom{a-1}{\frac{a-1}{2}}\right)$$

for any sufficiently large n. For $a \geq 1$ even with $(a,3)=1$ and n sufficiently large n, we have that

$$d(a3^n) \equiv (-1)^{a/2}\binom{a}{\frac{a}{2}} \pmod{3^{\nu_3\left(\binom{a}{\frac{a}{2}}\right)+1}};$$

and thus,

$$\nu_3(d(a3^n)) = \nu_3\left(\binom{a}{\frac{a}{2}}\right).$$

Note that (3) and (4) of Theorem 5 follow by applying Theorem 10 with $a=1$.

We can also use identity (11) to prove the next two theorems. Theorem 11 generalizes Theorem 8 and Remark 3.

Theorem 11. *For $b > 0$ and $k = 0$ or 1, we set*

$$c = c(b,k) = \begin{cases} \log_3(b-k)+1, & \text{if } b-k = 3^m \text{ for some integer } m \geq 0, \\ \lceil \log_3(b-k) \rceil, & \text{otherwise.} \end{cases}$$

For $a \geq 1$ odd with $(a,3)=1$, $b \geq 1$ odd, and n sufficiently large, we have that $k=0$ provides the smallest 3-adic order in (11) and

$$\nu_3(d(a3^n+b)) = n - c + \nu_3\left(\binom{3^c+b}{\frac{3^c+b}{2}}\binom{a-1}{\frac{a-1}{2}}\right).$$

For $a \geq 2$ even with $(a,3)=1$, $b \geq 2$ even, and n sufficiently large, we have that $k=0$ provides the smallest 3-adic order in (11) and

$$\nu_3(d(a3^n+b)) = \nu_3\left(\binom{b}{\frac{b}{2}}\binom{a}{\frac{a}{2}}\right).$$

For $a \geq 2$ even with $(a,3)=1$, $b \geq 1$ odd, and n sufficiently large, we have that $k=1$ provides the smallest 3-adic order in (11) and

$$\nu_3(d(a3^n+b)) = 1 + \nu_3(b) + \nu_3\left(\binom{b-1}{\frac{b-1}{2}}\binom{a}{\frac{a}{2}}\right).$$

For $a \geq 1$ odd with $(a,3) = 1$, $b \geq 2$ even, and n sufficiently large, we have that $k = 1$ provides the smallest 3-adic order in (11) and

$$\nu_3(d(a3^n + b)) = n - c + 1 + \nu_3(b) + \nu_3\left(\binom{3^c + b - 1}{\frac{3^c+b-1}{2}}\binom{a-1}{\frac{a-1}{2}}\right).$$

The next theorem generalizes Theorems 6 and 7 for other negative values of b in $d(a3^n + b)$.

Theorem 12. *For $b < 0$ and $k = 0$ or 1, we set*

$$c = c(b,k) = \begin{cases} \log_3 |b-k| + 1, & \text{if } |b-k| = 3^m \text{ for some integer } m \geq 0, \\ \lceil \log_3 |b-k| \rceil, & \text{otherwise.} \end{cases}$$

For $a \geq 1$ odd with $(a,3) = 1$, $b < 0$ odd, and n sufficiently large, we have that $k = 0$ provides the smallest 3-adic order in (11) and

$$\nu_3(d(a3^n + b)) = \nu_3\left(\binom{3^c + b}{\frac{3^c+b}{2}}\binom{a-1}{\frac{a-1}{2}}\right).$$

For $a \geq 2$ even with $(a,3) = 1$, $b < 0$ even, and n sufficiently large, we have that $k = 0$ provides the smallest 3-adic order in (11) and

$$\nu_3(d(a3^n + b)) = n - c + \nu_3\left(\binom{2 \cdot 3^c + b}{\frac{2\cdot 3^c+b}{2}}\binom{a}{\frac{a}{2}}\right).$$

For $a \geq 2$ even with $(a,3) = 1$, $b < 0$ odd, and n sufficiently large, we have that $k = 1$ provides the smallest 3-adic order in (11) and

$$\nu_3(d(a3^n + b)) = n - c + 1 + \nu_3(b) + \nu_3\left(\binom{2 \cdot 3^c + b - 1}{\frac{2\cdot 3^c+b-1}{2}}\binom{a}{\frac{a}{2}}\right).$$

For $a \geq 1$ odd with $(a,3) = 1$, $b < 0$ even, and n sufficiently large, we have that $k = 1$ provides the smallest 3-adic order in (11) and

$$\nu_3(d(a3^n + b)) = 1 + \nu_3(b) + \nu_3\left(\binom{3^c + b - 1}{\frac{3^c+b-1}{2}}\binom{a-1}{\frac{a-1}{2}}\right).$$

Remark 5. In each case, we obtain the lower bound $\nu_3(d(a3^n + b)) \geq n - c$ (with c defined in Theorems 11 and 12) or $\nu_3(d(a3^n + b)) = c'$ with some constant $c' = c'(a,b) \geq 0$, for any sufficiently large n. In Theorems 11 and 12 we did not include supercongruences for $d(a3^n + b)$. They can be derived the same way as in Theorem 10. These theorems imply the upper bound $\nu_3(d(a3^n + b)) \leq n + c''$ with some constant $c'' = c''(a,b)$, for any sufficiently large n.

The next conjecture provides the exact 3-adic valuations in certain cases.

Conjecture 5. For $a \geq 1$ odd and $b = 0$, or $a = 2$ and $b = -1$, and n sufficiently large, we have that

$$\nu_3(d(a3^{n+1} + b) - 3d(a3^n + b)) = 3(n + 1).$$

For $a \geq 1$ odd and $b \geq 1$, or $a \geq 2$ even and $b < 0$, and n sufficiently large, we have that

$$\nu_3(d(a3^{n+1} + b) - 3d(a3^n + b)) = 2n + c''$$

with some integer constant $c'' = c''(a, b)$.

We generalize Theorems 5 and 9 for arbitrary odd prime p in the following theorem.

Theorem 13. *For any odd prime p, we have that*

$$d(p^n) \equiv 1 + 2^{p^n} \pmod{p^2}, \quad \textit{if } n \geq 1; \tag{12}$$

$$d(p^{n+1}) - d(p^n) \equiv 2^{p^n}(2^{(p-1)p^n} - 1) = 2^{p^n} Q_p(2) p^{n+1} \pmod{p^{n+2}}, \quad \textit{if } n \geq 0; \tag{13}$$

$$\nu_p(d(p^{n+1}) - d(p^n)) = \nu_p(2^{(p-1)p^n} - 1),$$

and ν_p equals $n+1$ if $\nu_p(Q_p(2)) = 0$ holds for the Fermat quotient $Q_p(2) = (2^{p-1} - 1)/p$ while it is at least $n + 2$ otherwise for $\nu_p(2^{(p-1)p^n} - 1) = n + 1 + \nu_p(Q_p(2))$.

Now we turn to the differences of the quantities $d(ap^n + b)$. We define

$$\begin{aligned}\Delta_n = \Delta_n(a, b, p) &= d(ap^{n+1} + b) - d(ap^n + b) \\ &= \sum_{k=0}^{ap^{n+1}+b} \binom{ap^{n+1} + b}{k}^2 2^k - \sum_{k=0}^{ap^n+b} \binom{ap^n + b}{k}^2 2^k\end{aligned}$$

with $(a, p) = 1$. The goal is to prove a congruential relationship between Δ_{n+1} and Δ_n. The next theorem takes care of the case with $a = 1$ and $b = 0$.

Theorem 14. *For $n \in \mathbb{N}$ and odd prime p, we have that*

$$\Delta_{n+1}(1, 0, p) \equiv p2^{(p-1)p^{n-1}} \Delta_n(1, 0, p) \pmod{p^{n+2}}.$$

We conclude this section with a conjecture on $\Delta_n(a, b, p)$ and its relation to $\Delta_{n+1}(a, b, p)$.

Conjecture 6. For any odd prime p, $a \geq 1$ with $(a, p) = 1$, $b \in \mathbb{Z}$, and n sufficiently large, we have that $\nu_p(\Delta_n(a, b, p))$ is a linear function of n and

$$\nu_p\left(\Delta_{n+1}(a, b, p) - p2^{(p-1)p^{n-1}} \Delta_n(a, b, p)\right) = 2n + 2.$$

3. Proofs

This section contains the proofs.

Proof of Theorem 3. For $n \geq 2$ and $1 \leq k \leq a2^n$ we have

$$\nu_2\left(\binom{a2^n}{k}^2 2^k\right) \geq 2n.$$

In fact, if $1 \leq k \leq 2^n$ then

$$\nu_2\left(\binom{a2^n}{k}^2 2^k\right) = 2(n - \nu_2(k)) + k, \tag{14}$$

which is at least $2n$, while if $2^n < k \leq a2^n$ then $\nu_2(2^k) > 2^n \geq 2n$; thus, $\nu_2(2^k) \geq 2n+2$ if k is even, while the same lower bound trivially follows if k is odd. Note that for $n \geq 2$ the proof and Conjecture 1 suggest that

$$\nu_2(d(a2^n) - 1) = \nu_2\left(\sum_{1 \leq k \leq 3n+2\log_2 a} \binom{a2^n}{k}^2 2^k\right);$$

moreover,

$$d(a2^n) - 1 \equiv \sum_{1 \leq k \leq 3n+2\log_2 a} \binom{a2^n}{k}^2 2^k \pmod{2^{3n+1}}.$$

By considering the terms $k = 1, 2$, and 4 we can easily improve the lower bound, $2n$. Note that the corresponding 2-adic orders are $2n+1, 2n$, and $2n$, respectively by (14), and it is easy to determine these terms modulo 2^{2n+3}. The other terms clearly have 2-adic orders at least $2n+2$. □

Proof of Theorem 4. We have

$$d(a2^n + b) - 1 = \sum_{k=1}^{a2^n+b} \binom{a2^n+b}{k}^2 2^k \equiv (a2^n+b)^2 2 = 2b^2 \pmod{4},$$

and clearly, $\nu_2(2b^2) = 1$. On the other hand, easy calculation shows that

$$d(a2^n+3) + 1 \equiv 2 + \sum_{k=1}^{6} \binom{a2^n+3}{k}^2 2^k \equiv 2^6\left(1 + \binom{a2^n+3}{6}^2\right) \equiv 2^6 \pmod{2^7}$$

since $\nu_2\left(\binom{a2^n+3}{6}\right) > 0$ for $n \geq 3$. □

Proof of Theorem 5. By identity (1) we obtain

$$d(p^n) = 1 + \sum_{k=1}^{p^n-1} \binom{p^n}{k}^2 2^k + 2^{p^n} \equiv 3 \pmod{p},$$

since $\nu_p\left(\binom{p^n}{k}^2 2^k\right) = 2(n - \nu_p(k)) \geq 2$ for $1 \leq k \leq p^n - 1$ and $2^{p^n} \equiv 2 \pmod{p}$ by repeated application of $2^p \equiv 2 \pmod{p}$.

For $p = 3$, we have

$$d(3^n) = \sum_{k=0}^{3^n} \binom{3^n}{k}^2 2^k = 1 + 2^{3^n} + \sum_{m=0}^{n-1} \sum_{\substack{t=1 \\ (t,3)=1}}^{3^{n-m}} \binom{3^n}{t3^m}^2 2^{t3^m} \tag{15}$$

and observe that $\nu_3\left(\binom{3^n}{t3^m}^2\right) = 2(n-m)$ for $(t,3) = 1$. In the very last summation

$$\sum_{\substack{t=1 \\ (t,3)=1}}^{3^{n-m}} \binom{3^n}{t3^m}^2 2^{t3^m} \tag{16}$$

of (15) we pair up the terms with $t3^m$ and $3^n - t3^m$ in the binomial coefficients to yield

$$\begin{aligned}
\nu_3\left(\binom{3^n}{t3^m}^2 2^{t3^m} + \binom{3^n}{3^n - t3^m}^2 2^{3^n - t3^m}\right) &= 2(n-m) + \nu_3\left(2^{t3^m} + 2^{3^n - t3^m}\right) \\
&= 2(n-m) + \nu_3\left(1 + 2^{3^n - 2t3^m}\right) \\
&= 2(n-m) + (1+m) = 2n - m + 1
\end{aligned} \tag{17}$$

if $t3^m < 3^n - t3^m$ (and similarly, if $t3^m > 3^n - t3^m$) by Lemma 1. We get that

$$\nu_3\left(\frac{1}{2}\left(\sum_{\substack{t=1 \\ (t,3)=1}}^{3^{n-m}} \binom{3^n}{t3^m}^2 2^{t3^m} + \binom{3^n}{3^n - t3^m}^2 2^{3^n - t3^m}\right)\right) \geq 2n - m + 1.$$

When we sum up the sums described in (16) for all $m : 0 \leq m \leq n-1$ we obtain that the 3-adic order of $\left(\sum_{k=0}^{3^n} \binom{3^n}{k}^2 2^k\right) - (1 + 2^{3^n})$ is at least $n+2$ by identity (15), which proves that $d(3^n) \equiv 1 + 2^{3^n} \pmod{3^{n+2}}$. Then (3) follows by binomial expansion and $n - \nu_3(k) + k \geq n + 2$ if $k \geq 2$ since

$$1 + (3-1)^{3^n} = \sum_{k=1}^{3^n} \binom{3^n}{k} 3^k (-1)^{3^n - k} \equiv 3^{n+1} \pmod{3^{n+2}}.$$

□

Remark 6. Note that typically, $\nu_p\left(1+2^{p^n-2tp^m}\right) = m+1$ does not apply for primes $p>3$ in (17); thus, we can only partially generalize Theorem 5 in Theorem 13.

In the next two proofs we use the first definition of $d(n)$ in (1).

Proof of Theorem 6. Clearly, $d(0)=1$. If $n \geq 1$ then we use the first summation in (1). The term with $k=0$ evaluates to 1 and $\binom{3^n-1}{k}\binom{3^n-1+k}{3^n-1} \equiv 0 \pmod 3$ if $1 \leq k \leq 3^n-1$ by the second factor. □

Proof of Theorem 7. Clearly, $d(1)=3$. If $k=0$ or 1 then we have the terms 1 and $\binom{3^n-2}{1}\binom{3^n-1}{3^n-2} = (3^n-2)(3^n-1) \equiv 2 \pmod{3^{n+1}}$ in the first summation of (1).

If $n \geq 2$ then we get that $\binom{3^n-2}{k}\binom{3^n-2+k}{3^n-2} \equiv 0 \pmod 3$ if $2 \leq k \leq 3^n-2$. However, we need the modulo 9 remainders. Let a be either 1 or 2. The remainder is $6 \pmod 9$ if $k = a3^{n-1}+b$ and $b=0$ and it is $3 \pmod 9$ if $b=1$. In fact, $\binom{3^n-2}{k} \equiv \binom{2}{a} \pmod 3$ by the Lucas' theorem and $\binom{3^n-2+k}{3^n-2}/3 \equiv 1 \pmod 3$ if $a+b$ is odd and $2 \pmod 3$ if $a+b$ is even since $\binom{3^n-2+k}{3^n-2}/3 \equiv -1/(2a!) \pmod 3$ if $b=0$ and $-1/a! \pmod 3$ if $b=1$ by Theorem 1.

For other values of k we immediately have $\binom{3^n-2+k}{3^n-2} \equiv 0 \pmod 9$ for the second factor. The result follows by combining the terms. □

Proof of Theorem 8. The statement is true for $n=1$. Therefore, we can assume that $n \geq 2$. We prove that

$$d(3^n+1) - d(3^n) \equiv 2 \cdot 3^n \pmod{3^{n+1}},$$

which fact, combined with the congruence (3), already guarantees that $d(3^n+1) \equiv 2 \cdot 3^n \pmod{3^{n+1}}$.

In a similar fashion to (15), we use identity

$$d(3^n+1) = \sum_{k=0}^{3^n+1} \binom{3^n+1}{k}^2 2^k = \sum_{m=0}^{n} \sum_{\substack{t=1 \\ (t,3)=1}}^{3^{n-m}+1} \binom{3^n+1}{t3^m}^2 2^{t3^m} \tag{18}$$

and consider the difference $d(3^n+1)-d(3^n)$ in different ranges, for $1 \leq m \leq n-1$, $m=n$, and $m=0$.

In the difference $d(3^n+1)-d(3^n)$ we focus on the terms

$$\begin{aligned} b_{(t,m)} &= \binom{3^n+1}{t3^m}^2 2^{t3^m} - \binom{3^n}{t3^m}^2 2^{t3^m} \\ &= \left(\binom{3^n+1}{t3^m} - \binom{3^n}{t3^m}\right)\left(\binom{3^n+1}{t3^m} + \binom{3^n}{t3^m}\right) 2^{t3^m} \\ &= \binom{3^n}{t3^m-1}\left(\binom{3^n+1}{t3^m} + \binom{3^n}{t3^m}\right) 2^{t3^m} \end{aligned} \tag{19}$$

with $1 \le t \le 3^{n-m}+1$, $(t,3)=1$, and $0 \le m \le n$. The 3-adic order of each term is at least $n+(n-m)=2n-m$ except if $t3^m=1$ or $t3^m=3^n+1$, corresponding to the cases with $t=1$, $m=0$ and $t=3^n+1$, $m=0$, respectively.

If $1 \le m \le n-1$ then the 3-adic order of (19) is at least $n+(n-m)=2n-m \ge 2n-(n-1)=n+1$.

If $m=n$ then $t=1$, $t3^m=3^n$ and $b_{(1,n)} = \left(\binom{3^n+1}{3^n}^2 - \binom{3^n}{3^n}^2\right)2^{3^n} = ((3^n+1)^2-1)2^{3^n} \equiv 2^{3^n+1}\cdot 3^n \pmod{3^{2n}}$; thus,

$$b_{(1,n)} \equiv -2\cdot 3^n \pmod{3^{n+1}}, \tag{20}$$

since $\phi(3^{n+1})=2\cdot 3^n$ with the Euler ϕ-function and 2 is a primitive root modulo any power of 3. Therefore,

$$2^{3^n} \equiv -1 \pmod{3^{n+1}}. \tag{21}$$

Now we assume that $m=0$. We have two cases with $t\equiv 1 \pmod 3$ and $t \equiv 2 \pmod 3$. For $d(3^n)$ we have the pairing in (17) which results in

$$\nu_3\left(\sum_{\substack{t=1\\(t,3)=1}}^{3^n}\binom{3^n}{t}^2 2^t\right) \ge 2n+1.$$

On the other hand, in the summation for $d(3^{n+1})$, if $t\equiv 2 \pmod 3$ then clearly the 3-adic order of each term in $\binom{3^n+1}{t}^2 2^t$ is at least $2n$. If $t\equiv 1 \pmod 3$ then for $d(3^{n+1})$ we use a slightly different pairing from that of (17) which involves the terms

$$\begin{aligned} a_t &= \binom{3^n+1}{t}^2 2^t + \binom{3^n+1}{3^n+2-t}^2 2^{3^n+2-t} \\ &= (3^n+1)^2 2^t\left(\frac{1}{t^2}\binom{3^n}{t-1}^2 + \frac{1}{(3^n+2-t)^2}\binom{3^n}{3^n+1-t}^2 2^{3^n+2-2t}\right) \end{aligned}$$

with $t \le (3^n+1)/2$. Since $t\equiv 1 \pmod 3$, we can write $t=c3^r+1$ with either $c\ge 1$, $(c,3)=1$, and $1\le r\le n-1$ (since $t\le (3^n+1)/2$) or with $c=0$, i.e., $t=1$, and observe that $1/t^2 \equiv 1-2c3^r \pmod{3^{2r}}$ and $1/(3^n+2-t)^2 \equiv 1+2c3^r \pmod{3^{2r}}$. Therefore, it follows that

$$a_t = (3^n+1)^2 2^t\binom{3^n}{t-1}^2\left(1+2^{3^n-2c3^r} \pmod{3^r}\right).$$

If $r\ge 1$ then we obtain that $\nu_3(a_t)\ge 2(n-r)+r=2n-r$ by Lemma 1, and it is at least $n+1$ since $r\le n-1$. However, if $c=0$, i.e., $t=1$, then

$$a_1 = (3^n+1)^2 2 + 2^{3^n+1} \equiv 4\cdot 3^n \pmod{3^{n+1}} \tag{22}$$

by (21).

Combining the congruences (20) and (22) of the cases with $m = n$ and $m = 0$, the result follows. $\square$

Proof of Theorem 9. We use the congruence (3) of Theorem 5 to yield $d(3^{n+1}) - d(3^n) \equiv 2 \cdot 3^{n+1} \pmod{3^{n+2}}$. In a similar fashion, we apply (3) twice:

$$3d(3^n) \equiv 3 \cdot 3^{n+1} \pmod{3^{n+3}}$$

and

$$d(3^{n+1}) \equiv 3^{n+2} \pmod{3^{n+3}},$$

and then take the difference.

The proofs involving $d(3^n + 1)$ are similar and based on (9). $\square$

We combine the proofs of three theorems below.

Proof of Theorems 10–12. All cases rely on identity (11) which has the terms

$$A_k = A(a, b, n, k) = \binom{a3^n + b}{k} 3^k \binom{a3^n + b - k}{\frac{a3^n+b-k}{2}} 2^{\frac{a3^n+b-k}{2}}$$

with $0 \le k \le a3^n + b$ and $a3^n + b - k$ even. We prove that the 3-adically unique dominant term has $k = 0$ or $k = 1$ depending on the parity of $a3^n + b$, i.e., that of $a + b$. We set

$$l = \begin{cases} 0, & \text{if } a + b \text{ is even,} \\ 1, & \text{if } a + b \text{ is odd.} \end{cases}$$

We assume that $k \equiv l \pmod 2$ and prove for $0 \le k \le 3^n - 3 + b$ that

$$\nu_3(A_{k+2}/A_k) = \begin{cases} 2 + 2\nu_3(b - k) - \nu_3((k + 1)(k + 2)), & \text{if } b \ne k, \\ 2 + 2n - \nu_3((k + 1)(k + 2)), & \text{if } b = k. \end{cases} \tag{23}$$

Indeed,

$$\frac{A_{k+2}}{A_k} = \frac{\binom{a3^n+b}{k+2}}{\binom{a3^n+b}{k}} 3^2 \frac{\binom{a3^n+b-k-2}{\frac{a3^n+b-k-2}{2}}}{\binom{a3^n+b-k}{\frac{a3^n+b-k}{2}}} \frac{2^{\frac{a3^n+b-k-2}{2}}}{2^{\frac{a3^n+b-k}{2}}} = 3^2 \frac{(a3^n + b - k)^2}{8(k + 1)(k + 2)}.$$

Clearly, $\nu_3(A_l) < \nu_3(A_{l+2})$. For $k \ge 2$ we write

$$A_k = A_l \prod_{\substack{m=l \\ m \equiv l \pmod 2}}^{k-2} \frac{A_{m+2}}{A_m};$$

and thus,

$$\nu_3(A_k) = \nu_3(A_l) + \sum_{\substack{m=l \\ m\equiv l \pmod 2}}^{k-2} \nu_3(A_{m+2}/A_m) \geq \nu_3(A_l) + \lfloor k/2 \rfloor,$$

where we used the identity (23) and the inequality $\nu_3(k!) < k/2$. It follows that $\nu_3(A_l) < \nu_3(A_k)$ for $l+2 \leq k \leq 3^n - 3 + b$. Otherwise, we observe that $\nu_3(A_k) \geq 3^n - 3 + b \geq 3n \geq \nu_3(A_l)$ for any sufficiently large n. Now

$$d(a3^n + b) \equiv A_l \pmod{3^{\nu_3(A_l)+1}} \tag{24}$$

follows for any sufficiently large n.

We can derive the exact 3-adic orders and corresponding congruences in the different cases given in Theorems 10–12 by using Theorem 1. $\square$

Proof of Theorem 13. We modify the proof of Theorem 5 by writing

$$d(p^n) = 1 + \sum_{k=1}^{p^n-1} \binom{p^n}{k}^2 2^k + 2^{p^n} = 1 + 2^{p^n} + \sum_{m=0}^{n-1} \sum_{\substack{t=1 \\ (t,p)=1}}^{p^{n-m}} \binom{p^n}{tp^m}^2 2^{tp^m}.$$

The congruence (12) immediately follows.

We provide the proof of (13) based on the proof of [6, Theorem 2.5] to obtain information on the p-adic order of the difference $d(p^{n+1}) - d(p^n)$. However, this time we are able to determine the exact p-adic order if $\nu_p(Q_p(2)) = 0$. This fact demonstrates that the technique originally outlined for Motzkin numbers might be effectively used to find exact p-adic orders for differences of other combinatorial quantities. We use the same infinite incongruent disjoint covering system that we used in the mentioned proof. We rewrite the difference as

$$\begin{aligned}
d(p^{n+1}) - d(p^n) &= \sum_{k=1}^{p^n} \left(\binom{p^{n+1}}{pk}^2 2^{pk} - \binom{p^n}{k}^2 2^k \right) + \sum_{i=1}^{p-1} \left(\sum_{k\equiv i \pmod p}^{p^{n+1}} \binom{p^{n+1}}{k}^2 2^k \right) \\
&= \sum_{q=0}^{n-1} \sum_{i=1}^{p-1} \left(\sum_{\substack{k=ip^q+Kp^{q+1} \\ 0\leq K \leq \frac{p^{n-q}-i}{p}}} \left(\binom{p^{n+1}}{pk}^2 2^{pk} - \binom{p^n}{k}^2 2^k \right) \right) \qquad (25) \\
&+ 2^{p^{n+1}} - 2^{p^n} \qquad (26) \\
&+ \sum_{i=1}^{p-1} \left(\sum_{k\equiv i \pmod p}^{p^{n+1}} \binom{p^{n+1}}{k}^2 2^k \right). \qquad (27)
\end{aligned}$$

The term in (25) can be expressed as

$$\begin{aligned}
&\sum_{q=0}^{n-1}\sum_{i=1}^{p-1}\sum_{\substack{k=ip^q+Kp^{q+1}\\ 0\le K\le \frac{p^{n-q}-i}{p}}}\left(\binom{p^{n+1}}{pk}^2 2^{pk}-\binom{p^n}{k}^2 2^k\right)\\
&=\sum_{q=0}^{n-1}\sum_{i=1}^{p-1}\sum_{\substack{k=ip^q+Kp^{q+1}\\ 0\le K\le \frac{p^{n-q}-i}{p}}}\left(\binom{p^{n+1}}{pk}^2(2^{pk}-2^k)+\left(\binom{p^{n+1}}{pk}^2-\binom{p^n}{k}^2\right)2^k\right).
\end{aligned}$$

Note that with the Fermat quotient $Q_p(2)=(2^{p-1}-1)/p$ we have $2^{p^{n+1}}-2^{p^n}=2^{p^n}(2^{(p-1)p^n}-1)\equiv 2^{p^n}Q_p(2)p^{n+1} \pmod{p^{n+2}}$ by $2^{p-1}\equiv 1+Q_p(2)p \pmod{p^2}$ and binomial expansion. It yields that $\nu_p(2^{(p-1)p^n}-1)=n+1+\nu_p(Q_p(2))$. It is $n+1$ if $\nu_p(Q_p(2))=0$.

We consider the p-adic order of the terms in the last summation. For the first part we obtain $A_k=\nu_p\big(\binom{p^{n+1}}{pk}^2(2^{pk}-2^k\big)=2((n+1)-(q+1))+\nu_p(2^{i(p-1)p^q}-1)\ge 2(n-q)+(q+1+\nu_p(Q_p(2)))\ge 2n-q+1\ge n+2$ unless $q=n$, i.e., $k=p^n$ and $A_{p^n}=\nu_p(2^{p^{n+1}}-2^{p^n})\ge n+1$, which case is represented in (26). In fact, $A_{p^n}=n+1$ if $\nu_p(Q_p(2))=0$.

For the second part we obtain that $\nu_p\big(\binom{p^{n+1}}{pk}^2-\binom{p^n}{k}^2\big)2^k=3(n+1)$ if $p\ge 5$ and $3n+2$ if $p=3$ by [6, Corallary 3.4] applied with the settings $a=1$, $m=2$, $k=ip^q+Kp^{q+1}$ and substituting $n+1$ for n. The corollary is proven by Theorem 2 in [6].

The p-adic order of every term in (27) is $2(n+1)$, and thus, (13) follows. $\square$

Proof of Theorem 14. The proof is a straightforward consequence of Theorem 13. $\square$

4. The Schröder Numbers

The (large) Schröder number $s(n)$ is defined as the number of paths from $(0,0)$ to (n,n) with steps (1,0), (1,1), and (0,1) which never cross the main diagonal. It is also the number of paths from $(0,0)$ to $(2n,0)$ using only steps $(1,1)$, $(2,0)$ and $(1,-1)$ that never go below the x-axis. With $n\ge 0$ the first few values are: 1, 2, 6, 22, 90, 394, 1806, cf. A006318, [8]. It is well-known (cf. [10]) that $s(0)=1$ and

$$\begin{aligned}
s(n)=\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}\binom{n+k}{k}&=\sum_{k=1}^{n}\frac{1}{k}\binom{n-1}{k-1}\binom{n}{k-1}2^k\\
&=\frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}\binom{n}{k-1}2^k,\ n\ge 1. \qquad (28)
\end{aligned}$$

Note that the last definition (28) implies that for $n \geq 1$ we have that

$$s(n) = [x^{n+1}]\frac{1}{n}\left((1+2x)^n - 1\right)(1+x)^n = [x^{n+1}]\frac{(1+3x+2x^2)^n}{n}, \tag{29}$$

which yields that

$$s(n) = \frac{1}{n} \sum_{\substack{k=0 \\ n+1-k \text{ even}}}^{n} \binom{n}{k} 3^k \binom{n-k}{\frac{n+1-k}{2}} 2^{\frac{n+1-k}{2}}.$$

Identity (29) shows the inherent similarities in the behaviors of the Delannoy and Schröder numbers. We mention a conjecture and three theorems with sketches of their proofs.

Theorem 15. *For $p = 2$, $a \geq 1$ odd, and $n \geq 1$, we have that*

$$\nu_2(s(a2^n) - 2) = \nu_2(s(a2^n + 1) - 2) = n + 1,$$
$$\nu_2\left(s(a2^n) - (2 - a2^{n+1})\right) = 2n + 1, \tag{30}$$

and if $n \geq 3$ then

$$\nu_2\left(s(a2^n + 1) - (2 + a2^{n+1})\right) = 2n + 2.$$

The proof is based on identity (28). We note that Cao and Pan proved the following theorem.

Theorem 16 (Theorem 2, [2])**.** *For any $N \geq 1$ and $n \geq 2$,*

$$s(N + 2^n) \equiv s(N) + (-1)^{\lfloor \frac{N-1}{2} \rfloor} 2^{n+1} \pmod{2^{n+3}}.$$

For $a \geq 3$ odd and $n \geq 1$, identity (30) of Theorem 15 improves this to

$$s(a2^n) \equiv s((a-1)2^n) - 2^{n+1} \pmod{2^{2n+1}},$$

and $2n + 1$ is the best possible exponent.

The next corollary follows by (30) and (28).

Corollary 2. *For $p = 2$ and $n \geq 2$ even, we have*

$$s(n) \equiv 2 - 2n \pmod{2^{2\nu_2(n)+1}},$$

and the exponent in the modulus is best possible. If $n \geq 1$ is odd then

$$s(n) \equiv 2 \pmod{4},$$

and 4 is the best possible modulus if $n \equiv 3 \pmod{4}$.

Now we consider the odd primes.

Theorem 17. *For $p = 3$ and $n \geq 1$, we have that*

$$s(3^n+1) \equiv 3^{n+1} \pmod{3^{n+2}}; \tag{31}$$

thus,

$$\nu_3\left(s(3^n+1)\right) = n+1.$$

In addition, if $a \geq 1$ odd and $(a,3) = 1$ then we have that

$$\begin{aligned} s(a3^n+1) &\equiv a\binom{a-1}{\frac{a-1}{2}}3^{n+1}(-1)^n 2^{\frac{a3^n-1}{2}} \\ &\equiv \begin{cases} a\binom{a-1}{\frac{a-1}{2}}3^{n+1} \pmod{3^{n+2+\nu_3\left(\binom{a-1}{\frac{a-1}{2}}\right)}}, & \text{if } a \equiv 1 \pmod 4, \\ 2a\binom{a-1}{\frac{a-1}{2}} \cdot 3^{n+1} \pmod{3^{n+2+\nu_3\left(\binom{a-1}{\frac{a-1}{2}}\right)}}, & \text{if } a \equiv 3 \pmod 4; \end{cases} \end{aligned}$$

thus,

$$\nu_3\left(s(a3^n+1)\right) = n+1+\nu_3\left(\binom{a-1}{\frac{a-1}{2}}\right).$$

The proof of Theorem 17 is based on identity (29). In fact,

$$s(a3^n+1) = \frac{1}{a3^n+1}\sum_{\substack{k=0 \\ a3^n+2-k \text{ even}}}^{a3^n+1}\binom{a3^n+1}{k}3^k\binom{a3^n+1-k}{\frac{a3^n+2-k}{2}}2^{\frac{a3^n+2-k}{2}},$$

and the 3-adically unique dominant term has $k = 1$. Therefore, we obtain the statements by Theorem 1 and noting that $(-1)^n\frac{1}{2!}2^{\frac{3^n+1}{2}} \equiv 1 \pmod 3$, and in a similar fashion to the proof of Theorems 10–12 by applying identity (11).

Theorem 18. *For any odd prime p and $n \geq 1$, we have that*

$$s(p^n) \equiv \sum_{\substack{k=0 \\ k\equiv 0 \text{ or } p-1 \pmod p}}^{p^n-1}\binom{p^n}{k}^2\frac{2^{k+1}}{k+1}\left(1-\frac{k}{p^n}\right) \pmod{p^n}. \tag{32}$$

The terms not included in the summation in (32), i.e., those with $k \equiv 1, 2, \dots, p-2 \pmod p$, have p-adic order of n. We note that it is easy to see that

$$s(n) = \sum_{k=0}^{n-1}\binom{n}{k}^2\frac{2^{k+1}}{k+1}\left(1-\frac{k}{n}\right).$$

The p-adic limit of $s(p^n)$ as $n \to \infty$ exists according to the following conjecture and Theorem 15. The calculation of the limit can be simplified by Theorem 18. We note that the proving technique outlined in the proof of Theorem 13 does not seem to work for (33). The p-adic limit of $\{s(ap^n+b)\}_{n\to\infty}$ also exists if p is an odd prime by the conjecture.

Conjecture 7. For any odd prime p and sufficiently large n, we have that

$$\nu_p(s(p^{n+1}) - s(p^n)) = n. \tag{33}$$

In addition, if $a \geq 1$ with $(a, p) = 1$ and $b \in \mathbb{Z}$ then we have that $\nu_p(s(ap^{n+1} + b) - s(ap^n + b)) = n + c'''$ with some constant $c''' = c'''(a, b)$.

Acknowledgement. The author wishes to thank the referee for careful reading, helpful comments and catching an error in the statement and proof of Theorem 13.

References

[1] H. Cohen, *Number Theory, vol 2: Analytic and Modern Tools*, Graduate Texts in Mathematics, Springer, 2007.

[2] H.-Q. Cao and H. Pan, A Stern–type congruence for the Schröder numbers, *Discrete Math.* **340**:4 (2017), 708—712.

[3] M. J. Coster, Supercongruences, in: Baldassarri F., Bosch S., Dwork B. (eds), *p-adic Analysis*, Lecture Notes in Mathematics, vol 1454, Springer, Berlin, Heidelberg, 194–204.

[4] M. J. Coster and L. van Hamme, Supercongruences of Atkin and Swinnerton-Dyer type for Legendre polynomials, *J. Number Theory* **38** (1991), 265–286.

[5] A. Granville, Arithmetic properties of binomial coefficients. I. Binomial coefficients modulo prime powers, in *Organic Mathematics* (Burnaby, BC, 1995), volume 20 of CMS Conf. Proc., 253–276, Amer. Math. Soc., Providence, RI, 1997. Electronic version (a dynamic survey): http://www.dms.umontreal.ca/~andrew/Binomial/

[6] T. Lengyel, On divisibility properties of some differences of Motzkin numbers, *Ann. Math. Inform.* **41** (2013), 121–136.

[7] J.-C. Liu, A supercongruence involving Delannoy numbers and Schröder numbers, *J. Number Theory* **168** (2016), 117–127.

[8] N.J.A. Sloane, editor, The On-Line Encyclopedia of Integer Sequences, published electronically at https://oeis.org.

[9] E. Rowland, IntegerSequences, https://people.hofstra.edu/Eric_Rowland/packages.html#IntegerSequences or https://ericrowland.github.io/packages.html.

[10] R.A. Sulanke, Bijective recurrences concerning Schröder paths, *Electron. J. Combin.* **5** (1998), #R47, 11 pp.

[11] R.A. Sulanke, Objects counted by the central Delannoy numbers, *J. Integer Seq.* **6** (2003), Article 03.1.5, 19 pp.