RECURRENCE RELATIONS FOR GENERALIZED LAGUERRE POLYNOMIALS

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Abstract
In this paper we establish several types of new recurrence relations for the generalized Laguerre polynomials associated with $\alpha > -1$. One of which is especially interesting in that it is possible to set up its form without any restriction on both the number of involved terms and a range of consecutive polynomial indices.

1. Introduction

Throughout this paper, we denote by $(z)_k$ the falling factorial function of $z \in \mathbb{C}$ of order k. That is, letting $(z)_0 := 1$,

$$(z)_k := z(z-1)\cdots(z-k+1) = \frac{\Gamma(z+1)}{\Gamma(z-k+1)} \quad (k \geq 1),$$

where Γ is the Gamma function defined by $\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}dt$ with $\mathrm{Re}(z) > 1$.

The generalized Laguerre polynomials $L_n^{(\alpha)}(x)$, $n = 0, 1, 2, \ldots$, associated with a real number $\alpha > -1$ are defined by means of the generating function

$$\mathbb{L}^{(\alpha)}(t,x) := \frac{1}{(1-t)^{\alpha+1}} e^{\frac{-xt}{1-t}} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n \quad (|t| < 1). \tag{1.1}$$

In particular, if $\alpha = 0$, then they are called the *ordinary* Laguerre polynomials.

The ordinary and generalized Laguerre polynomials play an important role in many branches of numerical mathematics, combinatorics, source coding theory, mathematical physics, quantum mechanics, etc. For example, it is a well-established fact in quantum mechanics that these polynomials as well as the Legendre polynomials arise from the radial part of the solution of Schrödinger's equation related to the wave function for the hydrogen-like atom (cf., e.g., [9, 14, 15]).

These polynomials can be expressed in various kinds of ways. Among them, the most important and significant one is the Rodrigues formula

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!} \frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}) = \frac{x^{-\alpha}}{n!}\left(\frac{d}{dx} - 1\right)^n x^{n+\alpha}, \tag{1.2}$$

which yields a well-known expansion formula in terms of confluent hypergeometric functions of the first order, namely

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{\alpha+n}{n-k} \frac{(-x)^k}{k!} = \binom{\alpha+n}{n} {}_1F_1\Big(\genfrac{}{}{0pt}{}{-n}{\alpha+1}; x\Big), \tag{1.3}$$

where $\binom{\alpha+n}{k} := \frac{(\alpha+n)_k}{k!}$ and ${}_1F_1\Big(\genfrac{}{}{0pt}{}{-n}{\alpha+1}; x\Big) := \sum_{k=0}^{n} \frac{(-n)_k}{(\alpha+1)_k}\frac{x^k}{k!}$ (for details the reader is referred to [8]). Further, they can be expressed by a contour integral such that

$$L_n^{(\alpha)}(x) = \frac{1}{2\pi i}\oint_C \frac{e^{-\frac{xt}{1-t}}}{(1-t)^{\alpha+1}\, t^{n+1}}\, dt,$$

where the contour circles the origin once in a counterclockwise direction, but not enclosing the point $z = 1$.

Among many known recurrence relations satisfied by these polynomials, the most fundamental one is the familiar three-term identity, namely

$$(n+1)L_{n+1}^{(\alpha)}(x) = (\alpha + 2n + 1 - x)L_n^{(\alpha)}(x) - (\alpha+n)L_{n-1}^{(\alpha)}(x), \tag{1.4}$$

which can be derived in various ways (for reference, see a short proof given in the next section). One may alternatively define them by means of (1.4) having the initial values $L_0^{(\alpha)}(x) = 1$ and $L_1^{(\alpha)}(x) = -x + \alpha + 1$. As is a well-known fact, the family of these polynomials constitutes a complete orthogonal system in the Sobolev space $L^2_{\varphi_\alpha}(\mathbb{R}^+)$ with the weight function $\varphi_\alpha(x) := x^\alpha e^{-x}$, precisely

$$\int_0^\infty L_m^{(\alpha)}(x) L_n^{(\alpha)} \varphi_\alpha(x) dx = \frac{\Gamma(n+1+\alpha)}{n!}\delta_{mn} \quad (m, n \geq 0), \tag{1.5}$$

where δ_{mn} is Kronecker's symbol (for more details, see, e.g., [17, 20, 21]).

Moreover, it should be mentioned that these polynomials satisfy the second-order linear differential equation

$$xy''(x) + (\alpha + 1 - x)y'(x) + ny(x) = 0 \quad (n \geq 0), \tag{1.6}$$

which is usually called the *Laguerre equation.* In addition, they are eigenfunctions of the corresponding Sturm-Liouville problem. As a matter of fact, with the above function $\varphi_\alpha(x)$ and the eigenvalues n, it follows that (cf. [10, 13])

$$\Big(x\varphi_\alpha(x) \cdot (L_n^{(\alpha)}(x))'\Big)' + n \cdot \varphi_\alpha(x) L_n^{(\alpha)}(x) = 0.$$

This paper is, to a large extent, motivated by special types of recurrence relations for classical Bernoulli, Euler, Genocchi, Hermite, and other related numbers and polynomials. Various kinds of *shortened* (or *incomplete*) recurrence relations for them have been studied over the years, which are, in a sense, curious and unusual, but remarkable in that some of the preceding numbers or polynomials are completely excluded from given identities (for instance, see, e.g., [1–5]). For example, as is commonly known, the Hermite polynomials $H_n(x)$, $n = 0, 1, 2, \ldots$, defined by the generating function

$$e^{-t^2+2xt} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \quad (|t| < \infty)$$

satisfy the three-term recurrence relation

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0 \quad (n \geq 1) \tag{1.7}$$

with the initial values $H_0(x) = 1$ and $H_1(x) = 2x$. For (1.7) and other related details, see, e.g., [6, 19, 20]. Besides of (1.7), it has been verified that they also satisfy for any integer $n \geq 0$,

$$\sum_{j=0}^{n+1} \left\{ \binom{n}{j} + \binom{n+1}{j} \right\} (-4x)^j H_{2n+1-j}(x) = 0 \quad \text{(cf. [4, (3.14)]).} \tag{1.8}$$

As one may see at a glance, the first n polynomials $H_j(x)$, $j = 0, 1, \ldots, n-1$, are fully missing from this identity. Therefore, we may state that (1.8) is one of typical examples of shortened recurrence relations, as well as (1.7).

The main aim of this paper is to investigate various types of recurrence relations for generalized Laguerre polynomials. In Section 2, as a preliminary, after giving a short proof of (1.4) just for future reference, we reproduce some basic and useful properties of these polynomials. In Section 3, as our main results, we establish several kinds of new recurrence relations involving also shortened ones by using the generating function method. One of which is especially interesting in that it is involved deeply with (1.4) and it is possible to set up freely as one's will without any restriction on both the number of involved terms and a range of consecutive polynomial indices (see Theorem 3.3 and Corollary 3.4 below). We conclude this paper, in Section 4, with some additional remarks on a certain differential property of these polynomials, which is applicable to searching for differential equations satisfied by them.

2. Preliminary

We start this section by giving a simple proof of (1.4) just for future reference. Of course various methods of proof are available, but perhaps the simplest being to use the power series for $\mathbb{L}^{(\alpha)}(t,x)$ as defined in (1.1).

Proof of (1.4). Let $D_t := \frac{d}{dt}$, a differential operator with respect to t. It is easy to show that the first derivative of $\mathbb{L}^{(\alpha)}(t,x)$ can be calculated as follows:

$$D_t\,\mathbb{L}^{(\alpha)}(t,x) = \Big(\frac{\alpha+1}{1-t} - \frac{x}{(1-t)^2}\Big)\mathbb{L}^{(\alpha)}(t,x). \tag{2.1}$$

Thus multiplying both sides by $(1-t)^2$, we get

$$(1-t)^2 D_t\mathbb{L}^{(\alpha)}(t,x) = \{(\alpha+1)(1-t) - x\}\,\mathbb{L}^{(\alpha)}(t,x),$$

that is, as power series expression,

$$\begin{aligned}&\sum_{n=0}^{\infty} nL_n^{(\alpha)}(x)t^{n-1} - 2\sum_{n=0}^{\infty} nL_n^{(\alpha)}(x)t^n + \sum_{n=0}^{\infty} nL_n^{(\alpha)}(x)t^{n+1}\\ &= (\alpha+1)\Bigg(\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n - \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^{n+1}\Bigg) - x\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n.\end{aligned}$$

Equating the coefficients of t^n on both sides, we have

$$\begin{aligned}&(n+1)L_{n+1}^{(\alpha)}(x) - 2nL_n^{(\alpha)}(x) + (n-1)L_{n-1}^{(\alpha)}(x)\\ &\qquad = (\alpha+1)\left(L_n^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x)\right) - xL_n^{(\alpha)}(x),\end{aligned}$$

which leads to (1.4) by gathering similar terms together. □

We will later reprove (1.4) in a different way from the above argument, to be accurate, as a consequence of a certain recurrence relation (see (3.4) below).

The following binomial sum of falling factorials may be almost obvious, but we wish to give a brief proof for the sake of completeness.

Lemma 2.1. *For any $z \in \mathbb{C}$ and integers $k, r \geq 1$ it follows that*

$$\begin{aligned}&\binom{k}{r}(z+k+1+r)(z+k)_{k-r} + \binom{k}{r-1}(z+k)_{k+1-r}\\ &\qquad = \binom{k+1}{r}(z+k+1)_{k+1-r}.\end{aligned} \tag{2.2}$$

Proof. Using the identity $\binom{k+1}{r} = \binom{k}{r} + \binom{k}{r-1}$, let us rewrite (2.2) in the form

$$\binom{k}{r}u_k(r) + \binom{k}{r-1}v_k(r) = 0, \tag{2.3}$$

where $u_k(r)$ and $v_k(r)$ are given explicitly by

$$u_k(r) = (z+k+1+r)(z+k)_{k-r} - (z+k+1)_{k+1-r} = r(z+k)_{k-r};$$

$$v_k(r) = (z+k)_{k+1-r} - (z+k+1)_{k+1-r} = (r-k-1)(z+k)_{k-r}.$$

respectively. So we see that (2.2) (and hence (2.3)) is equivalent to

$$\left\{ r\binom{k}{r} + (r-k-1)\binom{k}{r-1} \right\}(z+k)_{k-r} = 0,$$

which is, however, trivial from the identity

$$\begin{aligned} r\binom{k}{r} + (r-k-1)\binom{k}{r-1} &= r\left\{ \binom{k}{r} + \binom{k}{r-1} \right\} - (k+1)\binom{k}{r-1} \\ &= r\binom{k+1}{r} - r\binom{k+1}{r} = 0. \end{aligned}$$

So the proof of (2.2) is complete. □

Using (2.2) in part, we are able to derive the nth derivative formula for $\mathbb{L}^{(\alpha)}(t,x)$ by an inductive method, as shown below.

Proposition 2.2. *For an integer $n \geq 0$ we have*

$$D_t^n\, \mathbb{L}^{(\alpha)}(t,x) = w_n^{(\alpha)}(t,x)\mathbb{L}^{(\alpha)}(t,x), \tag{2.4}$$

where

$$w_n^{(\alpha)}(t,x) := \frac{1}{(1-t)^n} \sum_{i=0}^{n} \binom{n}{i} (\alpha+n)_i \left(\frac{-x}{1-t}\right)^{n-i}.$$

Proof. We prove the result by induction on n. It is easy to confirm that (2.4) is true for $n = 0, 1$. Suppose now that (2.4) holds for a certain $n \geq 2$. For simplicity denoting $T := 1/(1-t)$, a straightforward calculation yields

$$\begin{aligned} &D_t\, T^\nu = \nu T^{\nu+1} \quad (\nu \in \mathbb{R}, \nu \geq 0); \\ &D_t\, w_n^{(\alpha)}(t,x) = T^{n+1} \sum_{r=0}^{n} \binom{n}{r} (n+r)(\alpha+n)_{n-r}(-Tx)^r; \\ &D_t\, \mathbb{L}^{(\alpha)}(t,x) = ((\alpha+1)T - T^2 x)\mathbb{L}^{(\alpha)}(t,x) \quad \text{(the same as (2.1))}. \end{aligned}$$

Based on these derivative formulas, we obtain with the help of (2.2) that

$$\begin{aligned} &D_t^{n+1}\, \mathbb{L}^{(\alpha)}(t,x) = D_t\, w_n^{(\alpha)}(t,x) \cdot \mathbb{L}^{(\alpha)}(t,x) + w_n^{(\alpha)}(t,x) \cdot D_t\, \mathbb{L}^{(\alpha)}(t,x) \\ &= \left(T^{n+1} \sum_{r=0}^{n} \binom{n}{r} (n+r)(\alpha+n)_{n-r}(-Tx)^r \right) \mathbb{L}^{(\alpha)}(t,x) \\ &\quad + \left(T^n \sum_{r=0}^{n} \binom{n}{r} (\alpha+n)_{n-r}(-Tx)^r \right) ((\alpha+1)T - T^2 x)\mathbb{L}^{(\alpha)}(t,x) \end{aligned}$$

$$
\begin{aligned}
&= \left(T^{n+1}\sum_{r=0}^{n}\binom{n}{r}(\alpha+n+1+r)(\alpha+n)_{n-r}(-Tx)^{r}\right)\mathbb{L}^{(\alpha)}(t,x)\\
&\quad + \left(T^{n+1}\sum_{r=0}^{n}\binom{n}{r}(\alpha+n)_{n-r}(-Tx)^{r+1}\right)\mathbb{L}^{(\alpha)}(t,x)\\
&= (\alpha+n+1)(\alpha+n)_{n}T^{n+1}\mathbb{L}^{(\alpha)}(t,x)\\
&\quad + \left(T^{n+1}\sum_{r=1}^{n}\left\{\binom{n}{r}(\alpha+n+1+r)(\alpha+n)_{n-r}+\binom{n}{r-1}(\alpha+n)_{n+1-r}\right\}\right.\\
&\qquad \left.\times(-Tx)^{r}\right)\mathbb{L}^{(\alpha)}(t,x)+(\alpha+n)_{0}T^{n+1}(-Tx)^{n+1}\mathbb{L}^{(\alpha)}(t,x)\\
&= (\alpha+n+1)_{n+1}T^{n+1}\mathbb{L}^{(\alpha)}(t,x)\\
&\quad + \left(T^{n+1}\sum_{r=1}^{n}\binom{n+1}{r}(\alpha+n+1)_{n+1-r}(-Tx)^{r}\right)\mathbb{L}^{(\alpha)}(t,x)\\
&\quad + (\alpha+n+1)_{0}T^{n+1}(-Tx)^{n+1}\mathbb{L}^{(\alpha)}(t,x)\\
&= \left(T^{n+1}\sum_{r=0}^{n+1}\binom{n+1}{r}(\alpha+n+1)_{r}(-Tx)^{n+1-r}\right)\mathbb{L}^{(\alpha)}(t,x)\\
&= w_{n+1}^{(\alpha)}(t,x)\mathbb{L}^{(\alpha)}(t,x),
\end{aligned}
$$

which is just (2.4) replaced n with $n+1$ and the proof by induction is ended. □

Let $\widetilde{L}_n^{(\alpha)}(x) := n! L_n^{(\alpha)}(x)$ for convenience. A closed-form expression for this polynomial is immediately given from (2.4) by setting $t = 0$.

Corollary 2.3. *We have*

$$\widetilde{L}_n^{(\alpha)}(x) = \sum_{i=0}^{n}\binom{n}{i}(\alpha+n)_i(-x)^{n-i} \quad (n \geq 0). \tag{2.5}$$

Remark 1. Needless to say, the Rodrigues formula (1.2) itself is deeply involved with differential equations and orthogonality properties of systems of polynomials. This formula ostensibly coincides with (2.5) in view of the fact that $x^{-\alpha}D_x^k\, x^{\alpha+n} = (\alpha+n)_k x^{n-k}$ $(0 \leq k \leq n)$. Further, note that it is possible to deduce (1.3) directly from (2.5) using the identity $\frac{1}{n!}\binom{n}{k}(\alpha+n)_{n-k} = \binom{\alpha+n}{n-k}\frac{1}{k!}$.

Using (2.5), the first few generalized Laguerre polynomials are given by

$$
\begin{aligned}
&\widetilde{L}_0^{(\alpha)}(x) = 1;\ \ \widetilde{L}_1^{(\alpha)}(x) = -x + (\alpha+1)_1;\\
&\widetilde{L}_2^{(\alpha)}(x) = x^2 - 2(\alpha+2)_1 x + (\alpha+2)_2;\\
&\widetilde{L}_3^{(\alpha)}(x) = -x^3 + 3(\alpha+3)_1 x^2 - 3(\alpha+3)_2 x + (\alpha+3)_3;\\
&\widetilde{L}_4^{(\alpha)}(x) = x^4 - 4(\alpha+4)_1 x^3 + 6(\alpha+4)_2 x^2 - 4(\alpha+4)_3 x + (\alpha+4)_4.
\end{aligned}
$$

We next introduce some basic formulas, which are applicable for searching for further advanced properties of these polynomials.

Proposition 2.4. *Let $D_x := \frac{d}{dx}$. For an integer $k \geq 1$ we have*

$$\begin{aligned}
&\text{(i)} \quad L_n^{(\alpha+k)}(x) - L_n^{(\alpha)}(x) = \sum_{j=1}^{k} L_{n-1}^{(\alpha+j)}(x);\\
&\text{(ii)} \quad L_n^{(\alpha+k)}(x) = \sum_{i=0}^{n} \binom{n+k-1-i}{k-1} L_i^{(\alpha)}(x);\\
&\text{(iii)} \quad D_x^k L_n^{(\alpha)}(x) = (-1)^k L_{n-k}^{(\alpha+k)}(x) \quad (k \leq n);\\
&\text{(iv)} \quad x D_x L_n^{(\alpha)}(x) = n L_n^{(\alpha)}(x) - (\alpha+n) L_{n-1}^{(\alpha)}(x);\\
&\text{(v)} \quad x D_x L_n^{(\alpha)}(x) = (n+1) L_{n+1}^{(\alpha)}(x) - (n+\alpha+1-x) L_n^{(\alpha)}(x).
\end{aligned} \tag{2.6}$$

Proof. (i) Considering the identity

$$\begin{aligned}
\mathbb{L}^{(\alpha+k)}(t,x) - \mathbb{L}^{(\alpha)}(t,x) &= \left(\frac{1}{(1-t)^k} - 1\right)\mathbb{L}^{(\alpha)}(t,x)\\
= \frac{t}{1-t}\left\{\sum_{j=0}^{k-1} \frac{1}{(1-t)^j}\right\}\mathbb{L}^{(\alpha)}(t,x) &= t\sum_{j=1}^{k} \mathbb{L}^{(\alpha+j)}(t,x),
\end{aligned}$$

we differentiate both sides n times with respect to t and then set $t = 0$ to obtain

$$\widetilde{L}_n^{(\alpha+k)}(x) - \widetilde{L}_n^{(\alpha)}(x) = n\sum_{j=1}^{k} \widetilde{L}_{n-1}^{(\alpha+j)}(x),$$

which immediately leads to (i) by dividing the whole by $n!$.
(ii) Next we differentiate both sides of the identity

$$\mathbb{L}^{(\alpha+k)}(t,x) = \frac{1}{(1-t)^k}\mathbb{L}^{(\alpha)}(t,x)$$

n times with respect to t. Then, according to Leibniz's rule, we have

$$D_t^n\, \mathbb{L}_n^{(\alpha+k)}(t,x) = \sum_{i=0}^{n} \binom{n}{i}\left(D_t^{n-i}\frac{1}{(1-t)^k}\right)\left(D_t^i\, \mathbb{L}^{(\alpha)}(t,x)\right),$$

which provides, by setting $t = 0$,

$$\begin{aligned}
\widetilde{L}_n^{(\alpha+k)}(x) &= \sum_{i=0}^{n} \binom{n}{i}(n+k-1-i)_{n-i}\widetilde{L}_i^{(\alpha)}(x)\\
&= \sum_{i=0}^{n} \binom{n}{i}\binom{n+k-1-i}{n-i}(n-i)!\widetilde{L}_i^{(\alpha)}(x)
\end{aligned}$$

$$= n! \sum_{i=0}^{n} \binom{n+k-1-i}{k-1} \frac{\widetilde{L}_i^{(\alpha)}(x)}{i!},$$

which gives (ii) by dividing both sides by $n!$.
(iii) Since $\binom{n}{i}(n-i) = n\binom{n-1}{i}$, it follows that

$$\begin{aligned} D_x L_n^{(\alpha)}(x) &= -\frac{1}{n!} \sum_{i=0}^{n-1} \binom{n}{i} (n-i)(\alpha+n)_i (-x)^{n-i-1} \\ &= -\frac{1}{(n-1)!} \sum_{i=0}^{n-1} \binom{n-1}{i} ((\alpha+1)+(n-1))_i (-x)^{(n-1)-i} \\ &= -L_{n-1}^{(\alpha+1)}(x). \end{aligned}$$

Therefore, in order to derive (iii) we have only to repeat the same thing k times.
(iv) Using $\binom{n}{i}i = n\binom{n-1}{i-1}$ and $(\alpha+n)_i = (\alpha+n)(\alpha+n-1)_{i-1}$, we obtain

$$\begin{aligned} xD_x\, L_n^{(\alpha)}(x) &= \frac{1}{n!} \sum_{i=0}^{n-1} \binom{n}{i} (n-i)(\alpha+n)_i (-x)^{n-i} \\ &= \frac{n}{n!} \sum_{i=0}^{n-1} \binom{n}{i} (\alpha+n)_i (-x)^{n-i} - \frac{1}{n!} \sum_{i=1}^{n-1} \binom{n}{i} i(\alpha+n)_i (-x)^{n-i} \\ &= \frac{n}{n!} \left\{ \sum_{i=0}^{n} \binom{n}{i} (\alpha+n)_i (-x)^{n-i} - (\alpha+n)_n \right\} \\ &\quad - \frac{1}{(n-1)!} \left\{ (\alpha+n) \sum_{i=1}^{n} \binom{n-1}{i-1} (\alpha+n-1)_{i-1} (-x)^{n-i} - (\alpha+n)_n \right\} \\ &= nL_n^{(\alpha)}(x) - (\alpha+n) L_{n-1}^{(\alpha)}(x). \end{aligned}$$

(v) As is clear, combining (iv) with (1.4) leads to (v). □

The formulas stated in (2.6) are just a few examples from a large number of significant ones. In passing, it should be noted that various differential properties and differential equations satisfied by these polynomials have been discussed extensively by many authors using suitable operators defined on the Lie algebras of endomorphisms of a vector space. For reference, see, e.g., [7,11,12,16,18].

3. Main Results

In this section we concentrate on a study of recurrence relations for generalized Laguerre polynomials using the generating function method. As a result, we derive several kinds of new recurrence relations involving a very interesting shortened one as being intimately concerned with (1.4).

Taking formally $\alpha = -1$ and $x = 1$ in (2.4), we have

$$D_t^n\, e^{\frac{-t}{1-t}} = \frac{1}{(1-t)^n}\left\{\sum_{i=0}^{n}\binom{n}{i}(n-1)_i\Big(\frac{-1}{1-t}\Big)^{n-i}\right\}e^{\frac{-t}{1-t}} \quad (n \geq 1). \tag{3.1}$$

Based on this formula we define the sequence $\{c_n\}_{n\geq 0}$ by $c_0 := e^{\frac{-t}{1-t}}|_{t=0} = 1$ and

$$c_n := D_t^n\, e^{\frac{-t}{1-t}}|_{t=0} = \sum_{i=0}^{n-1}(-1)^{n-i}\binom{n}{i}(n-1)_i \quad (n \geq 1).$$

We present below the first few members of $\{c_n\}_{n\geq 0}$. Letting $c_0 = 1$, we have the following table for $n \geq 1$:

n	1	2	3	4	5	6	7	8	9	10	11
c_n	-1	-1	-1	1	19	151	1091	7841	56519	396271	2442359

As we shall see later, these numbers can be obtained recursively by using a certain recurrence relation (see (3.3) below). Incidentally, it is shown that c_n is always odd and its last digit is 1 or 9, but we omit the details.

Theorem 3.1. *With the above sequence $\{c_n\}_{n\geq 0}$ we have for $n \geq 0$,*

$$\sum_{i=0}^{n}\binom{n}{i}c_{n-i}\widetilde{L}_i^{(\alpha)}(x) = \widetilde{L}_n^{(\alpha)}(x+1), \tag{3.2}$$

or, equivalently, in its original form,

$$\sum_{i=0}^{n}\frac{c_{n-i}}{(n-i)!}L_i^{(\alpha)}(x) = L_n^{(\alpha)}(x+1).$$

Proof. Considering the identity

$$e^{\frac{-t}{1-t}}\mathbb{L}^{(\alpha)}(t,x) = \mathbb{L}^{(\alpha)}(t,x+1),$$

we differentiate both sides n times with respect to t and set $t = 0$. Then, according to Leibniz's rule, the left-hand sides yields

$$\begin{aligned} D_t^n\big(e^{\frac{-t}{1-t}}\mathbb{L}^{(\alpha)}(t,x)\big)\big|_{t=0} &= \sum_{i=0}^{n}\binom{n}{i}\big(D_t^{n-i}e^{\frac{-t}{1-t}}\big)\big(D_t^i\,\mathbb{L}^{(\alpha)}(t,x)\big)\big|_{t=0} \\ &= \sum_{i=0}^{n}\binom{n}{i}c_{n-i}\widetilde{L}_i^{(\alpha)}(x). \end{aligned}$$

Meanwhile, the right-hand sides yields $D_t^n\,\mathbb{L}^{(\alpha)}(t,x+1)|_{t=0} = \widetilde{L}_n^{(\alpha)}(x+1)$. So equating them leads to (3.2). The latter identity can be derived by dividing both sides of (3.2) by $n!$ and using the identity $\binom{n}{i}\frac{i!}{n!} = \frac{1}{(n-i)!}$. □

Corollary 3.2. *For an integer $n \geq 1$ we have*

$$\sum_{j=1}^{n} \frac{c_j}{j!} = \sum_{j=1}^{n} \frac{(-1)^j}{j!} \binom{n}{j}. \tag{3.3}$$

Proof. Taking $\alpha = 0$ and $x = 0$ in (3.2) yields

$$\sum_{i=0}^{n} \binom{n}{i} c_{n-i}(i)_i = \widetilde{L}_n^{(0)}(1) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (n)_i.$$

Noting that $\binom{n}{i}(i)_i = (n)_i = \frac{n!}{(n-i)!}$ and dividing both sides by $n!$, we get

$$\sum_{i=0}^{n} \frac{c_{n-i}}{(n-i)!} = \sum_{i=0}^{n} \frac{(-1)^{n-i}}{(n-i)!} \binom{n}{i},$$

which leads to (3.3), after cancelling the terms corresponding to $i = n$ from both sides and reversing the order of summation. □

For simplify we set $F(t,x) := e^{\frac{xt}{1-t}}$. Similar to (3.1), taking formally $\alpha = -1$ and replacing x with $-x$ in (2.3), we have

$$D_t^n\, F(t,x) = \frac{1}{(1-t)^n} \left\{ \sum_{i=0}^{n} \binom{n}{i} (n-1)_i \left(\frac{x}{1-t} \right)^{n-i} \right\} F(t,x) \quad (n \geq 0).$$

Based on this formula, we define the polynomial sequence $\{f_n(x)\}_{n\geq 0}$ by $f_0(x) := F(t,x)\big|_{t=0} = 1$ and for $n \geq 1$,

$$f_n(x) := D_t^n\, F(t,x)\big|_{t=0} = \sum_{i=0}^{n-1} \binom{n}{i} (n-1)_i x^{n-i}.$$

The first few polynomials in $\{f_n(x)\}_{n\geq 0}$ can be expressed as follows:

$$\begin{aligned}
&f_0(x) = 1; \quad f_1(x) = x; \quad f_2(x) = x^2 + 2x;\\
&f_3(x) = x^3 + 6x^2 + 6x; \quad f_4(x) = x^4 + 12x^3 + 36x^2 + 24x;\\
&f_5(x) = x^5 + 20x^4 + 120x^3 + 240x^2 + 120x;\\
&f_6(x) = x^6 + 30x^5 + 300x^4 + 1200x^3 + 1800x^2 + 720x.
\end{aligned}$$

As is clear, we have $f_n(-1) = c_n$. Moreover, from the functional relations

$$\begin{aligned}
&F(t,-1)F(t,x) = F(t,x-1); \quad F(t,-x)F(t,x-1) = F(t,-1);\\
&F^2(t,x) = F(t,2x); \quad F(t,x)F(t,-x) = 1,
\end{aligned}$$

we see that these polynomials satisfy for $n \geq 0$,

$$\sum_{i=0}^{n}\binom{n}{i}c_{n-i}f_i(x) = f_n(x-1); \quad \sum_{i=0}^{n}\binom{n}{i}f_i(-x)f_{n-i}(x-1) = c_n;$$

$$\sum_{i=0}^{n}\binom{n}{i}f_{n-i}(x)f_i(x) = f_n(2x); \quad \sum_{i=0}^{n}\binom{n}{i}f_{n-i}(x)f_i(-x) = \begin{cases} 1 & \text{if } n=0, \\ 0 & \text{if } n \geq 1. \end{cases}$$

Using the sequence $\{f_n(x)\}_{n\geq 0}$, we are able to state the following theorem.

Theorem 3.3. *For integers $n, k \geq 0$ we have*

$$\sum_{i=0}^{n}\binom{n}{i}f_{n-i}(x)\widetilde{L}_{k+i}^{(\alpha)}(x) = \sum_{r=0}^{k}\binom{k}{r}(\alpha+k)_r(\alpha+n+2k-r)_n(-x)^{k-r}. \tag{3.4}$$

In particular, we have

$$\sum_{i=0}^{n}\binom{n}{i}f_{n-i}(x)\widetilde{L}_{i}^{(\alpha)}(x) = (\alpha+n)_n. \tag{3.5}$$

Proof. Just to be sure, we let $T := 1/(1-t)$ and $F(t,x) := e^{\frac{xt}{1-t}}$ once again. Since $\mathbb{L}^{(\alpha)}(t,x)F(t,x) = T^{\alpha+1}$, based on (2.4) we consider the functional identity

$$\begin{aligned} &\left\{D_t^k\,\mathbb{L}^{(\alpha)}(t,x)\right\}F(t,x) \\ &= T^k\left\{\sum_{r=0}^{k}\binom{k}{r}(\alpha+k)_r(-xT)^{k-r}\right\}\left(\mathbb{L}^{(\alpha)}(t,x)F(t,x)\right) \\ &= \left\{\sum_{r=0}^{k}\binom{k}{r}(\alpha+k)_r(-xT)^{k-r}\right\}T^{\alpha+1+k} \\ &= \sum_{r=0}^{k}\binom{k}{r}(\alpha+k)_r(-x)^{k-r}T^{\alpha+1+2k-r}, \end{aligned} \tag{3.6}$$

which is valid for all $k \geq 0$. By differentiating both sides of (3.6) n times with respect to t and then setting $t=0$ we obtain

$$\begin{aligned} \sum_{i=0}^{n}\binom{n}{i}f_{n-i}(x)\widetilde{L}_{k+i}^{(\alpha)}(x) &= \sum_{r=0}^{k}\binom{k}{r}(\alpha+k)_r(-x)^{k-r}D_t^n\left(T^{\alpha+1+2k-r}\right)\Big|_{t=0} \\ &= \sum_{r=0}^{k}\binom{k}{r}(\alpha+k)_r(\alpha+2k+n-r)_n(-x)^{k-r}, \end{aligned}$$

which is what we desired in (3.4). The case $k=0$ in (3.4) leads to (3.5). □

As is easily seen, since $(z)_r = \binom{z}{r} r!$ ($z \in \mathbb{C}$) and $\binom{k}{r} r! = \frac{k!}{(k-r)!}$, we may rewrite the right-hand side of (3.4) in the form

$$k!n! \sum_{r=0}^{k} \binom{\alpha+k}{r} \binom{\alpha+n+2k-r}{n} \frac{(-x)^{k-r}}{(k-r)!}.$$

Remark 2. Since $\deg f_{n-i}(x) = n-i$ and $\deg \widetilde{L}_{k+i}^{(\alpha)}(x) = k+i$, all terms on the right-hand side of (3.4) have the same degree $k+n$, whereas the degree of the right-hand side of (3.4) is exactly k. So we may assert that if $m > k$, then the sum S_m of the coefficients of x^m gathered from all terms on the left-hand side must vanish identically. That is to say, it follows that

$$S_m = \sum_{i=0}^{n} \binom{n}{i} [x^m] \left(f_{n-i}(x) \widetilde{L}_{k+i}^{(\alpha)}(x) \right) = 0 \quad (m > k),$$

where $[x^m] (\phi(x))$ denotes the coefficient of x^m in a polynomial $\phi(x)$. As a result, the left-hand side of (3.4) is written as $\sum_{m=0}^{k} S_m x^m$ and it is shown that

$$S_m = (-1)^m \binom{k}{m} (\alpha+k)_{k-m} (\alpha+n+k+m)_n \quad (0 \le m \le k).$$

We would like to restate once more that the most significant feature of (3.4) is that both the number of involved terms and a range of consecutive polynomial indices can be set up freely as one likes by choosing two suitable integer values of k and n. In that sense, we may call (3.4) a '*protean*' recurrence relation taking on different forms as required.

Now we examine special situations of (3.4) when $n \ge 0$ is small enough. If $n = 0$, then we see that (3.4) reduces to (2.5) with k in place of n. So (3.4) includes an explicit expression for $\widetilde{L}_k^{(\alpha)}(x)$ as a special case. When $n = 1, 2$, we are able to deduce the following recurrence relations.

Corollary 3.4. *We have*

$$\text{(i)} \quad \widetilde{L}_{k+1}^{(\alpha)}(x) = (\alpha+2k+1-x) \widetilde{L}_k^{(\alpha)}(x) - (\alpha+k)k \widetilde{L}_{k-1}^{(\alpha)}(x);$$

$$\begin{aligned} \text{(ii)} \quad \widetilde{L}_{k+2}^{(\alpha)}(x) &= -2x \widetilde{L}_{k+1}^{(\alpha)}(x) + \left\{ (\alpha+2k+2)_2 - 2x - x^2 \right\} \widetilde{L}_k^{(\alpha)}(x) \\ &\quad - 2(\alpha+2k+1)(\alpha+k)k \widetilde{L}_{k-1}^{(\alpha)}(x) + (\alpha+k)_2 (k)_2 \widetilde{L}_{k-2}^{(\alpha)}(x). \end{aligned} \tag{3.7}$$

Proof. (i) Taking $n = 1$ in (3.4) provides us with

$$\widetilde{L}_{k+1}^{(\alpha)}(x) + x \widetilde{L}_k^{(\alpha)}(x) = \sum_{r=0}^{k} \binom{k}{r} (\alpha+k)_r (\alpha+2k+1-r)_1 (-x)^{k-r}. \tag{3.8}$$

Using the identities $(\alpha+k)_r = (\alpha+k)(\alpha+k-1)_{r-1}$ and $\binom{k}{r}r = k\binom{k-1}{r-1}$ for $r \geq 1$, we see from (2.5) that the right-hand side of (3.8) becomes

$$\begin{aligned}
&(\alpha+2k+1)\sum_{r=0}^{k}\binom{k}{r}(\alpha+k)_r(-x)^{k-r} - \sum_{r=1}^{k}\binom{k}{r}r(\alpha+k)_r(-x)^{k-r}\\
&= (\alpha+2k+1)\widetilde{L}_k^{(\alpha)}(x) - (\alpha+k)k\sum_{r=1}^{k}\binom{k-1}{r-1}(\alpha+k-1)_{r-1}(-x)^{k-r}\\
&= (\alpha+2k+1)\widetilde{L}_k^{(\alpha)}(x) - (\alpha+k)k\widetilde{L}_{k-1}^{(\alpha)}(x).
\end{aligned}$$

Substituting this into (3.8) and gathering the terms involving $\widetilde{L}_k^{(\alpha)}(x)$ together, we can deduce (i). Needless to say, (i) is the same as (1.4) with $n = k$.

(ii) Next we take $n = 2$ in (3.4) to obtain

$$\begin{aligned}
&\widetilde{L}_{k+2}^{(\alpha)}(x)+2x\widetilde{L}_{k+1}^{(\alpha)}(x) + (x^2+2x)\widetilde{L}_k^{(\alpha)}(x)\\
&=\sum_{r=0}^{k}\binom{k}{r}(\alpha+k)_r(\alpha+2k+2-r)_2(-x)^{k-r}.
\end{aligned} \tag{3.9}$$

Denoting the right-hand side of (3.9) by $A_k(x)$ and noting that

$$(\alpha+2k+2-r)_2 = (\alpha+2k+2)_2 - (2\alpha+4k+3)r + r^2,$$

we split up $A_k(x)$ into three parts such as $A_k(x) = P_1(x) + P_2(x) + P_3(x)$, where

$$\begin{aligned}
P_1(x) &:=(\alpha+2k+2)_2\sum_{r=0}^{k}\binom{k}{r}(\alpha+k)_r(-x)^{k-r};\\
P_2(x) &:=-(2\alpha+4k+3)\sum_{r=1}^{k}\binom{k}{r}r(\alpha+k)_r(-x)^{k-r};\\
P_3(x) &:=\sum_{r=1}^{k}\binom{k}{r}r^2(\alpha+k)_r(-x)^{k-r}.
\end{aligned}$$

Since $\binom{k}{r}r = k\binom{k-1}{r-1}$ $(r \geq 1)$, we see from (2.5) that

$$\begin{aligned}
P_1(x) &= (\alpha+2k+2)_2\widetilde{L}_k^{(\alpha)}(x);\\
P_2(x) &= -(2\alpha+4k+3)(\alpha+k)k\sum_{r=1}^{k}\binom{k-1}{r-1}(\alpha+k-1)_{r-1}(-x)^{k-r}\\
&= -(2\alpha+4k+3)(\alpha+k)k\widetilde{L}_{k-1}^{(\alpha)}(x).
\end{aligned}$$

Further, since

$$(\alpha+k)_r = (\alpha+k)(\alpha+k-1)(\alpha+k-2)_{r-2} \quad (r \geq 2),$$

using the easy identity $\binom{k}{r}r^2 = k\binom{k-1}{r-1} + (k)_2\binom{k-2}{r-2}$, where $\binom{x}{y} = 0$ if $y < 0$ by convention, it can be shown from (2.5) that

$$\begin{aligned}P_3(x) =&k\sum_{r=1}^{k}\binom{k-1}{r-1}(\alpha+k)_r(-x)^{k-r} + (k)_2\sum_{r=2}^{k}\binom{k-2}{r-2}(\alpha+k)_r(-x)^{k-r}\\ =&(\alpha+k)k\sum_{r=1}^{k}\binom{k-1}{r-1}(\alpha+k-1)_{r-1}(-x)^{k-r}\\ &+(\alpha+k)_2(k)_2\sum_{r=2}^{k}\binom{k-2}{r-2}(\alpha+k-2)_{r-2}(-x)^{k-r}\\ =&(\alpha+k)k\widetilde{L}_{k-1}^{(\alpha)}(x) + (\alpha+k)_2(k)_2\widetilde{L}_{k-2}^{(\alpha)}(x).\end{aligned}$$

Using these expressions of $P_1(x), P_2(x)$, and $P_3(x)$, we can rewrite $A_k(x)$ as

$$\begin{aligned}A_k(x) =&(\alpha+2k+2)_2\widetilde{L}_{k}^{(\alpha)}(x) - 2(\alpha+2k+1)(\alpha+k)k\widetilde{L}_{k-1}^{(\alpha)}(x)\\ &+(\alpha+k)_2(k)_2\widetilde{L}_{k-2}^{(\alpha)}(x).\end{aligned}$$

Thus, by substituting this into (3.9) and gathering the terms involving $\widetilde{L}_{k}^{(\alpha)}(x)$ in one place we finally obtain (ii), as desired. □

It is interesting that we were able to reprove (1.4) (i.e., (3.7) (i) with $k = n$) just as the special case of (3.4) for $n = 1$ without any use of the power series of the generating function or the orthogonality of polynomials. On the other hand, we cannot say for certain at this time, but the five-term recurrence relation (3.7) (ii) seems to be quite new and original to the best of our knowledge.

4. Additional Remarks

In this final section, as a supplement to the formulas stated in Proposition 2.4, we introduce a certain differential property, which is applicable to searching for linear differential equations satisfied by generalized Laguerre polynomials arising from given recurrence relations.

Proposition 4.1. *For integers $n, r, s \geq 0$ we have*

$$D_x^{r+s}L_{n+s}^{(\alpha)}(x) = \big(D_x - 1\big)^s\big(D_x^r L_n^{(\alpha)}(x)\big). \tag{4.1}$$

Proof. For (4.1) it suffices to prove that

$$D_x L_{n+1}^{(\alpha)}(x) = \big(D_x - 1\big)L_n^{(\alpha)}(x). \tag{4.2}$$

Indeed, one is able to arrive at (4.1) by repeatedly using (4.2) s times. Multiplying both sides of (4.2) by $(n+1)!$, we shall prove

$$D_x \widetilde{L}_{n+1}^{(\alpha)}(x) = (n+1)\big(D_x - 1\big)\widetilde{L}_n^{(\alpha)}(x). \tag{4.3}$$

Using (2.5) and the identity $\binom{n}{i}(n-i) = n\binom{n-1}{i} = \binom{n}{i+1}(i+1)$, it is shown that

$$\begin{aligned} D_x \widetilde{L}_n^{(\alpha)}(x) &= D_x\left(\sum_{i=0}^{n}\binom{n}{i}(\alpha+n)_i(-x)^{n-i}\right) \\ &= -\sum_{i=0}^{n-1}\binom{n}{i}(n-i)(\alpha+n)_i(-x)^{n-1-i} \\ &= -\sum_{i=0}^{n-1}\binom{n}{i+1}(i+1)(\alpha+n)_i(-x)^{n-1-i}. \end{aligned} \tag{4.4}$$

Let $I_m := [x^m]\big((D_x-1)\widetilde{L}_n^{(\alpha)}(x)\big)$ with the same notation as used in Remark 2. Then we have $I_n = (-1)^{n+1}$ and for $m = 0, 1, \ldots, n-1$,

$$\begin{aligned} I_m &= (-1)^{m+1}\binom{n}{m}(n-m)(\alpha+n)_{n-1-m} - (-1)^m\binom{n}{m}(\alpha+n)_{n-m} \\ &= (-1)^{m+1}\binom{n}{m}\{(n-m)(\alpha+n)_{n-1-m} + (\alpha+n)_{n-m}\} \\ &= (-1)^{m+1}\binom{n}{m}(\alpha+n+1)(\alpha+n)_{n-1-m}. \end{aligned}$$

On the other hand, referring to (4.4) shifted n to $n+1$, we have $J_n := [x^n]\left(\widetilde{L}_{n+1}^{(\alpha)}(x)\right) = (-1)^{n+1}(n+1)$ and for $m = 0, 1, \ldots, n-1$,

$$\begin{aligned} J_m := [x^m]\left(\widetilde{L}_{n+1}^{(\alpha)}(x)\right) &= (-1)^{m+1}\binom{n+1}{m}(n-m+1)(\alpha+n+1)_{n-m}. \\ &= (-1)^{m+1}(n+1)\binom{n}{m}(\alpha+n+1)(\alpha+n)_{n-1-m}. \end{aligned}$$

Thus, it was shown that $J_m = (n+1)I_m$ holds for all $m = 0, 1, \ldots, n$; and the proof of (4.3) (hence, of (4.1)) is complete. □

We now recall (1.4) replaced n by $n+1$ and differentiate it twice with respect to x in order to obtain

$$\begin{aligned} &(n+2)D_x^2 L_{n+2}^{(\alpha)}(x) - (\alpha+2n+3-x)D_x^2 L_{n+1}^{(\alpha)}(x) + 2D_x L_{n+1}^{(\alpha)}(x) \\ &+ (\alpha+n+1)D_x^2 L_n^{(\alpha)}(x) = 0. \end{aligned}$$

From (4.1) we see that this equation can be written in the form

$$(n+2)(D_x-1)^2 L_n^{(\alpha)}(x) - (\alpha+2n+3-x)(D_x-1)(D_x L_n^{(\alpha)}(x))$$

$$+2(D_x-1)L_n^{(\alpha)}(x)+(\alpha+n+1)D_x^2L_n^{(\alpha)}(x)=0,$$

which is exactly the same as

$$xD_x^2L_n^{(\alpha)}(x)+(\alpha+1-x)D_xL_n^{(\alpha)}(x)+nL_n^{(\alpha)}(x)=0. \tag{4.5}$$

Hence, it was shown that $L_n^{(\alpha)}(x)$ is the solution of the Laguerre equation (1.6).

Incidentally, by shifting n to $n+1$ in (2.6) (iv) and using (4.1) we get

$$(n+1)L_{n+1}^{(\alpha)}(x)=\big(xD_x+(\alpha+n+1-x)\big)L_n^{(\alpha)}(x), \tag{4.6}$$

which may be much more convenient than (1.4) as recursion formulas, because $L_{n+1}^{(\alpha)}(x)$ can be obtained directly from $L_n^{(\alpha)}(x)$ without any knowledge of $L_{n-1}^{(\alpha)}(x)$. By the way, differentiating both sides of (4.6) with respect to x and using (4.1), we can deduce the same equation as (4.5).

By the same strategy as the above, we replace k by $k+2$ in (3.7) (ii) and differentiate it four times with respect to x. Then, by using (4.1) it is also possible to obtain a linear differential equation of the fourth order satisfied by these polynomials with the form of

$$\sum_{i=0}^{4}a_i(x)y^{(i)}(x)=0.$$

Here note that $a_0(x)$ is a real number, but the others are linear or quadratic polynomial functions. Because of the reason that the expressions of these polynomial coefficients are rather lengthy and complicated to mention here, we wish to omit further details about this equation.

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