MAXIMUM DISTANCES IN THE FOUR-DIGIT KAPREKAR PROCESS

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Abstract

For natural numbers x and b, the classical Kaprekar function is defined as $K_b(x) = D - A$, where D is the rearrangement of the base-b digits of x in descending order and A is ascending. The bases b for which K_b has a 4-digit non-zero fixed point were classified by Hasse and Prichett, and for each base this fixed point is known to be unique. In this article, we determine the maximum number of iterations required to reach this fixed point among all four-digit base-b inputs, thus answering a question of Yamagami. Moreover, we also explore – as a function of b – the fraction of four-digit inputs for which iterating K_b converges to this fixed point.

1. Introduction

In 1949, Dattatreya Ramchandra Kaprekar [6] introduced the following process. We start with a base-b four-digit number x (allowing this to contain leading zeros such as $x = 0309$). Rearrange the digits of x to be in decreasing order and subtract from this the rearrangement of the digits written in increasing order (these operations being done on base-b integer representations). This yields another 4-digit base-b number, and we denote this output as $K_b(x)$. For instance, in base 10, we have

$$\begin{aligned}
K_{10}(3223) &= 3322 - 2233 = 889\\
K_{10}(0889) &= 9880 - 0889 = 8991\\
K_{10}(8991) &= 9981 - 1899 = 8082\\
K_{10}(8082) &= 8820 - 0288 = 8532\\
K_{10}(8532) &= 8532 - 2358 = 6174\\
K_{10}(6174) &= 7641 - 1467 = 6174.
\end{aligned}$$

From the above, we see 6174 is a fixed point of K_{10}. Exploring this further, Kaprekar discovered that in fact, if we take any 4-digit base-10 integer not divisible by 1111, then iterating K_{10} starting at that input will necessarily reach 6174 within at most 7 steps. (Multiples of 1111 are sent to 0, which is a trivial fixed point.) This result was subsequently popularized by Martin Gardner, who featured it in his March 1975 column of *Mathematical Games* [4] published in Scientific American.[1]

Although the study of this map began as recreational exploration, it has since been shown that the 2-digit version is closely related to Mersenne primes [15], and quite recently this procedure has found independent applications to cryptographic encoding schemes [9]. Several variations of the Kaprekar process have been studied as well [17, 18, 1], and we point to [15] and [10] for helpful overviews of the literature.

Many authors seeking to understand the behavior of this problem turned to searching for fixed points in various bases [13, 7, 14], and a lot has been published on the subject. To this end, Ludington [8] studied the behavior of K_b with b remaining fixed and a variable numer of digits, and she proved that for each fixed b, there are only finitely many r for which almost every r-digit base-b number is sent to the same fixed point. Similarly, the base-10 fixed points have been carefully studied in [11, 2]. Yamagami and Matsui [16] recently explored this for other bases as well, proving a lower bound for the number of base-b fixed points in terms of the number of non-trivial divisors of b.

A second approach has been to fix the number of digits and analyze the corresponding map for different bases. Among the literature most relevant for us, Hasse and Prichett [5] studied the four-digit version of this map to analyze for which bases there exists a fixed point of K_b, and they also characterized the bases for which almost every starting point converges to such a fixed point. Prichett [12] subsequently obtained similar results for 5-digit numbers.

For 3-digit numbers, Eldridge and Sagong [3] proved a base-b fixed point exists if and only if b is even. In this case, they showed that every input not divisible by 111 is eventually mapped to this fixed point. For b odd, they proved that repeated iteration of K for nearly all x results in a loop of period 2. Their analysis also determines the maximum number of iterations needed to reach this 3-digit fixed point.

Finally, Yamagami [15] discussed the behavior of K_b for a range of examples with particular focus on the 2-digit case. Yamagami also asked: *in the four-digit case, what is the furthest finite distance a point is away from a fixed point?*

In other words, let S_b denote the set of 4-digit base-b inputs for which $K_b^t(x)$ is eventually a non-zero constant (as t increases). We then define M_b to be the least m for which $K_b^m(x) = K_b^{m+1}(x)$ for all $x \in S_b$.

Previous literature has studied M_b in the case that inputs have either two [15]

[1] We also recommend the first puzzle presented in this column, concerning a worm traversing an ever-stretching rubber rope.

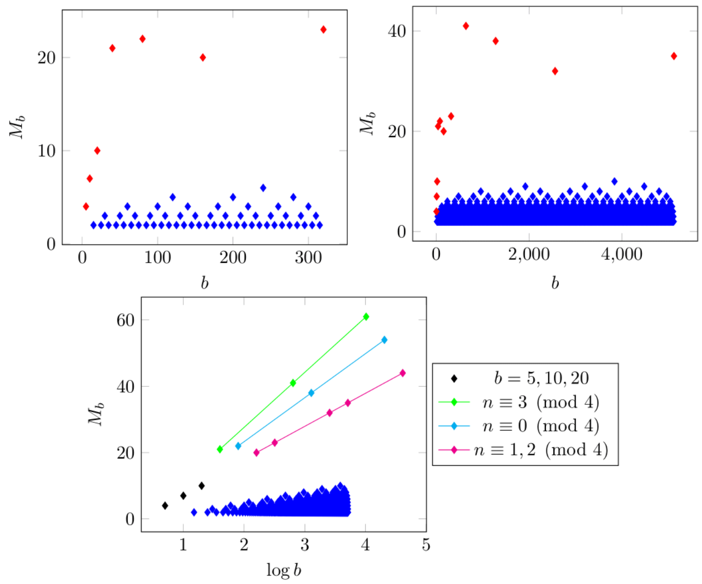

Figure 1: Plots of (computed) values of M_b shown at different scales.

or three [3] digits, and our first main result is a resolution of Yamagami's question for 4 digits. In this case, a result of [5] shows that a base-b non-trivial 4-digit fixed point exists if and only if $b \in \{2, 4\}$ or b is a multiple of 5. To determine M_b, we therefore need only consider these bases, which we accomplish as follows.

Theorem 1. *For each $b \in \{2, 4\} \cup \{5, 10, 15, 20, \ldots\}$, let M_b be the largest finite distance that a 4-digit base-b number is from a fixed point of the Kaprekar function. Then we have the following.*

(i) $M_2 = 1, M_4 = 3, M_5 = 4$, $M_{10} = 7$, *and* $M_{20} = 10$.

(ii) *If* $b = 5m \cdot 2^n$ *for some odd number* $m > 1$ *and some integer* $n \geq 0$, *then* $M_b = n + 2$.

(iii) *If* $b = 5 \cdot 2^n$ *and* $n \geq 3$, *then*

$$M_b = \begin{cases} 4n+6 & n \equiv 0 \pmod 4 \\ 3n+5 & n \equiv 1 \pmod 4 \\ 3n+5 & n \equiv 2 \pmod 4 \\ 5n+6 & n \equiv 3 \pmod 4. \end{cases}$$

The above result (and our proof) splits into several cases, the need for which is made evident when examining a plot of M_b, as in Figure 1. From this, we see most values of b exhibit a fractal-like pattern with several interspersed much larger values (namely those where $b = 5 \cdot 2^n$). Some version of (ii) (governing the fractal-like behavior) can be inferred from work present in [5], and we include a proof of this for completeness.

The results of [5] on the 4-digit Kaprekar function prove that for each base divisible by 5, there is exactly one non-trivial fixed point, the digits of which are $(3b/5)(b/5-1)(4b/5-1)(2b/5)$. Since this non-trivial fixed point is unique, it was a notable matter of interest to ask when (as in the base-10 case) every input not divisible by 1111 eventually reaches this fixed point. As before, let S_b denote the set of 4-digit inputs which eventually reach this fixed point. The main result of [5] was that $|S_b| = b^4 - b$ if and only if $b = 5 \cdot 2^n$ for $n = 0$ or n odd (the $-b$ term accounting for the b multiples of 1111). Our next result generalizes this to study $C_b = |S_b|/b^4$—the fraction of all input which eventually reach the non-trivial fixed point—for other bases b.

Theorem 2. *If $b = 5m \cdot 2^n$ for $m > 1$ odd and $n \geq 0$, then* $C_b = \dfrac{|S_b|}{b^4} = \dfrac{8 + 40 \cdot 4^n}{5 \cdot b^2} = \dfrac{8}{5 \cdot b^2} + \dfrac{8}{25m^2}$.

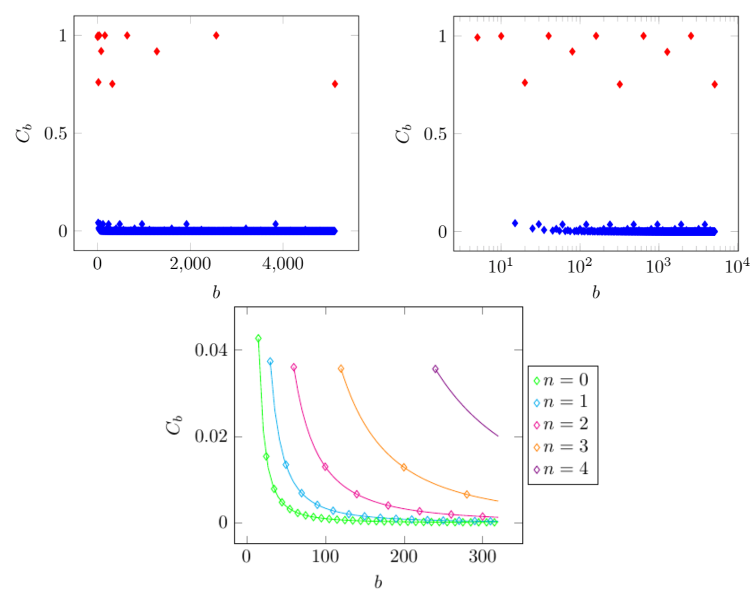

Figure 2: Plots of C_b. Points colored according to values of n.

In terms of Figure 2, the above theorem describes the behavior of all the points where C_b is relatively small (i.e., those points clustered near $C_b = 0$ in the first two graphs). The last graph in the figure zooms in on this region and overlays our proven formula for these values. As discussed, [5] provides a formula for C_b when $b = 5 \cdot 2^n$ and n odd. For even n, the middle graph in Figure 2 seems to suggest there is some fractal-like self-similarity. Numerically, it seems like these values of C_b converge, and a somewhat bold conjecture might be that $C_b \to 3/4$ for $b = 5 \cdot 2^{4n}$ and that $C_b \to 15/16$ for $b = 5 \cdot 2^{4n+2}$. We are unsure if either of these are true (or how a proof might go), but we believe a resolution either way would be interesting.

1.1. Outline of the Paper

We begin in Section 2 by introducing a useful transformation originally due to [5] that we will be using throughout. In Section 3, we study bases of the form $b = 5m \cdot 2^n$ with $m > 1$ odd. In this same section, we prove Theorem 1(ii) as well as Theorem 2. In Section 4, we address bases of the form $5 \cdot 2^n$ and prove Theorem 1(iii). We conclude in Section 5 with a discussion of open problems.

2. Difference Pairs

We begin with the following transformation, which is very useful in studying the 4-digit Kaprekar function. If the four base-b digits of x are $a_3 \geq a_2 \geq a_1 \geq a_0$, then its (base-$b$) *difference pair* is given by $(a_3 - a_0, a_2 - a_1)$. With this notation, Hasse and Prichett proved that the fixed point when $5|b$ has difference pair $(3b/5,\ b/5)$. For a given difference pair $X = (d, d')$, we will write $cX = c(d, d')$ to mean (cd, cd'). Moreover, from the definition, we see that any difference pair (d, d') necessarily has $d \geq d'$. Therefore, as a slight (but convenient) abuse of notation, we will occasionally allow "difference pairs" of the form (x, y) where $x < y$ with the understanding that this is to be interpreted as (y, x).

There is an important distinction to be made between a fixed point of K and a fixed pair. It is possible for a number to have the same difference pair as a fixed point of K and not be a fixed point itself. (e.g. 6174 and 8532 have the same difference pair). That said, Hasse and Prichett [5] made the observation that $K(x)$ is completely determined by the difference pair of x, allowing us to extend the definition of Kaprekar's function to difference pairs. But first, we classify difference pairs into three types:

Definition. Let (d, d') be a difference pair in base b.
If $d > d'$ and $d + d' \neq b$, we say that (d, d') is *type (a)*.
If $d = d'$ or $d + d' = b$, we say that (d, d') is *type (b)*.
If $d > d'$ and $d' = 0$, we say that (d, d') is *type (c)*.

With this, we can define K for difference pairs as follows. $K((d,d')) = (d_1, d_1')$, where

$$\{d_1, d_1'\} = \begin{cases} \{|2d-b|, |2d'-b|\} & (d,d') \text{ is type (a)} \\ \{|2d-(b-1)|, |2d-(b+1)|\} & (d,d') \text{ is type (b)} \\ \{d-1, b-d\} & (d,d') \text{ is type (c)} \\ \{0\} & d = d' = 0. \end{cases}$$

Importantly, under this definition, if x has difference pair (d_0, d_0') and $y = K(x)$ has difference pair (d_1, d_1'). Then $K((d_0, d_0')) = (d_1, d_1')$. Moreover, if x and z have the same difference pair, then $K(x)$ and $K(z)$ do as well (in fact, we also have $K(x) = K(z)$ as integers).

It turns out to be quite useful to be able to list out all possible predecessors of a particular difference pair. Hasse and Prichett [5, Table 1] provided a table describing just that, and below we provide two tables.

Type	Predecessor Type	Immediate Predecessors	Required Conditions
(a)	(a)	$(\frac{b+d}{2}, \frac{b+d'}{2})$ $(\frac{b+d}{2}, \frac{b-d'}{2})$ $(\frac{b-d'}{2}, \frac{b-d}{2})$ $(\frac{b+d'}{2}, \frac{b-d}{2})$	$d \equiv d' \equiv b \pmod 2$
	(b)	$(\frac{b-1+d}{2}, \frac{b-1+d}{2})$ $(\frac{b-1+d}{2}, \frac{b+1-d}{2})$ $(\frac{b+1-d}{2}, \frac{b+1-d}{2})$	$d \equiv b+1 \pmod 2$, $d = d'+2$
	(c)	$(d+1, 0)$ $(d'+1, 0)$	$d + d' = b - 1$
(b)	(a)	$(\frac{b+d}{2}, \frac{b+d'}{2})$ $(\frac{b+d}{2}, \frac{b-d'}{2})$ $(\frac{b-d'}{2}, \frac{b-d}{2})$ $(\frac{b+d'}{2}, \frac{b-d}{2})$	$d + d' = b$, $d \neq d'$, $d \equiv d' \equiv 0 \pmod 2$
	(b)	$(\frac{3b}{4}, \frac{3b}{4})$ $(\frac{3b}{4}, \frac{b}{4})$ $(\frac{b}{4}, \frac{b}{4})$	$d = d'+2$, $b \equiv 0 \pmod 4$
		$(\frac{b}{2}, \frac{b}{2})$	$d = d' = 1$, $b \equiv 0 \pmod 2$
	(c)	$(\frac{b+2}{2}, 0)$	$d = d' = \frac{b+1}{2}$, $b \equiv 1 \pmod 2$
(c)	(a)	$(\frac{b+d}{2}, \frac{b}{2})$ $(\frac{b}{2}, \frac{b-d}{2})$	$b \equiv d \equiv 0 \pmod 2$
	(b)	$(\frac{b+1}{2}, \frac{b+1}{2})$ $(\frac{b+1}{2}, \frac{b-1}{2})$ $(\frac{b-1}{2}, \frac{b-1}{2})$	$d = 2$, $b \equiv 1 \pmod 2$
	(c)	$(1, 0)$	$d = b - 1$

Alternatively, we can condense the contents of the table even further as follows:

Let $4 < b$ be any base divisible by four. The comprehensive list of what precedes each difference pair is:

- $(0,0) \leftarrow (0,0)$ and $(1,1) \leftarrow (b/2, b/2)$ and also $(b-1,0) \leftarrow (1,0)$

(i) for $i \neq j$, we have $(2i, 2j) \leftarrow \left(\frac{b}{2} \pm i, \frac{b}{2} \pm j\right)$ (all four sign combinations)

(ii) for all k, we have $(2k+1, 2k-1) \leftarrow \left(\frac{b}{2} \pm k, \frac{b}{2} \pm k\right)$ (all three (nonequivalent) sign combinations)

(iii) For $k \notin \{0, b-1\}$, there are two immediate predecessors of $(k, b-1-k)$. Namely $(k, b-1-k) \leftarrow (k+1, 0)$ and also $(k, b-1-k) \leftarrow (b-k, 0)$

- Pairs not listed above have no predecessors (in particular (x,x) has predecessors if and only if $x \in \{0,1\}$)

We now explore the contrast of applying K to an integer as opposed to applying it to its difference pair.

Lemma 1. *For any $b \equiv 0 \pmod 5$, if x has difference pair $(\frac{3b}{5}, \frac{b}{5})$, then $K(x)$ is the fixed point in base b. Furthermore, if (d, d') is a first generation predecessor of the fixed pair, and y has difference pair (d, d'), then $K(y) \neq K(x)$.*

Proof. Suppose x has difference pair $(\frac{3b}{5}, \frac{b}{5})$ and has the digits $\overline{\left(\frac{3b}{5}+c\right)\left(\frac{b}{5}+d\right)(d)\,(c)}$, then $K(x)$ is

$$\begin{array}{rcccc} & (\frac{3b}{5}+c) & (\frac{b}{5}+d) & (d) & (c) \\ - & (c) & (d) & (\frac{b}{5}+d) & (\frac{3b}{5}+c) \\ \hline & (\frac{3b}{5}) & (\frac{b}{5}) & (\frac{-b}{5}) & (\frac{-3b}{5}) \\ = & (\frac{3b}{5}) & (\frac{b}{5}-1) & (\frac{4b}{5}-1) & (\frac{2b}{5}). \end{array}$$

The second statement follows by a similar argument for each of the immediate difference-pair predecessors of the fixed point: $\left(\frac{4b}{5}, \frac{3b}{5}\right), \left(\frac{4b}{5}, \frac{2b}{5}\right)$, and $\left(\frac{2b}{5}, \frac{b}{5}\right)$. □

3. Bases of the Form $b = 5m \cdot 2^n$, for Odd $m > 1$

We first turn our attention to bases of the form $5m \cdot 2^n$, with $m > 1$ odd. For such bases, the behavior of K is relatively easy to understand. This was already noted as early as [5], and in fact their argument—much like ours—was split into these exceptional bases and the comparatively more regular bases discussed in later sections. We first provide a proof of Theorem 1(ii), essentially following [5].

3.1. Distance to the Fixed Point, M_b

Proof of Theorem 1(ii). Let $b = m \cdot 5 \cdot 2^n$, and define F_b to be the set of difference pairs for which repeated application of K_b leads to the base-b fixed point.

We first claim that if (u, v) is of type (a) with both coordinates divisible by $m > 1$ (with m odd), then any immediate predecessors of (u, v) must also have these two properties. From the table above, we see that (u, v) cannot have any immediate predecessors of type (b) since that would require $u = v + 2$ (a contradiction mod m). Similarly (u, v) cannot have any type (c) immediate predecessors either since that would require $u + v = b - 1$ (also a contradiction mod m). Therefore, any immediate predecessor of (u, v) must be of type (a). Such a predecessor would be of the form $(\frac{b\pm u}{2}, \frac{b\pm v}{2})$ and since m is odd, this would have both of its coordinates divisible by m.

From this, we see that every element of F_b must be of type (a) and have both coordinates divisible by m (since the fixed point X has both of these properties).

As an inductive base case, consider $n = 0$ (i.e., b is odd). In this case, we know that our fixed point X is in F_b, and the above table shows the immediate predecessors of X are

$$X = \left(\frac{3b}{5}, \frac{b}{5}\right), \qquad \text{and also} \qquad \left(\frac{4b}{5}, \frac{3b}{5}\right), \left(\frac{4b}{5}, \frac{2b}{5}\right), \left(\frac{2b}{5}, \frac{b}{5}\right).$$

Because $b = 5m$, these can be rewritten as $(3m, m), (4m, 3m), (4m, 2m), (2m, m)$. The first of these is the fixed point, so we need not consider it any further. As for the other three, they each have at least one even component, which precludes them from having any type (a) predecessors (because b is odd). Therefore, by our previous reasoning these three elements have no immediate predecessors whatsoever, which shows us that when $n = 0$, F_b consists of precisely the above four elements.

We claim that for all $b = 5m2^n$ (with $m > 1$ odd), we have

$$\begin{aligned} F_{2b} &= \bigcup_{(u,v)\in F_b} K_{2b}^{-1}(2u, 2v) \\ &= \Big\{(b+u, b+v), (b+u, b-v), (b+v, b-u), (b-v, b-u) \ : \ (u, v) \in F_b\Big\}. \end{aligned}$$

For this, suppose (r, s) is a type (a) difference pair in base $2b$. Further suppose that $K_{2b}(r, s) = (2u, 2v)$ (valid since applying K_{2b} to a type (a) elements yields a pair with both coordinates even because $2b$ is even). We know $(r, s) \in F_{2b}$ if and only if $K_{2b}(r, s) \in F_{2b}$, and because element of F_{2b} must be of type (a), our above formula for F_{2b} will be proven by showing that $(2u, 2v) \in F_{2b}$ if and only if $(u, v) \in F_b$.

First, note that for all p, q, we know (p, q) is a type (a) difference pair in base b if and only if $(2p, 2q)$ is a type (a) difference pair in base $2b$. And for type (a) difference pairs, applying K shows us that

$$\text{for type (a) pairs,} \qquad K_b(p,q) = (p', q') \quad \text{if and only if} \quad K_{2b}(2p, 2q) = (2p', 2q'). \tag{1}$$

Letting X denote the difference pair for the base-b fixed point, we know that $2X$ is the base-$2b$ fixed point. If $(u, v) \in F_b$, then it (and all its subsequent K_b-successors) will be of type (a). Moreover, repeatedly applying K_b to (u, v) will eventually yield X. Since each K_b-successor of (u, v) is type (a), we can iteratively use Equation (1) to show that repeatedly applying K_{2b} starting at $(2u, 2v)$ must eventually yield $2X$. Thus, if $(u, v) \in F_b$, then $(2u, 2v) \in F_{2b}$. For the reverse implication, we assume $(2u, 2v) \in F_{2b}$ and then apply this same argument, where the application of Equation (1) is valid since $(2u, 2v)$ and all of its subsequent K_{2b}-successors will be of type (a) with both coordinates even.

Finally, we prove the formula $M_b = n + 2$ by induction on n. When $n = 0$, our explicit base case worked above shows all four elements of F_b and Lemma 1 then gives us that $M_b = 2$. The induction step then follows immediately from our recursive formula for F_{2b} in terms of F_b, where we see the structures are essentially identical except that we need to take precisely one more generation of immediate predecessors. Thus, we see that $M_{2b} = 1 + M_b$, completing the proof. □

3.2. Convergence Rate

Our proof of Theorem 2 uses the notion of difference pairs, but the statement is about the number of inputs in S_b. Thus, we need the following, which relates how many four-digit numbers have a given difference pair.

Lemma 2. *For a type (a) difference pair (d, d') with $d \neq d'$ there are $N_b((d, d')) := 24(b - d)(d - d')$ four-digit base-b numbers with difference pair (d, d').*

Proof. Suppose x, which has type (a) difference pair (d, d'), has digits $a_3 \geq a_2 \geq a_1 \geq a_0$. Notice that $\{a_3, a_0\}$ can take values from $\{d, 0\}, \cdots, \{b-1, b-d-1\}$, $b-d$ possibilities. Similarly, $\{a_2, a_1\}$ can take values from $\{a_3, a_3 - d'\}, \cdots, \{a_0 + d', a_0\}$, $d - d' + 1$ possibilities. Since x is type (a), $d' \neq 0$ and $d' \neq d$. This means that the only situation where not all digits are distinct is when either $a_2 = a_3$ and $a_1 \neq a_0$ or $a_1 = a_0$ and $a_2 \neq a_3$. In both of these cases, x has exactly one pair of duplicate digits. Then we can compute how many x have type (a) difference pair (d, d') to be

$$4!(b - d)(d - d' - 1) + \frac{4!}{2}(b - d) \cdot 2 = 24(b - d)(d - d' - 1) + 24(b - d),$$

which equals $24(b - d)(d - d')$. □

Now we are equipped to prove Theorem 2.

Proof of Theorem 2. We let $m > 1$ be fixed and proceed by induction on n just as in our proof for M_b, and we let F_b denote the same set as before. Letting $A_n = C_b b^4$, we will prove that (i) $A_n = 40 \cdot 4^n \cdot m^2(1+5\cdot 4^n)$ and (ii) F_b can be partitioned into 4^n sets of the form $H = \{(x,y),(x,b-y),(y,b-x),(b-y,b-x)\}$ where $x > y > b-x$.

As an inductive base case, we consider $n = 0$. As before, the fixed point has only four difference-pair predecessors (including itself):

$$\left(\frac{3b}{5},\frac{b}{5}\right),\left(\frac{4b}{5},\frac{3b}{5}\right),\left(\frac{4b}{5},\frac{2b}{5}\right),\left(\frac{2b}{5},\frac{b}{5}\right).$$

Since m and b are odd, none of these has any additional predecessors. Setting $(x,y) = (\frac{4b}{5},\frac{3b}{5})$, we see this set is of the form required in claim (ii). Finally,

$$\begin{aligned}
N_b\left(\frac{4b}{5},\frac{3b}{5}\right) &= 24\left(b-\frac{4b}{5}\right)\left(\frac{4b}{5}-\frac{3b}{5}\right) = 24(2^n\cdot m)(2^n\cdot m)\\
N_b\left(\frac{4b}{5},\frac{2b}{5}\right) &= 24\left(b-\frac{4b}{5}\right)\left(\frac{4b}{5}-\frac{2b}{5}\right) = 24(2^n\cdot m)(2\cdot 2^n\cdot m)\\
N_b\left(\frac{2b}{5},\frac{b}{5}\right) &= 24\left(b-\frac{2b}{5}\right)\left(\frac{2b}{5}-\frac{b}{5}\right) = 24(3\cdot 2^n m)(2^n\cdot m)\\
N_b\left(\frac{3b}{5},\frac{b}{5}\right) &= 24\left(b-\frac{3b}{5}\right)\left(\frac{3b}{5}-\frac{b}{5}\right) = 24(2\cdot 2^n\cdot m)(2\cdot 2^n\cdot m),
\end{aligned}$$

and summing these (recalling that $n = 0$) gives us $A_0 = 240m^2$ as desired.

For our induction step, suppose our claims hold for some $n \geq 0$ and let $b = 5m2^n$. First notice that

$$N_b(x,y) = 24(b-x)(x-y), \quad N_b(x,b-y) = 24(b-x)(x+y-b),$$

$$N_b(b-y,b-x) = 24(y)(x-y), \quad N_b(y,b-x) = 24(b-y)(x+y-b).$$

Summing these four together gives us

$$N_b(H) := \sum_{(p,q)\in H} N_b(p,q) = 24(4bx+2by-2b^2-2x^2-2y^2).$$

For each set H in our partition of F_b, we define the set H' as

$$H' = \{(2x,2y),(2x,2b-2y),(2b-2y,2b-2x),(2y,2b-2x)\}.$$

By our recursive structural equation for F_{2b}, we see in fact that F_{2b} can be partitioned into $|F_b| = 4^{n+1}$ sets of the form $K_{2b}^{-1}(2u,2v)$, where $(u,v) \in F_b$. This immediately gives us part (ii) of our desired induction claim (taking $(x,y) = (b+u,b+v)$ to show the set $K_{2b}^{-1}(2u,2v)$ is of the desired form). We now need only prove (i). For this, notice from our partition that $A_{n+1} = \sum_H N_{2b}(K_{2b}^{-1}(H'))$, where the sum is over the sets H partitioning F_b. For every element h of H', the following table computes $N_{2b}(K_{2b}^{-1}(h))$, keeping in mind that we are now working in base $2b$.

h	Base $2b$ Immediate Predecessors	$N_{2b}(K_{2b}^{-1}(h))$
$(2x, 2y)$	$(b+x, b+y)$ $(b+x, b-y)$ $(b-y, b-x)$ $(b+y, b-x)$	$24(4bx - 2x^2 - 2y^2)$
$(2x, 2y)$	$(b+x, b+y)$ $(b+x, b-y)$ $(b-y, b-x)$ $(b+y, b-x)$	$24(4bx - 2x^2 - 2y^2)$
$(2x, 2b-2y)$	$(b+x, 2b-y)$ $(b+x, y)$ $(y, b-x)$ $(2b-y, b-x)$	$24(4bx + 4by - 2b^2 - 2x^2 - 2y^2)$
$(2b-2y, 2b-2x)$	$(2b-y, 2b-x)$ $(2b-y, x)$ (x, y) $(2b-x, y)$	$24(4bx - 2x^2 - 2y^2)$
$(2y, 2b-2x)$	$(b+y, 2b-x)$ $(b+y, x)$ $(x, b-y)$ $(2b-x, b-y)$	$24(4bx + 4by - 2b^2 - 2x^2 - 2y^2)$

We sum to obtain $N_{2b}(K_{2b}^{-1}(H')) = 24(16bx + 8by - 4b^2 - 8x^2 - 8y^2)$. Thus, for any given H and its corresponding H', we have $N_{2b}(K_{2b}^{-1}(H')) = 4N_b(H) + 24 \cdot 4b^2$. This then gives us

$$\begin{aligned} A_{n+1} &= \sum_H N_{2b}(K_{2b}^{-1}(H')) = \sum_H \left[4N_b(H) + 24 \cdot 4b^2\right] \\ &= 4\sum_H N_b(H) + 24\sum_H 4b^2 = 4A_n + 24 \cdot 4^{n+1}b^2 \\ &= 40 \cdot 4^{n+1} \cdot m^2 \left(1 + 5 \cdot 4^{n+1}\right), \end{aligned}$$

which completes the proof. □

4. Bases of the Form $b = 5 \cdot 2^n$

For bases of the form $5 \cdot 2^n$, a key observation of [5] was that repeatedly applying K to difference pairs will necessarily eventually result in a difference pair where both coordinates are divisible by $2^n = b/5$. After this, we need only consider trajectories of difference pairs of this type. For this, see Tables 1 and 3 of the Appendix, both of which have versions appearing in [5].

Thus, to understand the behavior of K, we need only understand how many iterations of K are required before reaching a difference pair where both coordinates are divisible by 2^n. This depends on which difference pair is ultimately reached, as discussed as follows.

Proposition 1. *Let $b = 5 \cdot 2^n$ with $n \geq 2$. Suppose (u, v) is any fixed difference pair, and let $L \geq 0$ be the least integer for which $K^L(u, v)$ is of the form $(p2^n, q2^n)$, where $0 \leq q \leq p < 5$. Then we have*

$$L \leq \begin{cases} 0 & \text{if } (p,q) = (3,1) \text{ or } p = q \\ n & \text{if } (p,q) \in \Big\{(4,1),(3,0),(4,0)\Big\} \\ 2n & \text{if } (p,q) \in \Big\{(4,2),(2,0)\Big\} \\ 2n+2 & \text{if } (p,q) \in \Big\{(1,0),(2,1),(3,2),(4,3)\Big\}. \end{cases}$$

Moreover, if $n \geq 5$, then for each (p, q) there is a pair (u, v) for which the above upper bound on L is attained.

We temporarily postpone a proof of the upper bound in the above proposition, but we show the equality case simply by explicitly constructing a pair (u, v) for each pair (p, q). Note that we need not consider the cases $p = q$ or $(p, q) = (3, 1)$. This is summarized in Table 2 of the Appendix, and each row is easily verified.

4.1. Derivation of Theorem 1(iii) from Proposition 1

Table 1 provides all the information required to know how many iterations are needed to reach the fixed point (or to loop) provided the initial input has a difference pair of the form $2^n(p, q)$. Moreover, Proposition 1 provides the exact bound for how many steps are possible before reaching any input whose difference sequence is of the form $2^n(p, q)$.

Together, we can combine these two pieces of information into Tables 3 and 4 of the Appendix. For each, we are determining exactly how many iterations of K are needed until arriving at the four digit fixed point (not merely a point whose difference pair is $(3b/5, b/5)$). Thus, in order to prove Theorem 1(iii) for $n \geq 5$, we need only find the largest value in each given column of Table 2. Bases $5 \cdot 2^n$ for $n \leq 4$ are each proven by an easy exhaustive computer search.

4.2. Lemmas to Prove Proposition 1

Lemma 3. *Suppose $4 < b$ is divisible by 2^n and $1 \leq t \leq n$. Further suppose that $K^t(u, v) = (2^t c, 2^t d)$. Then one of the coordinates of (u, v) is of the form $bi/2^t \pm c$ and the other is of the form $bj/2^t \pm d$, where i and j are positive odd integers each at most 2^t (with these two $\pm$ signs not necessarily being equal).*

Proof. We know from the second table in Section 2 that $K(u', v') = (2^t c, 2^t d)$ implies one coordinate of (u', v') is of the form $b/2 \pm 2^{t-1}c$ and the other of the form $b/2 \pm 2^{t-1}d$. If $t = 1$, this completes the proof. Otherwise, since $t \le n$, both of these are divisible by 2^{t-1}, so we may apply induction on t to assert that since $K^{t-1}(u, v) = (u', v')$, the coordinates of (u, v) must be of the form

$$\begin{aligned} \frac{bi}{2^{t-1}} \pm \left(\frac{b}{2^t} \pm c\right) &= \frac{b(2i \pm 1)}{2^t} \pm c, \qquad \text{and} \\ \frac{bj}{2^{t-1}} \pm \left(\frac{b}{2^t} \pm d\right) &= \frac{b(2j \pm 1)}{2^t} \pm d. \end{aligned}$$

And since $1 \le i \le 2^{t-1}$ is odd, we have that $2i \pm 1$ is odd as well and $1 \le 2i \pm 1 \le 2^t$, as desired. □

Lemma 4. *Suppose $b = 5 \cdot 2^n$ for $n \ge 2$. Suppose $(x, 0)$ is a difference pair such that 2^n does not divide x. If $K^L(u, v) = (x, 0)$, then $L \le n + 1$.*

Proof. First assume that $L \ge n + 1$ (otherwise, there is nothing to prove), and for each $0 \le t \le n + 1$, define $K^{L-t}(u, v) = (x_t, y_t)$ (so $K^t(x_t, y_t) = (x, 0) = (x_0, y_0)$).

Write $x = 2^m c$ for some integer $m < n$ and some odd number $c \ge 1$. By the second table of Section 2, if $m = 0$, then $(x, 0)$ has a predecessor only if $x = b - 1$. This implies $(x_1, y_1) = (1, 0)$, which has no predecessors. Thus $L \le 1 < n + 1$.

Now assume $m \ge 1$. Using Lemma 3, we know that (x_m, y_m) must be of the form $(bi/2^m \pm c, bj/2^m)$ or $(bj/2^m, bi/2^m \pm c)$, where i and j are positive odd integers at most 2^m. Since i, j and c are odd and since $m < n$, we have that one of the coordinates of (x_m, y_m) is even and the other is odd. Therefore, since (by assumption) $m < n < L$, we know that (x_m, y_m) has a predecessor (x_{m+1}, y_{m+1}), which in turn has another predecessor (x_{m+2}, y_{m+2}). But since $x_m \not\equiv y_m \pmod 2$ (and (x_m, y_m) has a predecessor), we see from the second table of Section 2 that $x_m + y_m = b - 1$ and therefore (x_{m+1}, y_{m+1}) is of the form either $(bj/2^m + 1, 0)$ or $(b - bj/2^m, 0)$. We consider these two cases separately.

We first consider the case that $(x_{m+1}, y_{m+1}) = (bj/2^m + 1, 0)$. Since this has a predecessor, we need $bj/2^m + 1 = b - 1$ and $(x_{m+2}, y_{m+2}) = (1, 0)$. And since $(1, 0)$ has no predecessors, this implies $L \le m + 2 \le n + 1$.

We now consider the case that $(x_{m+1}, y_{m+1}) = (b - bj/2^m, 0) = (5 \cdot 2^{n-m}(2^m - j), 0)$. By applying Lemma 3, since $K^{n-m}(x_{n+1}, y_{n+1}) = (x_{m+1}, y_{m+1})$, we have that the coordinates of (x_{n+1}, y_{n+1}) are of the form

$$bi'/2^{n-m} \pm 5 \cdot (2^m - j), \qquad \text{and} \qquad bj'/2^{n-m}.$$

As before, since $0 < m < n$, one of these coordinates is even and the other is odd. Thus, (x_{n+1}, y_{n+1}) has a predecessor only if $x_{n+1} + y_{n+1} = b - 1$, but this is not possible since $x_{n+1} \equiv y_{n+1} \equiv b \equiv 0 \pmod 5$. Thus, (x_{n+1}, y_{n+1}) cannot have any predecessors, which implies $L \le n + 1$. □

Lemma 5. *Suppose $b = 5 \cdot 2^n$ for $n \geq 2$, and suppose $(x, b - x)$ is a difference pair such that 2^n does not divide x. If $K^L(u, v) = (x, b - x)$, then $L \leq n - 1$.*

Proof. As in the previous proof, assume $L \geq n - 1$, and for each $0 \leq t \leq n - 1$ define $K^{L-t}(u, v) = (x_t, y_t)$. Also, write $x = 2^m c$ for $m < n$ and c odd. By Lemma 3, (x_m, y_m) must be of the form $(bi/2^m + \varepsilon_1 c, b(j + \varepsilon_2)/2^m - \varepsilon_2 c)$ or $(b(j + \varepsilon_2)/2^m - \varepsilon_2 c, bi/2^m + \varepsilon_1 c)$, where $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$ and i, j are positive odd integers each less than 2^m. Therefore, $x_m \equiv y_m \equiv 1 \pmod 2$, so if (x_{m+1}, y_{m+1}) exists, then we need either $(x_m, y_m) = (1, 1)$ or $x_m - y_m = 2$ (again by the second table of Section 2).

Handling these two cases separately, let us first suppose $(x_m, y_m) = (1, 1)$. But for $1 \leq t \leq n + 1$, we have $K^t(1, 1) = (b - 2^{t-1}, b - 3 \cdot 2^{t-1})$, and none of these are of the form $(x, b - x)$. So it's not possible to have $(x_m, y_m) = (1, 1)$ since $m < n$.

Moving to the second case, we now suppose $x_m - y_m = 2$. Then we need

$$\frac{b}{2^m}(i - j - \varepsilon_2) + c(\varepsilon_1 + \varepsilon_2) = \pm 2.$$

Looking at this mod 5, we see that if $L \geq m + 1$, then $c(\varepsilon_1 + \varepsilon_2) \equiv \pm 2 \pmod 5$, and thus $\varepsilon_1 = \varepsilon_2$. Therefore by the second table of Section 2, we have (x_{m+1}, y_{m+1}) is of the form

$$\left(\frac{b}{2} \pm \frac{b}{2^{m+2}}(i + j + \varepsilon_2), \frac{b}{2} \pm \frac{b}{2^{m+2}}(i + j + \varepsilon_2)\right).$$

Thus, $(x_{m+1}, y_{m+1}) = (5 \cdot 2^{n-m-2} p, 5 \cdot 2^{n-m-2} q)$, for odd integers p, q. By Lemma 3, the coordinates of (x_{n-1}, y_{n-1}) must therefore be of the form

$$bi'/2^{n-m-2} \pm 5p, \qquad \text{and} \qquad bj'/2^{n-m-2} \pm 5q,$$

for some odd integers $i', j' \geq 1$. And since both of these coordinates are odd and both are divisible by 5, the pair (x_{n-1}, y_{n-1}) cannot have any predecessors, and thus $L \leq n - 1$. □

4.3. Proof of Proposition 1

Proof. As noted before, the equality case is shown by cases as summarized in Table 2. For the remainder of our proof, we split the argument into four main cases. Note that the proofs for $L \leq n$ appear in cases (iii) and (iv). For $0 \leq t \leq L$, define $K^{L-t}(u, v) = (x_t, y_t)$ (so that $K^t(x_t, y_t) = (x_0, y_0) = (p2^n, q2^n)$). Throughout, we heavily rely on the second table of Section 2.

For case (i), we suppose either $(1, 1)$ or $(b - 1, 0)$ appears as some (x_t, y_t).

- If $(1, 1)$ is some (x_t, y_t), then $(u, v) \in \Big\{(1, 1), (b/2, b/2)\Big\}$ since $(1, 1)$ has only $(b/2, b/2)$ as a predecessor, and $(b/2, b/2)$ has none. For all $2 \leq t \leq n + 2$, we have $K^t(b/2, b/2) = (b - 2^{t-2}, b - 3 \cdot 2^{t-2})$, and since $K^{n+2}(b/2, b/2) = (4b/5, 2b/5)$ we need $(p, q) = (4, 2)$ and $L \leq n + 2$.

- Similarly, if $(b-1,0)$ is some (x_t,y_t), then $(u,v) \in \Big\{(b-1,0),(1,0)\Big\}$. For all $3 \le t \le n+2$ we have $K^t(1,0) = (b-2^{t-2}, b-2^{t-1})$, which implies $(p,q) = (4,3)$ and $L \le n+2$.

Thus, we may assume neither $(1,1)$ nor $(b-1,0)$ appears as any (x_t,y_t).

As case (ii), we suppose $p = q$ or that $(p,q) = (3,1)$. If $p = q \neq 0$, then $(p2^n, p2^n)$ has no predecessors. Moreover, the only predecessor of $(0,0)$ is $(0,0)$, and every immediate predecessor of $(3b/5, b/5)$ has both coordinates divisible by $b/5 = 2^n$. Thus, we have proven the claim for $p = q$ and for $(p,q) = (3,1)$.

For each of the next two cases, suppose $L \ge n$, and as before, we know that the coordinates of (x_n,y_n) are of the form

$$\frac{b}{2^n} i \pm p = 5i \pm p, \qquad \text{and} \qquad \frac{b}{2^n} j \pm q = 5j \pm q,$$

for positive odd integers i,j each at most 2^n.

For case (iii), suppose $p \equiv q \pmod 2$. The case $p \equiv q \equiv 1 \pmod 2$ was handled in case (ii), so we need only consider $p \equiv q \equiv 0 \pmod 2$. In this case, we see that both coordinates of (x_n,y_n) are odd, so if (x_n,y_n) has a predecessor, then $x_n = y_n + 2$ (since we have already ruled out the possibility that $(x_n,y_n) = (1,1)$).

Thus, if $L > n$, then $x_n - y_n = 2$, implying $\pm p \pm q \equiv \pm 2 \pmod 5$. If $(p,q) = (4,0)$, this is impossible, so in that case we need $L \le n$ as desired. But in general, if $L > n$, then (x_{n+1}, y_{n+1}) must be of the form $(b/2 \pm k, b/2 \pm k)$, where $k = (x_n + y_n)/4$. And since $(x_{n+1},y_{n+1}) \neq (1,1)$, if (x_{n+1},y_{n+1}) has any predecessors we need $(x_{n+1},y_{n+1}) = (b/2+k, b/2-k)$. By assumption, these coordinates are not both divisible by 2^n, so we may apply Lemma 5 to say that $L-(n+1) \le n-1$, proving $L \le 2n$.

As case (iv), we suppose $p \equiv q+1 \pmod 2$ so that $x_n \equiv y_n + 1 \pmod 2$. If $L > n$, then $x_n + y_n = b-1$, which implies $5(i+j) \pm p \pm q = 5 \cdot 2^n - 1$. Viewing this mod 5, this has no solutions if $(p,q) \in \Big\{(4,1),(3,0)\Big\}$, so in those cases we have $L \le n$. In general, we have $(x_{n+1},y_{n+1}) = (x_{n+1},0)$. Since x_{n+1} is not divisible by 2^n, Lemma 4 implies $L-(n+1) \le n+1$ as desired. □

5. Concluding Remarks

We have already discussed the open problem of determining C_b for all b, but there are several other interesting open questions as well. Perhaps the most obvious direction for future work would be to extend this analysis to the case of inputs with more than 4 digits, and [12] might be a good starting point.

As another example, now that M_b is understood, a natural next question might be to study how many points are t iterations away from the fixed point. We provide

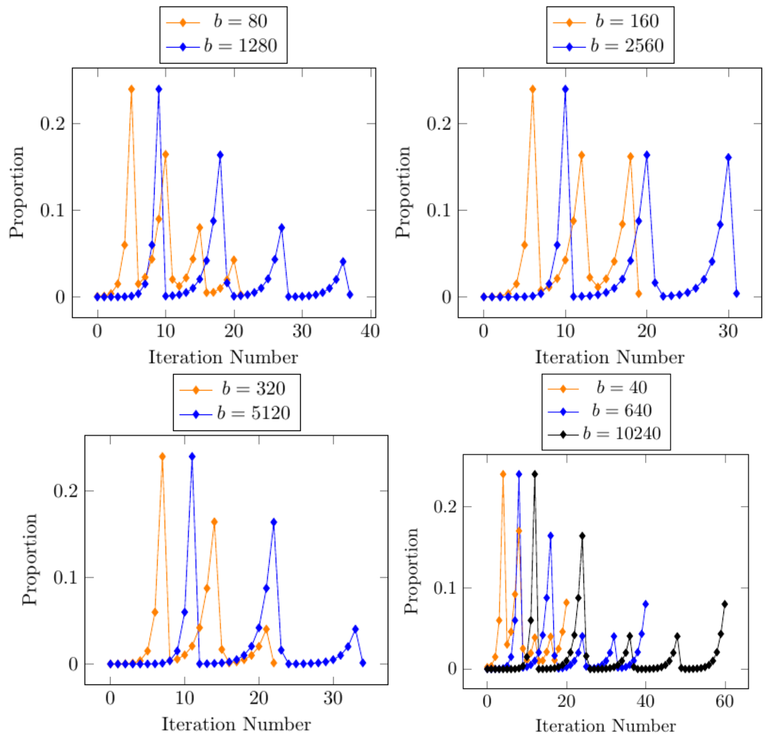

Figure 3: Distribution for how many points are t iterations from the fixed point

the following charts regarding these distributions for several given bases. Doing so reveals several intriguing features of these distributions.

It seems likely that the symmetry in these distributions can be explained in terms of Proposition 1—each spike consists of roughly $n+1$ points, which grow roughly like 4^t until most of these predecessors die out at the same time (as we no longer have any type (a) predecessors). We believe this heuristic could perhaps be made rigorous, but we anticipate that getting an exact understanding would be delicate.

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Appendix: Tables

Starting point	After exactly $n+1$ steps, we arrive at $n=4k$	$n=4k+1$	$n=4k+2$	$n=4k+3$
2^n(1,0)	2^n(1,0)	2^n(4,3)	2^n(2,1)	2^n(3,2)
2^n(2,0)	2^n(4,3)	2^n(2,1)	2^n(3,2)	2^n(1,0)
2^n(3,0)	2^n(3,2)	2^n(1,0)	2^n(4,3)	2^n(2,1)
2^n(4,0)	2^n(2,1)	2^n(3,2)	2^n(1,0)	2^n(4,3)
2^n(4,1)	2^n(4,2)	2^n(2,0)	2^n(4,2)	2^n(2,0)
2^n(1,1)	2^n(4,2)	2^n(2,0)	2^n(4,2)	2^n(2,0)
2^n(4,4)	2^n(4,2)	2^n(2,0)	2^n(4,2)	2^n(2,0)
2^n(3,2)	2^n(2,0)	2^n(4,2)	2^n(2,0)	2^n(4,2)
2^n(2,2)	2^n(2,0)	2^n(4,2)	2^n(2,0)	2^n(4,2)
2^n(3,3)	2^n(2,0)	2^n(4,2)	2^n(2,0)	2^n(4,2)

Table 1: (Derived in [5]) Trajectories of difference pairs where each coordinate is divisible by $2^n = b/5$. Note the behavior depends on the residue of n mod 4. For the 5 pairs not listed above, we have $K(0,0) = (0,0)$ and also $K(2b/5, b/5) = K(3b/5, b/5) = K(4b/5, 2b/5) = K(4b/5, 3b/5) = (3b/5, b/5)$.

Starting here	After this many steps	First pair with both coordinates divisible by 2^n: $n=4k$	$n=4k+1$	$n=4k+2$	$n=4k+3$
(4,1)	n	$2^n(4,1)$	$2^n(4,1)$	$2^n(4,1)$	$2^n(4,1)$
(5,1)	n	$2^n(4,0)$	$2^n(4,0)$	$2^n(4,0)$	$2^n(4,0)$
(5,2)	n	$2^n(3,0)$	$2^n(3,0)$	$2^n(3,0)$	$2^n(3,0)$
(9,1)	$2n$	$2^n(2,0)$	$2^n(4,2)$	$2^n(2,0)$	$2^n(4,2)$
(7,3)	$2n$	$2^n(4,2)$	$2^n(2,0)$	$2^n(4,2)$	$2^n(2,0)$
$(b/2,\ 5)$	$2n+2$	2^n(3,2)	2^n(1,0)	2^n(4,3)	2^n(2,1)
$(b/4,\ 5)$	$2n+2$	2^n(2,1)	2^n(3,2)	2^n(1,0)	2^n(4,3)
$(b/8,\ 5)$	$2n+2$	2^n(4,3)	2^n(2,1)	2^n(3,2)	2^n(1,0)
$(b/16,\ 5)$	$2n+2$	2^n(1,0)	2^n(4,3)	2^n(2,1)	2^n(3,2)

Table 2: Table showing the equality case in Proposition 1. In each of the last four rows, we need that the first coordinate be stricty larger than 5, whereas for the rest of the table, we only need $n \geq 2$.

Starting point	Number of steps until we reach the fixed point			
	$n=4k$	$n=4k+1$	$n=4k+2$	$n=4k+3$
2^n(2,1)	2	2	2	2
2^n(4,2)	2	2	2	2
2^n(4,3)	2	2	2	2
2^n(1,0)	Cycles (n/a)	$n+3$	$n+3$	$2n+4$
2^n(2,0)	$n+3$	$n+3$	Cycles (n/a)	$3n+5$
2^n(3,0)	$2n+4$	$2n+4$	$n+3$	$n+3$
2^n(4,0)	$n+3$	$2n+4$	$2n+4$	$n+3$
2^n(4,1)	$n+3$	$2n+4$	$n+3$	$4n+6$
2^n(1,1)	$n+3$	$2n+4$	$n+3$	$4n+6$
2^n(4,4)	$n+3$	$2n+4$	$n+3$	$4n+6$
2^n(3,2)	$2n+4$	$n+3$	Cycles (n/a)	$n+3$
2^n(2,2)	$2n+4$	$n+3$	Cycles (n/a)	$n+3$
2^n(3,3)	$2n+4$	$n+3$	Cycles (n/a)	$n+3$

Table 3: Trajectories derived from Table 1

First pair $2^n(p,q)$ encountered	Tight bound on total # steps to reach fixed point			
	$n=4k$	$n=4k+1$	$n=4k+2$	$n=4k+3$
2^n(2,1)	$2n+4$	$2n+4$	$2n+4$	$2n+4$
2^n(4,2)	$2n+2$	$2n+2$	$2n+2$	$2n+2$
2^n(4,3)	$2n+4$	$2n+4$	$2n+4$	$2n+4$
2^n(1,0)	$[2n+3]$*	$3n+5$	$3n+5$	$4n+6$
2^n(2,0)	$3n+3$	$3n+3$	$[2n+1]$*	$5n+5$
2^n(3,0)	$3n+4$	$3n+4$	$2n+3$	$2n+3$
2^n(4,0)	$2n+3$	$3n+4$	$3n+4$	$2n+3$
2^n(4,1)	$2n+3$	$3n+4$	$2n+3$	$5n+6$
2^n(1,1)	$n+3$	$2n+4$	$n+3$	$4n+6$
2^n(4,4)	$n+3$	$2n+4$	$n+3$	$4n+6$
2^n(3,2)	$4n+6$	$3n+5$	$[2n+3]$*	$3n+5$
2^n(2,2)	$2n+4$	$n+3$	[2]*	$n+3$
2^n(3,3)	$2n+4$	$n+3$	[2]*	$n+3$
2^n(0,0)	[1]*	[1]*	[1]*	[1]*
2^n(3,1)	1	1	1	1

Table 4: The tightness of these bounds requires $n \geq 5$. Numbers written as $[a]$* indicate that this does not lead to the fixed point associated with $2^n(3,1)$ but instead enters a loop. In this case, the number indicates the total number of steps until the process first reaches a value x for which $K^L(x) = x$ for some L (this is counting the number of steps to reach this value x, not merely a value having the same difference pair).