THE GAME OF FLIPPING COINS

Anthony Bonato[1]
Department of Mathematics, Ryerson University, Toronto, Ontario, Canada
abonato@ryerson.ca

Melissa A. Huggan[2]
Department of Mathematics and Computer Science, Mount Allison University, Sackville, New Brunswick, Canada
mhuggan@mta.ca

Richard J. Nowakowski[3]
Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia, Canada
r.nowakowski@dal.ca

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Abstract

We consider flipping coins, a partizan version of the impartial game turning turtles, played on lines of coins. We show that the values of this game are numbers, and these are found by first applying a reduction, then decomposing the position into an iterated ordinal sum. This is unusual since moves in the middle of the line do not eliminate the rest of the line. Moreover, when G is decomposed into lines H and K, then $G = (H : K^R)$. This is in contrast to hackenbush strings, where $G = (H : K)$.

1. Introduction

In Winning Ways Volume 3 [3], Berlekamp, Conway, and Guy introduced turning turtles and considered many variants. Each game involves a finite row of turtles, either on feet or backs, and a move is to turn one turtle over onto its back, with the option of flipping a number of other turtles, to the left, each to the opposite of its current state (feet or back). The number depends on the rules of the specific game. The authors moved to playing with coins as playing with turtles is cruel.

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These games can be solved using the Sprague-Grundy theory for impartial games [2], but the structure and strategies of some variants are interesting. The strategy for: MOEBIUS (flip up to five coins) played with 18 coins, involves *Möbius transformations*; MOGUL (flip up to seven coins) on 24 coins, involves the *miracle octad generator* developed by R. Curtis in his work on the Mathieu group M_{24} and the Leech lattice [6, 7]; TERNUPS [3] (flip three equally spaced coins) requires ternary expansions; and TURNING CORNERS [3], a two-dimensional version where the corners of a rectangle are flipped, needs nim-multiplication.

We consider a simple partizan version of TURNING TURTLES, also played with coins. We give a complete solution and show that it involves ordinal sums. This is somewhat surprising since moves in the middle of the line do not eliminate moves at the end. Compare this with HACKENBUSH STRINGS [2], and DOMINO SHAVE [5].

We will denote heads by 0 and tails by 1. Our partizan version will be played with a line of coins, represented by a 0-1 sequence, $d_1 d_2 \ldots d_n$, where $d_i \in \{0,1\}$. To this position, we associate the binary number $\sum_{i=1}^{n} d_i 2^{i-1}$. Left moves by choosing some pair of coins d_i, d_j, $i < j$, where $d_i = d_j = 1$ and flips them over so that both coins are 0s. Right also chooses a pair d_k, d_ℓ, $k < \ell$, with $d_k = 0$ and $d_\ell = 1$, and flips them over. If j is the greatest index such that $d_j = 1$, then d_k, $k > j$, will be deleted. For example,

$$1011 = \{0001, 001, 1 \mid 1101, 111\}.$$

The game eventually ends since the associated binary number decreases with every move. We call this game FLIPPING COINS.

Another way to model FLIPPING COINS is to consider tokens on a strip of locations. Left can remove a pair of tokens, and Right is able to move a token to an open space to its left. We use the coin flipping model for this game to be consistent with the literature.

The game is biased to Left. If there are a non-zero even number of 1s in a position, then Left always has a move; that is, she will win. Left also wins any non-trivial position starting with 1. However, there are positions that Right wins. The two-part method to find the outcomes and values of the remaining positions can be applied to all positions. First, apply a modification to the position (unless it is all 1s), which reduces the number of consecutive 1s to at most three. After this reduction, build an iterated ordinal sum, by successively deleting everything after the third last 1, this deleted position determines the value of the next term in the ordinal sum. As a consequence, the original position is a Right win, if the position remaining at the end is of the form $0\ldots01$, and the value is given by the ordinal sum.

The necessary background for numbers is in Section 2. Section 3 contains results about outcomes, and it also includes our main results. First, we show that the values are numbers in Theorem 3.2. Next, an algorithm to find the value of a position is presented, and Theorem 3.3 states that the value given by the algorithm is correct.

The actual analysis is in Section 4. It starts by identifying the best moves for both players in Theorem 4.2. This leads directly to the core result Lemma 4.5, which shows that the value of a position is an ordinal sum. The ordinal sum decomposition of G is found as follows. Let G^L be the position after the Left move that removes the rightmost 1s. Let H be the string $G \setminus G^L$; that is, the substring eliminated by Left's move. Let H^R be the result of Right's best move in H. Now, we have that $G = G^L : H^R$. In contrast, the ordinal sums for HACKENBUSH STRINGS and DOMINO SHAVE [5], involve the value of H not H^R.

The proof of Theorem 3.3 is given in Section 4.1. The final section includes a brief discussion of open problems.

Finally we pose a question for the reader, which we answer at the end of Section 4.1: Who wins $0101011111 + 1101100111 + 0110110110111$ and how?

2. Numbers

All the values in this paper are numbers and this section contains all the necessary background to make the paper self-contained. For further details, consult [1, 8]. Positions are written in terms of their options; that is, $G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$.

Definition 2.1 ([1, 2, 8])**.** Let G be a number whose options are numbers and let G^L, G^R be the Left and Right options of the canonical form of G.

1. If there is an integer k, $G^L < k < G^R$, or if either G^L or G^R does not exist, then G is the integer, say n, closest to zero that satisfies $G^L < n < G^R$.
2. If both G^L and G^R exist and the previous case does not apply, then $G = \frac{p}{2^q}$, where q is the least positive integer such that there is an odd integer p satisfying $G^L < \frac{p}{2^q} < G^R$.

The properties of numbers required for this paper are contained in the next three theorems.

Theorem 2.2 ([1, 2, 8])**.** *Let G be a number whose options are numbers and let G^L, G^R be the Left and Right options of the canonical form of G. If G' and G'' are any Left and Right options respectively, then*

$$G' \leqslant G^L < G < G^R \leqslant G''.$$

Theorem 2.2 shows that if we know that the string of inequalities holds, we need to only consider the unique best move for both players in a number.

We include the following examples to further illustrate these ideas.

(a) $0 = \{\,\mid\,\} = \{-9 \mid\,\} = \{-\frac{1}{2} \mid \frac{7}{4}\}$;

(b) $-2 = \{\,|\,-1\} = \{-\frac{5}{2} \mid -\frac{31}{16}\}$;

(c) $1 = \{0\,|\,\} = \{0 \mid 100\}$;

(d) $\frac{1}{2} = \{0 \mid 1\} = \{\frac{3}{8} \mid \frac{17}{32}\}$.

For games G and H, to show that $G \geqslant H$, we need to show that $G - H \geqslant 0$. Meaning, we need to show that $G - H$ is a Left win moving second. For more information, see Sections 5.1, 5.8, and 6.3 of [1].

Let G and H be games. The *ordinal sum* of G, the *base*, and H, the *exponent*, is

$$G : H = \{G^{\mathcal{L}}, G : H^{\mathcal{L}} \mid G^{\mathcal{R}}, G : H^{\mathcal{R}}\}.$$

Intuitively, playing in G eliminates H but playing in H does not affect G. For ease of reading, if an ordinal sum is a term in an expression, then we enclose it in brackets.

Note that $x : 0 = x = 0 : x$ since neither player has a move in 0. We demonstrate how to calculate the values of other positions with the following examples.

(a) $1 : 1 = \{1\,|\,\} = 2$;

(b) $1 : -1 = \{0 \mid 1\} = \frac{1}{2}$;

(c) $1 : \frac{1}{2} = \{0, (1 : 0) \mid (1 : 1)\} = \{0, 1 \mid \{1\,|\,\}\} = \{1 \mid 2\} = \frac{3}{2}$;

(d) $\frac{1}{2} : 1 = \{0, (\frac{1}{2} : 0) \mid 1\} = \{0, \frac{1}{2} \mid 1\} = \{\frac{1}{2} \mid 1\} = \frac{3}{4}$;

(e) $(1 : -1) : \frac{1}{2} = (\frac{1}{2} : \frac{1}{2}) = \{0, (\frac{1}{2} : 0) \mid 1, (\frac{1}{2} : 1)\} = \{0, \frac{1}{2} \mid 1, \frac{3}{4}\} = \{\frac{1}{2} \mid \frac{3}{4}\} = \frac{5}{8}$.

Note that in all cases, when base and exponent are numbers, the players prefer to play in the exponent. In the remainder of this paper all the exponents will be positive.

One of the most important results about ordinal sums was first reported in *Winning Ways*.

Theorem 2.3 (Colon Principle [2])**.** *If $K \geqslant K'$, then $G : K \geqslant G : K'$.*

The Colon Principle helps prove inequalities that will be useful in this paper.

Theorem 2.4. *Let G and H be numbers all of whose options are also numbers, and let $H \geqslant 0$.*

1. *If $H = 0$, then $G : H = G$. If $H > 0$, then $(G : H) > G$.*
2. *$G^L < (G : H^L) < (G : H) < (G : H^R) < G^R$.*

Proof. For item (1), the result follows immediately by Theorem 2.3.

For item (2), if $H \geqslant 0$ and all the options of G and H are numbers, then $G^L < G = (G : 0) \leqslant (G : H^L) < (G : H) < (G : H^R)$. The second, third, and fourth inequalities hold since H is a number and thus $0 \leqslant H^L < H < H^R$ and by applying the Colon Principle. To complete the proof, we need to show that $(G : H^R) < G^R$. To do so we check that $G^R - (G : H^R) > 0$, in words, we check that Left can always win. Left moving first can move in the second summand to $G^R - G^R = 0$ and win. Right moving first has several options:

1. Moving to $G^R - G^L > 0$, since G and its options are numbers. Hence Left wins.

2. Moving to $G^R - (G : H^{RL}) > 0$, by induction.

3. Moving to $G^{RR} - G : H^R$ but Left can respond to $G^{RR} - G^R > 0$ since G and its options are numbers.

In all cases, Left wins moving second. The result follows. □

To prove that all the positions are numbers, we use results from [4]. A set of positions from a ruleset is called a *hereditarily closed set of positions of a ruleset* if it is closed under taking options. This game satisfies ruleset properties introduced in [4]. In particular, the properties are called the *F1 property* and the *F2 property*, which both highlight the notion of *First-move-disadvantage* in numbers, and are defined formally as follows.

Definition 2.5 ([4])**.** Let S be a hereditarily closed ruleset. Given a position $G \in S$, the pair $(G^L, G^R) \in G^{\mathcal{L}} \times G^{\mathcal{R}}$ satisfies the *F1 property* if there is a $G^{RL} \in G^{R\mathcal{L}}$ such that $G^{RL} \geqslant G^L$ or there is a $G^{LR} \in G^{L\mathcal{R}}$ such that $G^{LR} \leqslant G^R$.

Definition 2.6 ([4])**.** Let S be a hereditarily closed ruleset. Given a position $G \in S$, the pair $(G^L, G^R) \in G^{\mathcal{L}} \times G^{\mathcal{R}}$ satisfies the *F2 property* if there are $G^{LR} \in G^{L\mathcal{R}}$ and $G^{RL} \in G^{R\mathcal{L}}$ such that $G^{RL} \geqslant G^{LR}$.

As proven in [4], if given any position $G \in S$, all pairs $(G^L, G^R) \in G^{\mathcal{L}} \times G^{\mathcal{R}}$ satisfy one of these properties, then the value of all positions are numbers. Furthermore, satisfying the F2 property implies satisfying the F1 property, and it was shown that all positions $G \in S$ are numbers if and only if for any $G \in S$, all pairs $(G^L, G^R) \in G^{\mathcal{L}} \times G^{\mathcal{R}}$ satisfy the F1 property. Combining these results gives the following theorem.

Theorem 2.7 ([4])**.** *Let S be a hereditarily closed ruleset. All positions $G \in S$ are numbers if and only if for any position $G \in S$, all pairs $(G^L, G^R) \in G^{\mathcal{L}} \times G^{\mathcal{R}}$ satisfy either the F1 or the F2 property.*

3. Main Results

Before considering the values and associated strategies, we consider the outcomes; that is, we partially answer the question: "Who wins the game?" The full answer requires an analogous analysis to finding the values.

Theorem 3.1. *Let $G = d_1 d_2 \dots d_n$. If $d_1 d_2 \dots d_n$ contains an even number of* 1*s, or if $d_1 = 1$ and there are least two* 1*s, then Left wins G.*

Proof. A Right move does not decrease the number of 1s in the position. Thus, if in G, Left has a move, then she still has a move after any Right move in G. Consequently, regardless of d_1, if there are an even number of 1s in G, it will be Left who reduces the game to all 0s. Similarly, if $d_1 = 1$ and there are an odd number of 1s, Left will eventually reduce G to a position with a single 1; that is, to $d_1 = 1$ and $d_i = 0$ for $i > 1$. In this case, Right has no move and loses. □

The remaining case, $d_1 = 0$ and an odd number of 1s, is more involved. The analysis of this case is the subject of the remainder of the paper. We first prove the following.

Theorem 3.2. *All* FLIPPING COINS *positions are numbers.*

Proof. Let G be a FLIPPING COINS position. If only one player has a move, then the game is an integer. Otherwise, let L be the Left move to change (d_i, d_j) from $(1, 1)$ to $(0, 0)$. Let R be the Right move to change (d_k, d_ℓ) from $(0, 1)$ to $(1, 0)$. No other digits are changed. If all four indices are distinct, then both L and R can be played in either order. In this case $G^{LR} = G^{RL}$. Thus, the F2 property holds. If there are only three distinct indices, then two of the bits are ones. If Left moves first, then $d_i = d_j = d_k = 0$. If Right moves first, then there are still two ones remaining after his move. After Left moves, we have $d_i = d_j = d_k = 0$ and hence, $G^L = G^{RL}$. The F1 property holds.

There are no more cases since there must be at least three distinct indices. Since every position satisfies either the F1 or F2 property it follows that, by Theorem 2.7, every position is a number. □

Given a position G, the following algorithm returns a value.

Algorithm: Let G be a FLIPPING COINS position. Let $G_0 = G$.

1. Set $i = 0$.
2. Reductions: Let α and β be binary strings, and either can be empty.
   (a) If $G_0 = \alpha 01^{3+j}\beta$, $j \geqslant 1$, then set $G_0 = \alpha 101^j\beta$.
   (b) If $G_0 = \alpha 01^3\beta$, and β contains an even number of 1s, then set $G_0 = \alpha 10\beta$.

(c) Repeat until neither case applies, then go to Step 3.

3. If G_i is 0^r1, $r \geqslant 0$ or $1^a0^{p_i}10^{q_i}1$, $a \geqslant 0$ and $p_i + q_i \geqslant 0$; then go to Step 5. Otherwise, $G_i = \alpha 01^a0^{p_i}10^{q_i}1$, $p_i + q_i \geqslant 1$, $a > 0$ and some α. Set

$$\begin{aligned} Q_i &= 0^{p_i}10^{q_i}1, \\ G_{i+1} &= \alpha 01^a. \end{aligned}$$

Go to Step 4.

4. Set $i = i + 1$. Go to Step 3.

5. If $G_i = 0^r1$, then set $v_i = -r$. If $G_i = 1^a0^{p_i}10^{q_i}1$, then set $v_i = \lfloor \frac{a}{2} \rfloor + \frac{1}{2^{2p_i+q_i}}$. Go to Step 6.

6. For j from $i - 1$ down to 0, set $v_j = v_{j+1} : \frac{1}{2^{2p_j+q_j-1}}$.

7. Return the number v_0.

The algorithm implicitly returns two different results:

1. For Step 3, the substrings, $Q_0, Q_1, \ldots, Q_{i-1}, G_i$, partition the reduced version of G;

2. The value v_0.

First we illustrate the algorithm with the following example. Consider the position $G = 100111101101101110111110011$. We highlight at each step which reduction is being applied to the underlined digits; 2(a) is denoted by †, while 2(b) is denoted by ‡. The algorithm gives that:

$$\begin{aligned} 100111101101101110111110011 &= 10011110110110111\underline{0111}10011(\dagger) \\ &= 1001111011011\underline{0111}1010011(\dagger) \\ &= 1001111011\underline{0111}01010011(\ddagger) \\ &= 1001111\underline{0111}001010011(\ddagger) \\ &= 10\underline{0111}110001010011(\dagger) \\ &= 1010110001010011. \end{aligned}$$

Step 3 partitions the last expression into 101(011)(000101)(0011) so that the ordinal sum is given by

$$\begin{aligned} v_0 &= \left(\left(\frac{1}{2} : \frac{1}{2} \right) : \frac{1}{64} \right) : \frac{1}{8} \\ &= \frac{10257}{16348}. \end{aligned}$$

Now let $H = 01001110110111011101$. The reductions give that:

$$\begin{aligned} 01001110110111011101 &= 0100111011\underline{0111}011101 \\ &= 010011\underline{1011}10011101 \\ &= 010\underline{0111}100011101 \\ &= 01010100011101. \end{aligned}$$

The last expression partitions into $01(0101)(00011)(101)$ so that

$$\begin{aligned} v_0 &= \left(\left(-1 : \frac{1}{4}\right) : \frac{1}{32}\right) : 1 \\ &= -\frac{893}{1024}. \end{aligned}$$

The next theorem is the main result of the paper.

Theorem 3.3 (Value Theorem)**.** *Let G be a* FLIPPING COINS *position. If v_0 is the value obtained by the algorithm applied to G, then $G = v_0$.*

In the next section, we derive several results that will be used to prove Theorem 3.3. The proof of Theorem 3.3 will appear in Section 4.1.

4. Best Moves and Reductions

The proofs in this section use induction on the options. An alternate but equivalent approach is to regard the techniques as induction on the associated binary number of the positions. The proofs require detailed examination of the positions and we will use notation suitable to the case being considered. Often, a typical position will be written as a combination of generic strings and the substring under consideration. For example, 111011000110101 might be parsed as (11101)(100011)(0101), and written $\alpha100011\beta$ or more compactly as $\alpha10^31^2\beta$.

We require several results before being able to prove Theorem 3.3. We begin by proving a simplifying reduction, followed by the best moves for each player, and then the remaining reductions used in the algorithm.

As an immediate consequence of Theorems 3.2 and 2.2 we have the following.

Corollary 4.1. *Let α, β, and γ be arbitrary binary strings. We then have that $\alpha1\beta0\gamma > \alpha0\beta1\gamma$. Moreover, for an integer $r \geqslant 0$ we have that $\beta10^r1 > \beta$.*

Proof. Recall that by Theorem 3.2 all FLIPPING COINS positions are numbers. Thus, Theorem 2.2 applies.

A Right option of $\alpha0\beta1\gamma$ is $\alpha1\beta0\gamma$ and so we have that $\alpha1\beta0\gamma > \alpha0\beta1\gamma$. Similarly, a Left option of $\beta10^r1$ is β and so we have that $\beta10^r1 > \beta$. $\square$

Next we prove the best moves for each player. Right wants to play the zero furthest to the right and the 1 adjacent to it. Left wants to play the two ones furthest to the right.

Theorem 4.2. *Let G be a* FLIPPING COINS *position, where in G, r and n, $r \neq n$, are the greatest indices such that $d_r = d_n = 1$. Let s be the greatest index such that $d_s = 0$. Left's best move is to play (d_r, d_n), and Right's best move is to play (d_s, d_{s+1}).*

Proof. We prove this theorem by induction on the options. Note that we use the equivalent binary representation of the game position. If there are three or fewer bits, then, by exhaustive analysis, the theorem is true.

Let G be $d_1 d_2 \ldots d_n$. We begin by proving Left's best moves. Let r and n be the two largest indices, where $d_r = d_n = 1$, thus $d_k = 0$ for $r < k < n$. Let i and j, $i < j$, be two indices with $d_i = d_j = 1$. We use the notation $G(d_i, d_j, d_r, d_n)$ to highlight the salient bits. The claimed best Left move is from $G(1, 1, 1, 1)$ to $G(1, 1, 0, 0)$. This must be compared to any other Left move, represented by moving from $G(1, 1, 1, 1)$ to $G(0, 0, 1, 1)$. That is, we need to show that $G(1, 1, 0, 0) - G(0, 0, 1, 1) \geqslant 0$.

For the moves to be different, at least three of i, j, r, n are distinct. We first assume the four indices are distinct. In this case, we have that $i < j < r < n$. By applying Corollary 4.1 twice, we have that

$$G(1, 1, 0, 0) > G(1, 0, 0, 1) > G(0, 0, 1, 1).$$

We may assume then, without loss of generality, that $j = r$ or $j = n$. If $j = n$ then $i < r$, since there are two distinct moves. Now consider $G(d_i, d_r, d_n) = G(1, 1, 1)$. By Corollary 4.1, we have that if $j = r$, $G(1, 0, 0) > G(0, 0, 1)$, and if $j = n$, $G(1, 0, 0) > G(0, 1, 0)$.

We now prove Right's best move. There are more cases to consider. Let s be the largest index such that $d_s = 0$ and therefore $d_{s+1} = 1$. Let i, j, $i < j$ be indices with $d_i = 0$ and $d_j = 1$. The claimed best move is d_s, d_{s+1} and this must be compared to the arbitrary Right move d_i, d_j. For the moves to be different, there must be at least three distinct indices.

The original position is either

$$G(d_i, d_j, d_s, d_{s+1}) = G(0, 1, 0, 1), \quad i < s$$

or

$$G(d_s, d_{s+1}, d_j) = G(0, 1, 1), \quad i = s, j > s + 1.$$

We need to show either $D = G(1, 0, 0, 1) - G(0, 1, 1, 0) \geqslant 0$ or $D = G(1, 1, 0) - G(1, 0, 1) \geqslant 0$, respectively. Suppose Right plays in the first summand of D. Note that, by induction, the best moves of Left and Right are known.

1. First, suppose $j < s$. By induction, Right's best move in the first summand of D, is to $D' = G(1,0,1,0) - G(0,1,1,0)$. Since $i < j$, it follows that $G(1,0,1,0)$ is a Right option of $G(0,1,1,0)$ and thus, D' is positive by Corollary 4.1.

2. If $j = s+1$, then there are only three distinct indices. The original game is $G(d_i, d_s, d_{s+1}) = G(0,0,1)$ and $D = G(1,0,0) - G(0,1,0)$. Since $G(1,0,0)$ is a Right option of $G(0,1,0)$, it follows that D is positive by Corollary 4.1.

3. Suppose $j > s+1$.

   If $i < s$ then the original game is of the form

$$G = \alpha d_i \beta d_s d_{s+1} 1^a d_j 1^b = \alpha 0 \beta 011^a 11^b, \quad a \geqslant 0, b \geqslant 0$$

   and

$$D = \alpha 1 \beta 011^a 01^b - \alpha 0 \beta 101^a 11^b.$$

   Two applications of Corollary 4.1 (applied to the highlighted terms) give

$$\alpha \underline{1} \beta \underline{0} 11^a 01^b \geqslant \alpha 0 \beta 1 \underline{1} 1^a \underline{0} 1^b \geqslant \alpha 0 \beta 101^a 11^b.$$

   If $i = s$ then

$$G = \alpha d_s d_{s+1} 1^a d_j 1^b = \alpha 011^a 11^b, \quad a \geqslant 0, b \geqslant 0$$

   and

$$D = \alpha 111^a 01^b - \alpha 101^a 11^b.$$

   One application of Corollary 4.1 (relevant terms again highlighted) gives

$$\alpha 1 \underline{1} 1^a \underline{0} 1^b \geqslant \alpha 101^a 11^b.$$

   Thus $D \geqslant 0$.

Next, we consider Right moving in the second summand of $D = G(1,0,0,1) - G(0,1,1,0)$. Note that by the choices of the subscripts, $d_\ell = 1$ if $n \geqslant \ell \geqslant s+1$.

1. If $n > s+2$, then Right's best move in the second summand is to change d_{n-1}, d_n from $(1,1)$ to $(0,0)$. Left copies this move in the first summand and the resulting difference game is non-negative by induction.

2. Suppose $n = s+2$.

   i. If $j < s+1$, then $G(d_i, d_j, d_s, d_{s+1}, d_{s+2}) = G(0,1,0,1,1)$ and $D = G(1,0,0,1,1) - G(0,1,1,0,1)$. Right's best move is to $G(1,0,0,1,1) - G(0,1,0,0,0)$. Left moves to $G(1,0,0,0,0) - G(0,1,0,0,0)$. This is positive by Corollary 4.1 and Left wins.

For the next two sub-cases, exactly two 1s will occupy two of the four indexed positions. Since Right is moving in the second summand, he is changing two 1s to two 0s. Thus, Left's best response for each case is to move in the first summand, bringing the game to $G(0,0,0,0) - G(0,0,0,0) = 0$, and she wins. For these cases, we only list the original position. The strategy for both cases is as just described.

ii. If $j = s{+}1$, then $G(d_i, d_s, d_{s+1}, d_{s+2}) = G(0,0,1,1)$ and $D = G(1,0,0,1) - G(0,1,0,1)$.

iii. If $j = s{+}2$, then $G(d_i, d_s, d_{s+1}, d_{s+2}) = G(0,0,1,1)$ and $D = G(1,0,1,0) - G(0,1,0,1)$.

3. Now suppose $n = s + 1$.

i. If $j < s + 1$, then let $\ell < s + 1$ be the largest index such that $d_\ell = 1$.

If $j < \ell$, then we have $G(d_i, d_j, d_\ell, d_s, d_{s+1}) = G(0,1,1,0,1)$ and $D = G(1,0,1,0,1) - G(0,1,1,1,0)$. Right's best move is to $G(1,0,1,0,1) - G(0,1,0,0,0)$. Left moves to $G(1,0,0,0,0) - G(0,1,0,0,0)$ which is positive since $G(1,0,0,0,0)$ is a Right option of $G(0,1,0,0,0)$.

If $j = \ell$, then $G(d_i, d_j, d_s, d_{s+1}) = G(0,1,0,1)$ and $D = G(1,0,0,1) - G(0,1,1,0)$. Right's best move is to $G(1,0,0,1) - G(0,0,0,0)$. Left moves to $G(0,0,0,0) - G(0,0,0,0) = 0$, and Left wins.

ii. If $j = s + 1$, then $G(d_i, d_s, d_{s+1}) = G(0,0,1)$ and $D = G(1,0,0) - G(0,1,0)$. This is positive by Corollary 4.1.

In all cases, Left wins D moving second, proving the result. □

Suppose in a position that the bits of the best Right move are different from those of the best Left move. The next lemma essentially says that the position before and after one move by each player are equal. It is phrased in a way that is useful for reducing the length of the position. Recall that a non-trivial position looks like, $G = \alpha 0 1^a 0^p 1 0^q 1 \beta$, where a, p, and q are non-negative integers and α and β are arbitrary binary strings. For the algorithm, it suffices to prove the result for β being empty. However, it is useful, certainly for a human, to reduce the length of the position as much as possible.

Lemma 4.3. *Let α be an arbitrary binary string. If $a \geqslant 0$, then we have that $\alpha 0 1 1 1 1^a = \alpha 1 0 1^a$.*

Proof. Let $H = \alpha 0 1 1 1 1^a - \alpha 1 0 1^a$. We need to show $H = 0$. To simplify the proof, in some cases the second player will play sub-optimal moves. We have several cases to consider.

1. If $a \geqslant 2$, then playing the same move in the other summand is a good response. After two such moves we have either

$$\alpha 01111^{a-2} - \alpha 101^{a-2} = 0, \quad \text{by induction,}$$

or

$$\alpha 10111^{a} - \alpha 1101^{a-1} = \alpha 1101^{a-1} - \alpha 1101^{a-1} = 0, \quad \text{by induction.}$$

2. If $a = 1$, then $H = \alpha 01111 - \alpha 101$. The cases are:

   i. Left plays in the first summand to $\alpha 011 - \alpha 101$, then Right moves to $\alpha 101 - \alpha 101 = 0$.

   ii. Right plays in the second summand to $\alpha 01111 - \alpha$, then Left moves to $\alpha 011 - \alpha$. Since $(\alpha 011)^L = \alpha$, we have $\alpha 011 > \alpha$.

   iii. Right plays in the first summand to $\alpha 10111 - \alpha 101$, then Left responds to $\alpha 101 - \alpha 101 = 0$.

   iv. Left plays in the second summand to $\alpha 01111 - \alpha 11$, then Right moves to $\alpha 10111 - \alpha 11 = \alpha 11 - \alpha 11 = 0$, by induction.

3. If $a = 0$, then $H = \alpha 0111 - \alpha 1$. There are several cases to consider.

   i. If Left or Right play in the first summand, then the response is in the first summand giving $\alpha 1 - \alpha 1 = 0$.

   ii. If Left plays in the second summand, then since there is a Left move, we have $\alpha = \beta 01^b$, $b \geqslant 0$. If $b > 0$, we have that $\beta 01^b 0111 - \beta 01^b 1$ and Left moves to $\beta 01^b 01^3 - \beta 101^b$. Here, Right responds to $\beta 101^{b-1} 01^3 - \beta 101^b$, which by induction is equal to $\beta 101^{b-1} 1 - \beta 101^b = 0$. If $b = 0$ we have that $\beta 01^b 0111 - \beta 01^b 1 = \beta 00111 - \beta 01$ and we want to show that Right can win moving second. Left plays to $\beta 00111 - \beta 10$ and Right can respond to $\beta 01110 - \beta 1$ which, by induction, is equal to $\beta 1 - \beta 1 = 0$.

   iii. Right plays in the second summand. For a Right move to exist, then $\alpha = \beta 10^a$, $a \geqslant 0$. Thus, $H = \beta 10^a 0111 - \beta 10^a 1$, and Right moves to $\beta 10^a 0111 - \beta$. Left responds by moving to $\beta 00^a 011 - \beta$. We then have that $(\beta 00^a 011)^L = \beta$; thus, $\beta 00^a 011 > \beta$. Hence, we find that $\beta 00^a 011 - \beta > 0$.

In all cases, the second player wins H thereby proving the result. □

There are reductions that can be applied to the middle of the position, but extra conditions are needed.

Lemma 4.4. *Let α and β be arbitrary binary strings where either (a) β starts with a* 1*, or (b) β starts with* 0 *and has an even number of* 1*s. We then have that*

$$\alpha 0111\beta = \alpha 10\beta.$$

Proof. Let $H = \alpha 0111\beta - \alpha 10\beta$. We need to show that $H = 0$. We have several cases to consider.

1. If β is empty or $\beta = 1^a$, then $H = 0$ by Lemma 4.3. Therefore, we may assume that β has at least one 1 and one 0.
2. If $\beta = 1\gamma 1$ (β must end in a 1), then the best moves, in both summands, are pairs of bits in β and $-\beta$. If each player copies the opponent's move in the other summand, then this leads to

$$\alpha 0111\beta - \alpha 10\beta \to \alpha 0111\beta' - \alpha 10\beta'$$

   and the latter expression is equal to 0, by induction.
3. If $\beta \neq 1\gamma 1$, then $\beta = 0\gamma 1$ and $\gamma 1$ has at least two 1's. The best moves are in β and $-\beta$ and are the best responses to each other. We then derive that

$$\alpha 0111\beta - \alpha 10\beta \to \alpha 0111\beta' - \alpha 10\beta' = 0, \quad \text{by induction.}$$

In all cases $H = 0$, and this concludes the proof. □

In Lemma 4.4, the conditions are necessary. An example is:

$$3/8 = 011101 \neq 1001 = 1/4.$$

Here, β starts with a 0 and has an odd number of 1s.

These reduction lemmas are important in evaluating a position. The reduced positions will end in 011 or 01. By considering the exact end of the string, specifically, if there are at least two 0s (in one special case three 0s), then we can find an ordinal sum decomposition. The decomposition is determined by where the third rightmost 1 is situated.

The next result is the start of the ordinal sum decomposition of a position. The exponent is the value of the Right option of the substring being removed.

Lemma 4.5. *Let α be an arbitrary binary string. If $a \geqslant 1$ and p and q are non-negative integers such that $p + q \geqslant 1$, then*

$$\alpha 01^a 0^p 10^q 1 = \alpha 01^a : \frac{1}{2^{2p+q-1}}.$$

Proof. We prove that

$$\alpha 01^a 0^p 10^q 1 - \left(\alpha 01^a : \frac{1}{2^{2p+q-1}}\right) = 0.$$

Note that in Theorem 2.4 we have that playing in the base of $\alpha 01^a : \frac{1}{2^{2p+q-1}}$ is worse than playing in the exponent. We have two cases to consider.

1. Left plays first in the first summand, and Right responds in the second summand. Or Right plays first in the second summand, and Left responds in the first summand. In either case, Right has a move in the exponent (moves to 0) since $2p + q - 1 \geqslant 0$. In either order, the final position is given by:

$$\alpha 01^a - (\alpha 01^a : 0) = \alpha 01^a - \alpha 01^a = 0.$$

2. Right plays first in the first summand and Left responds in the second summand. Or Left plays first in the second summand and Right responds in the first summand. In either case, we consider,

$$\alpha 01^a 0^p 10^q 1 - \left(\alpha 01^a : \frac{1}{2^{2p+q-1}}\right).$$

   We have two sub-cases.

   i. Assume $2p + q - 1 \neq 0$. After the two moves we have the position

   $$\alpha 01^a 0^r 10^s 1 - \left(\alpha 01^a : \frac{1}{2^{2p+q-2}}\right),$$

   where $2r + s = 2p + q - 1$. By induction, we have that

   $$\begin{aligned}\alpha 01^a 0^r 10^s 1 &= \alpha 01^a : \frac{1}{2^{2r+s-1}}\\ &= \alpha 01^a : \frac{1}{2^{2p+q-2}}.\end{aligned}$$

   Thus, $\alpha 01^a 0^r 10^s 1 - \left(\alpha 01^a : \frac{1}{2^{2p+q-2}}\right) = 0$.

   ii. Assume $2p + q - 1 = 0$, that is, $q = 1$, $p = 0$. The original position is

   $$\alpha 01^a 101 - (\alpha 01^a : 1).$$

   After the two moves we have the position $\alpha 01^a 11 - \alpha 101^{a-1}$ (note that Left has no move in the exponent). By Lemma 4.3, $\alpha 01^a 11 = \alpha 101^{a-1}$. Hence, we have that $\alpha 01^a 11 - \alpha 101^{a-1} = 0$ and the result follows.

□

The values of the positions not covered by Lemma 4.5 are given next.

Lemma 4.6. *Let a, p, and q be non-negative integers. We then have that*

$$0^p1 = -p, \quad \text{and} \quad 1^a0^p10^q1 = \left\lfloor \frac{a}{2} \right\rfloor + \frac{1}{2^{2p+q}}.$$

Proof. Let $G = 0^p1$. Left has no moves and Right has p. Note that in 1^a, Left has $\lfloor \frac{a}{2} \rfloor$ moves and Right has none.

Now, let $G = 1^a0^p10^q1$. We proceed by induction on $p+q$. In all cases, Left's move is to 1^a, that is, to $\lfloor \frac{a}{2} \rfloor$. If $p = 0$ and $q = 0$ then $G = 1^a11$, which has value $\lfloor \frac{a}{2} \rfloor + \frac{1}{2^0} = \lfloor \frac{a}{2} \rfloor + 1$. Assume that $p + q = k$, $k > 0$. If $q > 0$, then $G = \{\lfloor \frac{a}{2} \rfloor \mid 1^a0^p10^{q-1}1\}$. By induction, we have that

$$G = \left\{ \left\lfloor \frac{a}{2} \right\rfloor \middle| \left\lfloor \frac{a}{2} \right\rfloor + \frac{1}{2^{2p+q-1}} \right\} = \left\lfloor \frac{a}{2} \right\rfloor + \frac{1}{2^{2p+q}}.$$

If $q = 0$, then $G = \{\lfloor \frac{a}{2} \rfloor \mid 1^a0^{p-1}10^11\}$. By induction, we have that

$$G = \left\{ \left\lfloor \frac{a}{2} \right\rfloor \middle| \left\lfloor \frac{a}{2} \right\rfloor + \frac{1}{2^{2(p-1)+1}} \right\} = \left\lfloor \frac{a}{2} \right\rfloor + \frac{1}{2^{2p}},$$

and the result follows. □

4.1. Proof of the Value Theorem

We now have all of the tools to prove Theorem 3.3.

Proof of Theorem 3.3. Let G be a FLIPPING COINS position. Step 2 reduces the binary string. The reductions in Step 2(a) are those of Lemma 4.3 and Lemma 4.4 part(a). The reductions in Step 2(b) are those of Lemma 4.4 part(b). In all cases, these lemmas show that each new reduced position is equal to G.

In Step 3, we claim $G_i \neq \beta 1^3$ for any β. This is true for $i = 0$ by Lemma 4.3. If $i > 0$, then at each iteration of Step 3, the last two 1s are removed from G_{i-1}. Now, the original reduced position would be $G_0 = \beta 1^3 \gamma$, where γ has an even number of 1s. Lemma 4.4 part(b) would apply eliminating the three consecutive 1s. Now either G_i is one of 0^r1, $r \geqslant 0$ or $1^a0^{p_i}10^{q_i}1$, $a \geqslant 0$ and $p_i + q_i \geqslant 0$, or $G_i = \alpha 01^a0^{p_i}10^{q_i}1$, $p_i + q_i \geqslant 1$, $a > 0$. In the latter case, the index is incremented and the algorithm goes back to Step 3.

Step 5 applies when Step 3 no longer applies, i.e., G_i is one of 0^r1, $r \geqslant 0$ or $1^a0^{p_i}10^{q_i}1$, $a \geqslant 0$ and $p_i + q_i \geqslant 0$. Now, v_i is the value of G_i, as given in Lemma 4.6.

Lemma 4.5 shows that for each $j < i$, $G_j = G_{j+1} : \frac{1}{2^{2p_j+q_j}}$, the evaluation in Step 6. Thus, the value of G is v_0, and the theorem follows. □

The question: "Who wins $0101011111+1101100111+0110110110111$ and how?" from Section 1 can now be answered.

First, we have that

$$\begin{aligned}
0101011111 &= 01011011 = \left(01011 : \frac{1}{2}\right) = \left(\left(01 : \frac{1}{2}\right) : \frac{1}{2}\right) \\
&= \left(\left(-1 : \frac{1}{2}\right) : \frac{1}{2}\right) = -\frac{11}{16} \\
1101100111 &= 1101101 = (1101 : 1) = \left(\frac{1}{2} : 1\right) = \frac{3}{4} \\
0110110110111 &= 0110110111 = 0110111 = 0111 = 0.
\end{aligned}$$

Thus, we have that

$$0101011111 + 1101100111 + 0110110110111 = -\frac{11}{16} + \frac{3}{4} + 0 = \frac{1}{16}.$$

Left's only winning move is to

$$01010111 + 1101100111 + 0110110110111 = -\frac{3}{4} + \frac{3}{4} + 0 = 0.$$

Her best moves in the second summand gives a sum of $-\frac{11}{16} + \frac{5}{8} + 0 = -\frac{1}{16}$, and in the third yields $-\frac{11}{16} + \frac{3}{4} - \frac{1}{8} = -\frac{1}{16}$. Left loses both times.

5. Future Directions

Natural variants of FLIPPING COINS involve increasing the number of coins that can be flipped from two to three or more. A brief computer search suggests that the only version where the values are numbers is the game in which Left flips a subsequence of all 1s and Right a subsequence of 0s ended by a 1. We conjecture that a similar ordinal sum structure will arise in these variants. Other variants have values that include *switches, tinies, minies,* and other three-stop games. However, some variants, when the reduced canonical values are considered, only seem to consist of numbers and switches. A more thorough investigation should shed light on their structures.

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