# EXCEPTIONAL TOTIENT NUMBERS

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*Received: 4/28/21, Accepted: 8/17/21, Published: 8/27/21*

**Abstract**

A positive integer $n$ is called an exceptional totient number if the set $R_e(n) = \{x \in \mathbb{Z} : 1 \le x < n, \gcd(n,x) = \gcd(n,x-1) = 1\}$ can be partitioned into two disjoint subsets of equal sum. Take, for example, $R_e(7) = \{2,3,4,5,6\}$. This can be partitioned into the subsets $\{2,3,5\}$ and $\{4,6\}$ whose elements each add to 10. In this article, we provide a complete classification of exceptional totient numbers.

## 1. Introduction

Let $n$ be a positive integer and let $R(n)$ be the set of positive integers less than $n$ that are relatively prime to $n$. In 2017, Ali and Mahmood defined $n$ to be *super totient* if $R(n)$ can be partitioned into two disjoint sets of equal sum [1]. In 2020, Harrington and Wong classified all super totient numbers with the following theorem [5].

**Theorem 1.** *A positive integer $n$ is super totient if and only if it is not in the set $\{1,2,4,6,18\} \cup \{p^r : r \in \mathbb{N}$ and $p$ is prime with $p \equiv 3 \pmod{r}\}$.*

Recognizing $R(n)$ as the set of units modulo $n$, in this article we let $R_e(n) = \{x \in R(n) : \gcd(n,x-1) = 1\}$: the set of exceptional units modulo $n$. Exceptional units were introduced by Nagell [7] in 1969. The interested reader is directed to articles [2, 3, 6, 8, 9] for examples of recent research on exceptional units.

If $R_e(n)$ can be partitioned into two disjoint subsets of equal sum, we say that $n$ is an *exceptional totient number*. The following theorem, which we prove in Section 3, provides a complete classification of all exceptional totient numbers and is the main result of this paper.

**Theorem 2.** *A positive integer $n$ is an exceptional totient number if and only if it is not in the set* $\{3, 15\} \cup \{x : x \equiv 1 \pmod 4\}$.

## 2. Preliminary Results, Notation, and Lemmas

It is well known that Euler's $\varphi$-function counts the number of elements of $R(n)$. In 2010, Harrington and Jones introduced the function $\varphi_e$, which counts the number of elements of $R_e(n)$, and provided the following lemma [4].

**Lemma 1.** *For an integer $n > 1$, let $n = p_1^{t_1} \cdot p_2^{t_2} \cdots p_k^{t_k}$ be the prime factorization of $n$. Then*

$$\varphi_e(n) = \prod_{j=1}^{k} p_j^{t_j-1}(p_j - 2).$$

We note that Lemma 1 implies that $|R_e(n)| = \varphi_e(n) = 0$ for even values of $n$. In this case, $R_e(n) = \emptyset = \emptyset \cup \emptyset$ can be partitioned into two disjoint subsets, each of sum 0. Hence, the following theorem follows immediately from Lemma 1.

**Theorem 3.** *All even positive integers are exceptional totient numbers.*

With Theorem 3 in mind, our main focus through the remainder of this section will be on odd positive integers. The following lemma provides a lower bound on $\varphi_e(n)$ when $n$ is odd.

**Lemma 2.** *For an odd positive integer $n$, $\varphi_e(n) < 6$ if and only if*

$$n \in \{1, 3, 5, 7, 9, 15, 21\}.$$

*Proof.* The theorem can be checked computationally for $n \leq 21$. So, the proof will follow by showing that $\varphi_e(n) \geq 6$ when $n > 21$. If $n$ is divisible by a prime $p \geq 11$, then $\varphi_e(n) \geq 9$ by Lemma 1. So, suppose that $n = 3^t \cdot 5^r \cdot 7^s$, where $t$, $r$, and $s$ are non-negative integers. If $t \geq 3$, $r \geq 2$, or $s \geq 2$, then $\varphi_e(n) \geq 9$. Thus, we are left to show that the theorem holds when $t \in \{0, 1, 2\}$, $r \in \{0, 1\}$, and $s \in \{0, 1\}$. Since $n > 21$, the proof is finished by noticing that $\varphi_e(35) = 15$, $\varphi_e(45) = 9$, $\varphi_e(63) = 15$, $\varphi_e(105) = 15$, and $\varphi_e(315) = 45$. $\square$

We now turn our attention to showing that for a fixed odd positive integer $n$, $R_e(n)$ contains several key elements.

**Lemma 3.** *Let $n$ be a positive integer. If $a \in R_e(n)$, then $n + 1 - a \in R_e(n)$.*

*Proof.* Let $a \in R_e(n)$. Then $\gcd(a, n) = \gcd(a-1, n) = 1$. Suppose $p$ is a prime divisor of $n$. Then

$$\begin{aligned} n+1-a &\equiv 1-a \pmod{p} \\ &\equiv -(a-1) \pmod{p}. \end{aligned}$$

Since $p$ divides $n$ and $\gcd(a-1, n) = 1$, we deduce that $n+1-a$ is not divisible by $p$. Therefore, $\gcd(n+1-a, n) = 1$. Similarly,

$$(n+1-a) - 1 \equiv -a \pmod{p}$$

implies that $\gcd((n+1-a)-1, n) = 1$. Hence, $n+1-a \in R_e(n)$. □

**Lemma 4.** *Let $n$ be an odd positive integer. Then $(n+1)/2 \in R_e(n)$.*

*Proof.* Suppose $p_1$ is a prime divisor of $(n+1)/2$. Then there exists an integer $t_1$ such that $n = 2p_1t_1 - 1$. Thus, $n \equiv -1 \pmod{p_1}$, and therefore $p_1$ does not divide $n$. Hence, $\gcd(n, (n+1)/2) = 1$. Similarly, if $p_2$ is a prime that divides $(n+1)/2 - 1$, then there exists some integer $t_2$ such that $n = 2p_2t_2 + 1$. We deduce from this that $\gcd(n, (n+1)/2 - 1) = 1$. Hence, $(n+1)/2 \in R_e(n)$. □

**Lemma 5.** *Let $n \equiv 3 \pmod{4}$ be a positive integer not divisible by 3. Then $(n+1)/4 \in R_e(n)$*

*Proof.* Suppose $p_1$ is a prime divisor of $(n+1)/4$. Then there exists an integer $t_1$ such that $n = 4p_1t_1 - 1$. Thus, $n \equiv -1 \pmod{p_1}$, and therefore $p_1$ does not divide $n$. Hence, $\gcd(n, (n+1)/4) = 1$. Similarly, if $p_2$ is a prime that divides $(n+1)/4 - 1$, then there exists some integer $t_2$ such that $n = 4p_2t_2 + 3$, and $n \equiv 3 \pmod{p_2}$. We deduce from this that $\gcd(n, (n+1)/4 - 1) = 1$ when 3 does not divide $n$. Hence, $(n+1)/4 \in R_e(n)$ when $n$ is not divisible by 3. □

**Lemma 6.** *Let $n \equiv 3 \pmod{4}$ be a positive integer that is divisible by 3. If $n \notin \{3, 15\}$, then there exist distinct elements $x$ and $y$ of $R_e(n)$ such that $x + y = (n+1)/4$.*

*Proof.* Let $p$ be a prime divisor of $n$ and notice that $(n+1)/4 - 2 = (n-7)/4$ is divisible by $p$ only if $p = 7$. Similarly, $(n+1)/4 - 3 = (n-11)/4$ is divisible by $p$ only if $p = 11$. Thus, if $n$ is not divisible by 7 or 11, then

$$\gcd(n, 2) = \gcd(n, 1) = \gcd\left(n, \frac{n+1}{4} - 2\right) = \gcd\left(n, \frac{n+1}{4} - 3\right) = 1.$$

Further, since $(n+1)/4 - 2$ is positive when $n > 7$ and since $n \equiv 3 \pmod{4}$ and $n \neq 3$, we deduce that 2 and $(n+1)/4 - 2$ are elements of $R_e(n)$ when $n$ is not divisible by 7 or 11. Since $n \neq 15$, we know that $(n+1)/4 - 2 \neq 2$. Hence, if $n$ is not divisible by 7 or 11, then we may let $x = 2$ and $y = (n+1)/4 - 2$.

Let $p_1, \ldots, p_t$ be the prime divisors of $n$ with $3 = p_1 < p_2 < \cdots < p_t$. For each $p_j$, we will choose $c_j \in \mathbb{Z}$ to satisfy the following congruence conditions:

$$\begin{aligned} c_j &\not\equiv 0 \pmod{p_j} \\ c_j &\not\equiv 1 \pmod{p_j} \\ c_j &\not\equiv 4^{-1} \pmod{p_j} \\ c_j &\not\equiv -3 \cdot 4^{-1} \pmod{p_j}. \end{aligned} \tag{1}$$

We let $c_1 \equiv 2 \pmod 3$ and since (1) imposes no more than four restrictions on the congruence class of $c_j$ modulo $p_j$, it is clear that there exists such a $c_j$ for each $p_j \geq 5$. Notice that the conditions in (1) ensure that if $x \equiv c_j \pmod{p_j}$ for each $j \in \{1, 2, \ldots, t\}$, then $x \in R_e(n)$ and $y = (n+1)/4 - x \in R_e(n)$ with $x + y = (n+1)/4$. Thus, our proof will be complete by showing that $x$ can be chosen to satisfy $0 < x < (n+1)/4$.

If $n$ is divisible by 7 or 11 and $n$ is not divisible by a prime greater than 19, then we may choose $0 < x < (n+1)/4$ satisfying the conditions in (1) as shown in Table 1.

We now assume that $n$ is divisible by a prime greater 19. For each $j \in \{1, 2, \ldots, t-1\}$, let $c_j$ satisfy the conditions in (1). Using the Chinese remainder theorem, we let $C$ satisfy $C \equiv c_j \pmod{p_j}$ for all $j \in \{1, 2, \ldots, t-1\}$, and further ensure that $0 \leq C < p_1 p_2 \cdots p_{t-1}$. Again, since (1) imposes no more than four restrictions on the congruence class of $c_t$ modulo $p_t$, there exists a $k \in \{0, 1, 2, 3, 4\}$ so that

$$c_t = C + k \cdot \prod_{j=1}^{t-1} p_j$$

satisfies the conditions in (1) when $j = t$. Letting $x = c_t$ ensures that $x \equiv c_j \pmod{p_j}$ for each $j \in \{1, 2, \ldots, t\}$ and since $p_t \geq 23$, we have that

$$x = C + k \cdot \prod_{j=1}^{t-1} p_j \leq C + 4 \prod_{j=1}^{t-1} p_j < 5 \prod_{j=1}^{t-1} p_j \leq \frac{p_t}{4} \cdot \prod_{j=1}^{t-1} p_j = \frac{\prod_{j=1}^{t} p_j}{4} \leq \frac{n}{4} < \frac{n+1}{4}.$$

$\square$

## 3. Proof of Theorem 2

In this section we use the results established in Section 2 to prove Theorem 2.

*Proof of Theorem 2.* Let $T = \{3, 15\} \cup \{x : x \equiv 1 \pmod 4\}$. We begin our proof by showing that if $n \in T$, then $n$ is not an exceptional totient number. It is easy to

| Prime divisors of $n$ | $x$ | | Prime divisors of $n$ | $x$ |
|---|---|---|---|---|
| 3,7 | 5 | | 3,11 | 5 |
| 3,5,7 | 17 | | 3,7,11 | 38 |
| 3,5,11 | 17 | | 3,11,13 | 59 |
| 3,7,13 | 17 | | 3,11,17 | 125 |
| 3,7,17 | 38 | | 3,11,19 | 59 |
| 3,7,19 | 59 | | 3,5,7,11 | 17 |
| 3,5,11,13 | 422 | | 3,7,11,13 | 290 |
| 3,5,7,13 | 17 | | 3,7,13,17 | 626 |
| 3,11,13,17 | 158 | | 3,5,11,17 | 257 |
| 3,7,11,17 | 752 | | 3,5,7,17 | 122 |
| 3,7,17,19 | 920 | | 3,11,17,19 | 1940 |
| 3,7,13,19 | 899 | | 3,11,13,19 | 158 |
| 3,5,11,19 | 257 | | 3,7,11,19 | 59 |
| 3,5,7,19 | 122 | | 3,5,7,11,13 | 3062 |
| 3,5,7,13,17 | 2537 | | 3,7,11,13,17 | 7451 |
| 3,5,11,13,17 | 587 | | 3,5,7,11,17 | 3062 |
| 3,5,7,17,19 | 4847 | | 3,7,11,17,19 | 3062 |
| 3,5,11,17,19 | 3062 | | 3,11,13,17,19 | 12599 |
| 3,7,13,17,19 | 11273 | | 3,5,7,13,19 | 3902 |
| 3,7,11,13,19 | 4448 | | 3,5,11,13,19 | 9167 |
| 3,5,7,11,19 | 3062 | | 3,5,7,11,13,17 | 58502 |
| 3,5,7,11,17,19 | 3062 | | 3,5,11,13,17,19 | 56357 |
| 3,7,11,13,17,19 | 187631 | | 3,5,7,13,17,19 | 99452 |
| 3,5,7,11,13,19 | 28472 | | 3,5,7,11,13,17,19 | 749192 |

Table 1: Values of $x$ for Lemma 6 when $n$ is divisible by 7 or 11 and not divisible by a prime greater than 19.

see that $n = 3$ and $n = 15$ are not exceptional totient numbers since $R_e(3) = \{2\}$ and $R_e(15) = \{2, 8, 14\}$. Now suppose that $n \equiv 1 \pmod 4$. We know by Lemma 3 that if $a \in R_e(n)$ then $n + 1 - a \in R_e(n)$, and by Lemma 4 that $(n + 1)/2 \in R_e(n)$. Further, notice that $n + 1 - a = a$ if and only if $a = (n + 1)/2$. Thus, there are $(\varphi_e(n) - 1)/2$ pairs $\{a, n + 1 - a\}$, such that $a$ and $n + 1 - a$ are distinct elements of $R_e(n)$. Consequently,

$$\sum_{\substack{a \in R_e(n) \\ a \neq (n+1)/2}} a = \frac{\varphi_e(n) - 1}{2} \cdot (n + 1),$$

and therefore,

$$\sum_{a\in R_e(n)} a = \frac{\varphi_e(n)-1}{2}\cdot(n+1)+\frac{n+1}{2}$$
$$= \varphi_e(n)\cdot\frac{n+1}{2}.$$

Notice that if $n \equiv 1 \pmod 4$, then this sum is odd. Hence, the elements of $R_e(n)$ cannot be partitioned into two disjoint subsets of equal sum.

We are now left to show that if $n \notin T$, then $n$ is an exceptional totient number. By Theorem 3, we may restrict our attention to odd values of $n$.

For the remainder of the proof, assume that $n \equiv 3 \pmod 4$ with $n \notin \{3, 15\}$. We will construct sets $A$ and $B$ so that $A \subseteq R_e(n)$, $B \subseteq R_e(n)$, $A \cap B = \emptyset$, and

$$\sum_{a\in A} a = \sum_{b\in B} b.$$

To begin, we first construct sets $A_0$ and $B_0$ by distributing the pairs $\{a, n+1-a\}$ as evenly as possible so that

$$|A_0| = \begin{cases} |B_0| & \text{if } \varphi_e(n) \equiv 1 \pmod 4; \\ |B_0|+2 & \text{if } \varphi_e(n) \equiv 3 \pmod 4, \end{cases}$$

and therefore

$$\sum_{a\in A_0} a = \begin{cases} \displaystyle\sum_{b\in B_0} b & \text{if } \varphi_e(n) \equiv 1 \pmod 4; \\[2ex] n+1+\displaystyle\sum_{b\in B_0} b & \text{if } \varphi_e(n) \equiv 3 \pmod 4. \end{cases}$$

Further, if $n$ is not divisible by 3, then by Lemma 5 we ensure that $(n+1)/4 \in A_0$; if $n$ is divisible by 3, then we use Lemma 6 to find $x, y \in R_e(n)$ such that $x+y = (n+1)/4$, and since $n \neq 7$, and therefore $\varphi_e(n) \geq 6$ by Lemma 2, we can ensure that both $x$ and $y$ are in $A_0$. Next, we place $(n+1)/2$ into the appropriate set so that

$$\sum_{a\in A_0} a = \frac{n+1}{2} + \sum_{b\in B_0} b.$$

Note that this can be achieved by putting $(n+1)/2$ in $A_0$ if $\varphi_e(n) \equiv 1 \pmod 4$ or placing it in $B_0$ if $\varphi(n) \equiv 3 \pmod 4$. To finish our construction, we let

$$A = \begin{cases} A_0 \setminus \{\frac{n+1}{4}\} & \text{if } 3 \nmid n; \\ A_0 \setminus \{x, y\} & \text{if } 3 \mid n; \end{cases}$$

and we let

$$B = \begin{cases} B_0 \cup \left\{ \frac{n+1}{4} \right\} & \text{if } 3 \nmid n; \\ B_0 \cup \{x, y\} & \text{if } 3 \mid n. \end{cases}$$

Hence, if $n \equiv 3 \pmod 4$ and $n \notin \{3, 15\}$, then $n$ is an exceptional totient number. $\square$

## References


[1] S. Ali and M. Mahmood, A novel labeling algorithm on several classes of graphs, *Punjab Univ. J. Math. (Lahore)* **49** (1027), no. 2, 23–35.

[2] Anand, J. Chattopadhyay, and B. Roy, On sums of polynomial-type exceptional units in $\mathbb{Z}/n\mathbb{Z}$, *Arch. Math. (Basel)* **114** (2020), no. 3, 271–283.

[3] D. Dolžan, The sums fo exceptional units in a finite ring, *Arch. Math. (Basel)* **112** (2019), no. 6, 581–586.

[4] J. Harrington and L. Jones, On the iteration of a function related to Euler's $\phi$-function, *Integers* **10** (2010), A42, 497–515.

[5] J. Harrington and T. W. H. Wong, On super totient numbers and super totient labelings of graphs, *Discrete Math* **343** (2020), no. 2, 111670, 11pp.

[6] J. Houriet, Exceptional units and Euclidean number fields, *Arch. Math. (Basel)* **88** (2007), no. 5, 425–433.

[7] T. Nagell, Sur un type particulier d'unités algébriques (French), *Ark. Mat.* **8** (1969), 163–184.

[8] J. W. Sander, Sums of exceptional units in residue class rings, *J. Number Theory* **159** (2016), 1–6.

[9] Q. Yang and Q. Zhao, On the sumsets of exceptional units in $\mathbb{Z}_n$, *Monatsh. Math.* **182** (2017), no. 2, 489–493.