# SUMS OVER PRIMES

**Rusen Li**
*School of Mathematics, Shandong University, Jinan, China*
limanjiashe@163.com

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**Abstract**

In this paper, we give explicit asymptotic formulas for some sums over primes involving generalized hyperharmonic numbers. Analogous results for numbers with $k$-prime factors will also be considered.

## 1. Introduction and Preliminaries

Let $p_n$ be the sequence of prime numbers. Recall that the Prime Number Theorem states that

$$\sum_{p_n \le x} 1 \sim \frac{x}{\log x}\,. \tag{1}$$

This theorem is proved by Hadamard [6] and de la Vallée Poussin [2] independently and almost simultaneously in 1896. We recall $A(x) \sim B(x)$, that is, $A(x)$ is asymptotic to $B(x)$, which is equivalent to

$$\lim_{x\to\infty} \frac{A(x)}{B(x)} = 1\,.$$

Let $k \ge 1$ and consider a positive integer $n$ which is the product of just $k$ prime factors, i.e.,

$$n = p_1 p_2 \cdots p_k\,. \tag{2}$$

We write $\tau_k(x)$ for the number of such $n \le x$. If we impose the additional restriction that all the $p$ in Equation (2) shall be different, $n$ is squarefree. We write $\pi_k(x)$ for the number of these (squarefree) $n \le x$. It was proved by Landau [9] that

$$\pi_k(x) \sim \tau_k(x) \sim \frac{x(\log\log x)^{k-1}}{(k-1)!\log x} \quad (k \ge 2)\,. \tag{3}$$

For $k = 1$, this result would reduce to the Prime Number Theorem, if, as usual, we take $0! = 1$.

An exercise in a book on analytic number theory [5] states that $\sum_{p\le x} p \sim \frac{x^2}{2\log x}$. This result reminds the author that it would be interesting to consider sums over primes of types $\sum_{p_n\le x} p_n^{\alpha} f(n)^m$, where $f(n)$ is an arithmetical function. In fact, the summation $\sum_{p_n\le x} p_n^{\alpha}$ had been studied by some mathematicians. Šalát and Znám [14] obtained asymptotic formulas for $\sum_{p_n\le x} p_n^{\alpha}$, i.e., $\sum_{p_n\le x} p_n^{\alpha} \sim \frac{x^{1+\alpha}}{(1+\alpha)\log x}$. Jakimczuk [7, 8] extended this kind of summation to numbers with $k$ prime factors and functions of slow increase. With the help of the Prime Number Theorem with error terms, Gerard and Washington [4] also gave accurate estimates of $\sum_{p_n\le x} p_n^{\alpha} - \frac{x^{1+\alpha}}{(1+\alpha)\log x}$.

*The hyperharmonic numbers* were introduced by Conway and Guy [1] as

$$h_n^{(r)} := \sum_{j=1}^{n} h_j^{(r-1)} \quad (n, r \in \mathbb{N} := \{1, 2, 3, \cdots\}) \quad \text{with} \quad h_n^{(1)} = H_n.$$

Note that $H_n := \sum_{j=1}^{n} 1/j$ are the classical harmonic numbers. Starting from the classical generalized harmonic numbers $H_n^{(p,1)} = H_n^{(p)} := \sum_{j=1}^{n} 1/j^p$, Dil, Mező and Cenkci [3] introduced *the generalized hyperharmonic numbers*

$$H_n^{(p,r)} := \sum_{j=1}^{n} H_j^{(p,r-1)} \quad (n, p, r \in \mathbb{N}).$$

Ömür and Koparal [13] introduced the generalized hyperharmonic numbers $H_n^{(p,r)}$ independently and almost simultaneously, and defined two $n \times n$ matrices $A_n$ and $B_n$ with $a_{i,j} = H_i^{(j,r)}$ and $b_{i,j} = H_i^{(p,j)}$, respectively, and gave some interesting factorizations and determinant properties of the matrices $A_n$ and $B_n$. The author [10] proved that the generalized hyperharmonic numbers $H_n^{(p,r)}$ could be expressed as linear combinations of $n$'s power times generalized harmonic numbers.

Sums over primes involving arithmetic functions from analytic number theory (e.g. Mőbius function) have been studied extensively, however it seems that there are few papers on sums over primes involving arithmetic functions from combinatorial number theory. In this paper, we will make some progress toward this direction. The motivation of the present paper arises from an exercise in a book on analytic number theory [5] and the recent work [10] on generalized hyperharmonic numbers $H_n^{(p,r)}$. It seems that sums over primes involving generalized hyperharmonic numbers has not been considered. In this paper, we give explicit asymptotic formulas for sums over primes involving generalized hyperharmonic numbers of type $\sum_{p_n\le x} p_n^{\alpha} (H_n^{(p,r)})^m$. Analogous results for numbers with $k$-prime factors will also be considered.

## 2. Main Results

We now recall some lemmata.

**Lemma 1** ([11]). *For all $n \in \mathbb{N}$ and a fixed order $r \geq 1$, when $n \to \infty$, we have*

$$h_n^{(r)} \sim \frac{1}{(r-1)!} n^{r-1} \log(n) \, .$$

**Lemma 2** ([10]). *For $r, n, p \in \mathbb{N}$, we have*

$$H_n^{(p,r)} = \sum_{m=0}^{r-1} \sum_{j=0}^{r-1-m} a(r,m,j) n^j H_n^{(p-m)} \, .$$

*If $-p+m \geq 0$, then $H_n^{(p-m)}$ is understood to be the sum $\sum_{j=1}^{n} j^{-p+m}$. The coefficients $a(r,m,j)$ satisfy the following recurrence relations:*

$$\begin{aligned} a(r+1,r,0) &= -\sum_{m=0}^{r-1} a(r,m,r-m-1) \frac{1}{r-m} \, , \\ a(r+1,m,\ell) &= \sum_{j=\ell-1}^{r-1-m} \frac{a(r,m,j)}{j+1} \binom{j+1}{j-\ell+1} B_{j-\ell+1}^{+} \\ &\qquad (0 \leq m \leq r-1, 1 \leq \ell \leq r-m) \, , \\ a(r+1,m,0) &= -\sum_{y=0}^{m} \sum_{j=max\{0,m-y-1\}}^{r-1-y} a(r,y,j) D(r,m,j,y) \quad (0 \leq m \leq r-1) \, , \end{aligned}$$

*where*

$$D(r,m,j,y) = \sum_{\ell=max\{0,m-y-1\}}^{j} \frac{1}{j+1} \binom{j+1}{j-\ell} B_{j-\ell}^{+} \binom{\ell+1}{m-y} (-1)^{1+\ell-m+y}$$

*and Bernoulli numbers $B_n^+$ are determined by the recursive formula*

$$\sum_{j=0}^{k} \binom{k+1}{j} B_j^{+} = k+1 \quad (k \geq 0)$$

*or by the generating function*

$$\frac{t}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{+} \frac{t^n}{n!} \, .$$

*The initial value is given by $a(1,0,0) = 1$.*

**Lemma 3.** *For $r, n, p \in \mathbb{N}$ with $p \geq 2$, when $n \to \infty$, we have*

$$H_n^{(p,r)} \sim \frac{1}{(r-1)!} n^{r-1} \zeta(p) \,,$$

*where $\zeta(p) := \sum_{n=1}^{\infty} n^{-p}$ denotes the well-known Riemann zeta function.*

*Proof.* By using Lemma 2, we know that the main term of $H_n^{(p,r)}$ is $a(r, 0, r-1) n^{r-1} H_n^{(p)}$. Note that $a(r, 0, r-1)$ is independent of $p$ and therefore it can be evaluated using $p = 1$. For $p = 1$, we have $H_n^{(1,r)} = h_n^{(r)}$. From Lemma 1, we know that $h_n^{(r)} \sim \frac{1}{(r-1)!} n^{r-1} \log(n)$. Then it follows that $a(r, 0, r-1) = \frac{1}{(r-1)!}$. Thus we get the desired result. □

**Lemma 4** ([7, 8])**.** *Let $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ be two series of positive terms such that $\lim_{n\to\infty} \frac{a_i}{b_i} = 1$. Then if $\sum_{i=1}^{\infty} b_i$ is divergent, the following limit holds:*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} = 1 \,.$$

**Lemma 5.** *Let $p_{n,k}$ denote the nth squarefree number with just $k$ prime factors and $q_{n,k}$ denote the nth number with $k$ prime factors counted with multiplicity. Then the following asymptotic relations hold:*

$$p_{n,k} \sim q_{n,k} \sim (k-1)! \frac{n \log(n)}{(\log\log(n))^{k-1}} \,,$$
$$p_{n,k} (\log\log(p_{n,k}))^{k-1} \sim q_{n,k} (\log\log(q_{n,k}))^{k-1} \sim (k-1)! n \log(n) \,.$$

*For $k = 1$, we have $p_n \sim n \log(n)$.*

*Proof.* The first asymptotic relation can be found in the paper [7]. From Landau's Prime Number Theorem (3), we know that $\log(p_{n,k}) \sim \log(q_{n,k}) \sim \log(n)$, and thus we get the second asymptotic relation. □

**Lemma 6** ([8])**.** *Let the function $f(x)$ be of slow increase, i.e., $f(x)$ is a function defined on the interval $[a, \infty)$ such that $f(x) > 0$, $\lim_{x\to\infty} f(x) = \infty$ and with continuous derivative $f'(x) > 0$, and $\lim_{x\to\infty} \frac{\log(f(x))}{\log(x)} = 0$. Then the following limit holds:*

$$\lim_{x\to\infty} \frac{\int_a^x t^{\alpha} f(t)^{\beta} \mathrm{d}t}{\frac{x^{\alpha+1}}{\alpha+1} f(x)^{\beta}} = 1$$

*for all $\alpha > -1$ and for all $\beta$.*

It is not hard to verify that $(\log(x))^m$ and $\frac{(\log(x))^m}{(\log\log(x))^k}$ $(m, k \in \mathbb{N})$ are of slow increase.

**Lemma 7** ([12])**.** *Let $a$ and $b$ be integers with $a < b$, and let $f(x)$ be a monotonic function on the interval $[a, b]$. Then*

$$min\{f(a), f(b)\} \leq \sum_{k=a}^{b} f(k) - \int_a^b f(t)\mathrm{d}t \leq max\{f(a), f(b)\} .$$

**Lemma 8.** *For $m, n, k, x \in \mathbb{N}$, we have*

$$\sum_{\ell=1}^{x} \ell^m (\log(\ell))^n \sim \frac{x^{m+1}(\log(x))^n}{m+1} ,$$
$$\sum_{\ell=1}^{x} \frac{\ell^m (\log(\ell))^n}{(\log\log(\ell))^k} \sim \frac{x^{m+1}(\log(x))^n}{(m+1)(\log\log(x))^k} .$$

*Proof.* By using Lemma 7, we have

$$0 \leq \sum_{\ell=1}^{x} \ell^m (\log(\ell))^n - \int_1^x t^m (\log(t))^n \mathrm{d}t \leq x^m (\log(x))^n .$$

With the help of Lemma 6, we get the first asymptotic formula. The second asymptotic formula can be proved in a similar manner. □

Now we will prove our main theorems.

**Theorem 1.** *Define $P_k := \{p_{1,k}, p_{2,k}, \cdots\}$ and $Q_k := \{q_{1,k}, q_{2,k}, \cdots\}$. Let $c_{\ell,k}$ run through the elements of $P_k$ or through the elements of $Q_k$, and similarly for $d_{\ell,k}$ and $e_{\ell,k}$, independently of the choice of sets for the other two. For $m, k, n \in \mathbb{N}$, we have*

$$\sum_{\ell \leq \pi_k(x)} c_{\ell,k}^m (\log\log(d_{\ell,k}))^{m(k-1)} (\log(e_{\ell,k}))^n \sim \frac{x^{m+1}(\log(x))^{n-1}(\log\log(x))^{(m+1)(k-1)}}{(m+1)(k-1)!} ,$$
$$\sum_{\ell \leq \pi_k(x)} c_{\ell,k}^m (\log\log(d_{\ell,k}))^{m(k-1)} (H_\ell)^n \sim \frac{x^{m+1}(\log(x))^{n-1}(\log\log(x))^{(m+1)(k-1)}}{(m+1)(k-1)!} .$$

*Proof.* By using Lemma 4, Lemma 5 and Lemma 8, we have

$$\sum_{\ell \leq \pi_k(x)} c_{\ell,k}^m (\log\log(d_{\ell,k}))^{m(k-1)} (\log(e_{\ell,k}))^n \sim \sum_{\ell \leq \pi_k(x)} ((k-1)!)^m \ell^m (\log(\ell))^{m+n}$$
$$\sim \frac{x^{m+1}(\log(x))^{n-1}(\log\log(x))^{(m+1)(k-1)}}{(m+1)(k-1)!} .$$

The second asymptotic formula can be proved in a similar manner. □

**Theorem 2.** *For $\alpha, m, k, r \in \mathbb{N}$, we have*

$$\sum_{\ell \le x} p_{\ell,k}^{\alpha} (h_\ell^{(r)})^m \sim \frac{((k-1)!)^{\alpha} x^{\alpha+m(r-1)+1} (\log(x))^{\alpha+m}}{((r-1)!)^m (\alpha+m(r-1)+1)(\log\log(x))^{\alpha(k-1)}},$$

$$\sum_{p_{\ell,k} \le x} p_{\ell,k}^{\alpha} (h_\ell^{(r)})^m \sim \frac{x^{\alpha+m(r-1)+1} (\log\log(x))^{(m(r-1)+1)(k-1)}}{((k-1)!)^{m(r-1)+1} ((r-1)!)^m (\alpha+m(r-1)+1)} \times \frac{1}{(\log(x))^{m(r-2)+1}},$$

$$\sum_{p_{\ell,k} \le x} p_{\ell,k}^{\alpha} (\log(p_{\ell,k}))^m \sim \frac{x^{\alpha+1} (\log\log(x))^{k-1} (\log(x))^{m-1}}{((k-1)!)(\alpha+1)}.$$

*Proof.* By using Lemma 1, Lemma 4, Lemma 5 and Lemma 8, we have

$$\sum_{\ell \le x} p_{\ell,k}^{\alpha} (h_\ell^{(r)})^m \sim \sum_{\ell \le x} \frac{((k-1)!)^{\alpha} \ell^{\alpha+m(r-1)} (\log(\ell))^{\alpha+m}}{((r-1)!)^m (\log\log(\ell))^{\alpha(k-1)}}$$
$$\sim \frac{((k-1)!)^{\alpha} x^{\alpha+m(r-1)+1} (\log(x))^{\alpha+m}}{((r-1)!)^m (\alpha+m(r-1)+1)(\log\log(x))^{\alpha(k-1)}}.$$

The other two asymptotic formulas can be proved in a similar manner. □

**Theorem 3.** *For $\alpha, m, k, q, r, s, n \in \mathbb{N}$ with $q \ge 2$, we have*

$$\sum_{\ell \le x} p_{\ell,k}^{\alpha} (H_\ell^{(q,r)})^m \sim \frac{((k-1)!)^{\alpha} \zeta(q)^m x^{\alpha+m(r-1)+1} (\log(x))^{\alpha}}{((r-1)!)^m (\alpha+m(r-1)+1)(\log\log(x))^{\alpha(k-1)}},$$

$$\sum_{p_{\ell,k} \le x} p_{\ell,k}^{\alpha} (H_\ell^{(q,r)})^m \sim \frac{\zeta(q)^m x^{\alpha+m(r-1)+1} (\log\log(x))^{(m(r-1)+1)(k-1)}}{((k-1)!)^{m(r-1)+1} ((r-1)!)^m (\alpha+m(r-1)+1)} \times \frac{1}{(\log(x))^{m(r-1)+1}},$$

$$\sum_{\ell \le x} p_{\ell,k}^{\alpha} (H_\ell^{(q,r)})^m (h_\ell^{(s)})^n \sim \frac{((k-1)!)^{\alpha} \zeta(q)^m x^{\alpha+m(r-1)+n(s-1)+1}}{((r-1)!)^m ((s-1)!)^n} \times \frac{(\log(x))^{\alpha+n}}{(\alpha+m(r-1)+n(s-1)+1)(\log\log(x))^{\alpha(k-1)}},$$

$$\sum_{p_{\ell,k} \le x} p_{\ell,k}^{\alpha} (H_\ell^{(q,r)})^m (h_\ell^{(s)})^n \sim \frac{\zeta(q)^m x^{\alpha+m(r-1)+n(s-1)+1}}{((r-1)!)^m ((s-1)!)^n (\alpha+m(r-1)+n(s-1)+1)} \times \frac{(\log\log(x))^{(m(r-1)+n(s-1)+1)(k-1)}}{((k-1)!)^{m(r-1)+n(s-1)+1} (\log(x))^{m(r-1)+n(s-2)+1}}.$$

*Proof.* By using Lemma 3, Lemma 4, Lemma 5 and Lemma 8, we have

$$\sum_{\ell \le x} p_{\ell,k}^{\alpha} (H_n^{(q,r)})^m \sim \sum_{\ell \le x} \frac{((k-1)!)^{\alpha} \zeta(q)^m \ell^{\alpha+m(r-1)} (\log(\ell))^{\alpha}}{((r-1)!)^m (\log\log(\ell))^{\alpha(k-1)}}$$
$$\sim \frac{((k-1)!)^{\alpha} \zeta(q)^m x^{\alpha+m(r-1)+1} (\log(x))^{\alpha}}{((r-1)!)^m (\alpha+m(r-1)+1)(\log\log(x))^{\alpha(k-1)}}$$

The other three asymptotic formulas can be proved in a similar manner. □

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