MISÈRE DOMINEERING ON $2 \times N$ BOARDS

Aaron Dwyer[1]
School of Mathematics and Statistics, Carleton University, Canada
aarondwyer@cmail.carelton.ca

Rebecca Milley[2]
Computational Mathematics, Grenfell Campus, Memorial University, Canada
rmilley@grenfell.mun.ca

Michael Willette[3]
Department of Applied Mathematics, University of Waterloo, Canada
mwillett@uwaterloo.ca

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Abstract

Domineering is a well-studied tiling game in which one player places vertical dominoes and a second places horizontal dominoes, alternating turns until someone cannot place on their turn. Previous research has found game outcomes and values for certain rectangular boards under *normal play* (last move wins); however, nothing has been published about domineering under *misère play* (last move loses). We find optimal-play outcomes for all $2 \times n$ boards under misère play: these games are Right-win for $n \geqslant 12$. We also present algebraic results including sums, inverses, and comparisons in misère domineering.

1. Introduction

The game of domineering has two players alternately placing dominoes to tile a checkerboard or any other grid. The player called *Left* can only place dominoes in a vertical orientation, and the player called *Right* can only place horizontally. Domineering is a *combinatorial game* because there is perfect information and no chance, and it is *partizan* (as opposed to *impartial*) because the two players have different move options. In normal-play combinatorial games, the first player unable

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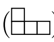
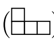
to move on their turn loses; under misère play, the first player unable to move is the winner. This paper considers domineering under misère play.

A game G is defined by the sets of *Left options* and *Right options* that the corresponding player can reach with a single move. We use $G_{m\times n}$ to denote a game of domineering on an empty $m \times n$ board. So, for example, $G_{2\times 2}$ has one Left option to $G_{2\times 1}$ and one Right option to $G_{1\times 2}$.

Given any game position G, the *outcome* $\mathrm{o}(G)$ is the winner under optimal play. There are four possibilities:

$$\mathrm{o}(G) = \begin{cases} \mathcal{L}, & \text{if Left wins } G \text{ whether she goes first or second;} \\ \mathcal{R}, & \text{if Right wins } G \text{ whether he goes first or second;} \\ \mathcal{N}, & \text{if the next player to move in } G \text{ wins;} \\ \mathcal{P}, & \text{if the previous player (i.e. not the next player) wins.} \end{cases}$$

We use $\mathrm{o}^-(G)$ to denote the outcome of G under misère play and $\mathrm{o}^+(G)$ for the outcome under normal play. For example, $\mathrm{o}^-($$) = \mathcal{N}$ and $\mathrm{o}^+($$) = \mathcal{R}$. The *zero game*, in which there are no moves for either player (e.g., a 1×1 board in domineering), has $\mathrm{o}^-(0) = \mathcal{N}$ and $\mathrm{o}^+(0) = \mathcal{P}$. The *negative* of a game G, denoted $-G$, is the game G with the roles of Left and Right swapped; in domineering, this is equivalent to rotating G by 90 degrees. The *disjunctive sum* of two games G and H is the game $G + H$ in which, on their turn, a player can choose to play in G or in H. In domineering, as players place pieces, a single connected board often breaks into a disjunctive sum of disjoint boards: for example, if Left plays in the third column of $G_{2\times 6}$, the new position is $G_{2\times 2} + G_{2\times 3}$.

Two games G and H are *equal* if they can be interchanged in any sum without affecting the outcome: that is, if $\mathrm{o}(G + X) = \mathrm{o}(H + X)$ for any sum of games X. Inequality is defined by $G \geqslant H$ if $\mathrm{o}(G+X) \geqslant \mathrm{o}(H+X)$ for all X, where outcomes are ordered according to preference by Left: $\mathcal{L} > \mathcal{N} > \mathcal{R}$ and $\mathcal{L} > \mathcal{P} > \mathcal{R}$, with $\mathcal{N}$ and $\mathcal{P}$ incomparable. Equality and inequality are dependent on the ending condition; games can be equal or comparable in normal play but not in misère play, etc. In normal play, $G + (-G) = 0$ for all games G.

Normal-play domineering has been the subject of numerous papers by mathematicians and computer scientists. Elwyn Berlekamp found normal-play outcomes and values for positions in $2 \times n$ and $3 \times n$ domineering in his 1988 paper [2]. Since that time, computer programs have been developed to find the normal-play outcome of rectangular boards: up to 9×9 was solved by the computer program developed in [3]; this was extended to 10×10 by [4], and finally to 11×11 by [10] . In [5], theoretical and computational techniques were used to determine outcomes of all $2 \times n$ boards under normal play: for $n \geqslant 28$, the boards are all Right-win.

What about misère play? The primary purpose of this paper is to find outcomes of all $2 \times n$ games of domineering under misère play. In general, misère play is much less studied; although the standard definitions of addition, negation, equality, and inequality can be applied, there are many problems with the algebra. For example, if $G \neq 0$, then G and $-G$ never sum to zero in general misère play [7], and even in *restricted* play (see Section 3), most games are not invertible. Another problem is that knowing the misère outcome of two games gives no information about the misère outcome of their sum [7]; in Section 3.1, we show that this property is true even when restricted to domineering positions. For these and other reasons, it is much more difficult to analyze misère games using the usual game theoretic techniques. Indeed, our solution for $2 \times n$ boards is purely combinatorial.

The remainder of the paper is structured as follows. Section 2 presents the solution for $2 \times n$ domineering. Section 3 considers a number of algebraic properties of misere domineering, including outcomes of sums (3.1), invertibility (3.2), and comparisons (3.3) of certain $2 \times n$ positions. Section 4 gives a summary and further discussion.

2. Misère Outcomes of $2 \times n$ Domineering

Let kG denote a disjunctive sum of k copies of the same position G. Consider the following two games:

$2G_{2\times 2}$ $G_{2\times 4}$

Note that in normal-play, $\boxplus$ is its own additive inverse and $\boxplus + \boxplus = 0$; in misère play, even restricting only to domineering positions, $\boxplus + \boxplus$ is not zero.[4] We claim that Right will always prefer the 2×4 board[5]. The intuition is this: if Right has a good strategy on $2G_{2\times 2}$, then when playing on $G_{2\times 4}$, Right can just pretend the board has been sliced down the middle, and follow his good strategy on $2G_{2\times 2}$. So Right will do at least as well on the 2×4 board as on two disjoint 2×2 boards. The intuition generalizes to more than two copies of $G_{2\times 2}$, but note that it is not obvious or immediate: who is to say that Left cannot force Right to play across the imaginary boundaries? We will show that Right can control the game in this way, when desired. To see when that might be, we first determine the misère outcome of multiple copies of $G_{2\times 2}$.

Lemma 1. *The misère outcome of $(2k)G_{2\times 2}$ is next-win, and the misère outcome of $(2k+1)G_{2\times 2}$ is previous-win.*

[4]To see $\boxplus + \boxplus \neq 0$ in misère play, we need a 'distinguishing' game X with $o(\boxplus + \boxplus + X) \neq o(0 + X)$. Let $X = 2G_{2\times 1}$. The misère outcome of $\boxplus + \boxplus + \boxminus + \boxminus$ is $\mathcal{P}$, while the misère outcome of $\boxminus + \boxminus$ is $\mathcal{R}$.

[5]'Right prefers $G_{2\times 4}$ over $G_{2\times 2} + G_{2\times 2}$' is equivalent to the inequality $G_{2\times 2} + G_{2\times 2} \geqslant G_{2\times 4}$. We will show this is true (modulo a restricted set of games) in Section 3.3, using a result from [6].

Proof. We show winning strategies for Right, and the strategies for Left follow by symmetry. Right playing first on an even sum of 2×2 boards should use his first k moves to 'claim' half of the boards, placing one piece in each of k different boards. Left cannot prevent this. Right should use the next k moves to play a second piece in each of those boards (i.e., Right plays in all the ▭ positions he just created). In total, Right places $2k$ pieces. During this time, there are exactly $2k$ moves available for Left among the other k boards. Left as the second player will get the last move, and so Right wins.

The same strategy works for Right playing second on an odd sum of 2×2 boards: this time, after Left and Right have each made $2k$ moves, there is an extra 2×2 board remaining (or possibly two 2×1 boards), and it is Left's turn next. Left is forced to move to ⊟, and Right wins. □

We now analyze $2 \times n$ boards. A program was written in Python to determine the outcome of $m \times n$ domineering boards under misère play (see the Appendix for other computational results with $m > 2$). The following outcomes were determined by hand and confirmed computationally:

n	0	1	2	3	4	5	6	7	8	9	10	11
$o^-(G_{2\times n})$	$\mathcal{N}$	$\mathcal{R}$	$\mathcal{P}$	$\mathcal{L}$	$\mathcal{N}$	$\mathcal{R}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{N}$

These initial cases do not indicate a pattern in the outcomes; fortunately, the next 12 (solved computationally) do:

n	12	13	14	15	16	17	18	19	20	21	22	23
$o^-(G_{2\times n})$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$

As in normal play, Right appears to have the advantage in a $2 \times n$ domineering board, for large enough n. Indeed, we will now show that for $n \geqslant 12$, all $2 \times n$ boards are Right-win. (Interestingly, in normal play, other outcomes are possible until $n \geqslant 28$.) The strategy for Right depends on the congruency of n modulo 4, and so we prove the result across four separate theorems (Theorems 1, 2, 3, 4).

To begin, we define some standards moves for Right in $2 \times n$ domineering (see Figure 1). Two Right pieces are *adjacent* if they are in the same row and occupy consecutive columns, *stacked* if they are in the same two columns of different rows, and *staggered* if they are in different rows and share exactly one column. We say "Right makes a stacked move" to mean Right places a piece that creates a pair of stacked pieces. In some of the strategies described below, Right places two adjacent pieces in order to guarantee that he can make a staggered move later in the game. We must show that Left cannot prevent Right from placing one or two pairs of adjacent pieces, as needed, as long as n is sufficiently large; this is done in Lemma 2.

Lemma 2. *In a game of domineering on an empty $2 \times n$ board:*

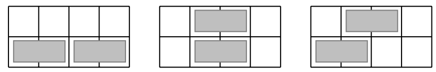


Figure 1: Adjacent, stacked, and staggered Right moves.

(i) Right moving first can place his first two pieces adjacent, if $n \geqslant 6$.

(ii) Right moving second can place his first two pieces adjacent, if $n \geqslant 12$.

(iii) Right moving first can place his first four pieces as two disconnected pairs of adjacent pieces, if $n \geqslant 19$.

(iv) Right moving second can place his first four pieces as two disconnected pairs of adjacent pieces, if $n \geqslant 24$.

Proof.

(i) If $n \geqslant 6$ then Right moving first can play in the middle of an empty 2×6 section of the board. Left can only reply on one side or the other, and then Right's second piece can be placed on the opposite side, adjacent to his first piece. See Figure 2.

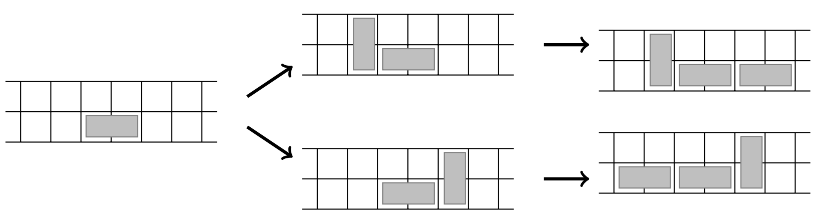


Figure 2: Right playing first in a 2×6 can place two pieces adjacent.

(ii) If $n \geqslant 12$ and Left plays first, then there is an empty 2×6 section on one side or the other of Left's first piece. By (i), Right can place two pieces adjacent in that section.

(iii) If $n \geqslant 19$ then Right's first move should be in the middle of the first 2×6 section of the board (i.e., across columns 3 and 4).

If Left's first move is within that 2×6 section, Right should immediately place the adjacent piece as in (i). Now, there is still at least an empty 2×12 section of the board starting after column 7, and so Right playing second from here can place another two adjacent pieces after column 7, by (ii). Note that Right may have to avoid the 7th column to ensure the pairs of adjacent pieces are not connected.

If Left's first move is not in the original 2×6 section, but rather somewhere in columns 7 to 19, then as in (ii), Right can play in an empty 2×6 section within the last 12 columns. Right can place his third and fourth pieces adjacent to his first and second pieces (or in the other order, if threatened by Left).

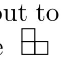
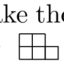
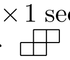
(iv) If $n \geqslant 24$ then Right moving second can place a piece in the middle of the first 2×6 section of the board, assuming (without loss of generality) that Left placed her first piece in the second half of the board.

If Left replies within that section, Right will place his adjacent piece as in (i). Left then makes a third move, with at most two in the latter $24 - 7 = 17$ columns of the board (Right will avoid the 7th column to ensure the pairs of adjacent pieces are not connected). At most two Left moves in a 2×17 section will necessarily leave an empty 2×6 section, with Right to move next, so Right can place another two adjacent pieces by (i).

If Left does not reply in the first 2×6 section, then after her second move, Left has placed two pieces in the rightmost 2×18 section of the board; this still guarantees an empty 2×6 section in the rightmost 2×17 section (avoiding column 7), in which Right can place his second piece. As in (iii), Right can place his third and fourth pieces adjacent to his first and second (or second and first, if necessary). □

As noted above, Right's strategy for $G_{2 \times n}$ will depend on the congruency of n modulo 4. In several cases, the strategy will lead to a position of the form shown in Figure 3: a $2 \times n$ position whose empty squares consist of an equal number of 2×1 and 1×4 sections, where the 1×4 sections are not adjacent to each other (but may be connected by one or more of the 2×1 sections). Lemma 3 shows that Right can always win these particular end-game positions.

Lemma 3. *If the empty squares in a $2 \times n$ domineering position consist of an equal number of 2×1 and 1×4 sections, with no two 1×4 sections adjacent, then Right can win this position playing first.*

Proof. Right should play in the middle of each 1×4 section; meanwhile, Left has no choice but to take the 2×1 sections one at a time. Right may temporarily create a piece like  or  or , but Left will play in the 2×1 section(s) of those before Right runs out of 1×4 middle moves, because there are the same number of (2×1)s as (1×4)s. When Left takes the last 2×1, there are no moves remaining and Right wins. □

We are now ready for the main results.

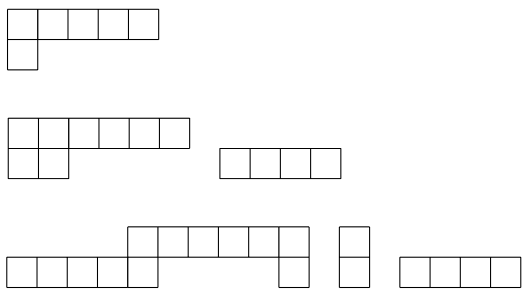


Figure 3: Positions in $2 \times n$ domineering whose empty squares consist of non-adjacent 1×4 pieces and the same number of (possibly connected or connecting) 2×1 pieces.

Theorem 1. *If $n \equiv 0 \pmod 4$ and $n \geqslant 12$ then a $2 \times n$ domineering board is Right-win under misère play. That is,*

$$o^{-}(G_{2\times 4k}) = \mathcal{R}, \text{ for } n = 4k \geqslant 12.$$

Proof. Assume $n = 4k \geqslant 12$. If Right plays first on a $2 \times n$ board, he should pretend that the board is cut into 2×2 pieces and play the winning next-player strategy (as per Lemma 1). Right can do this by first placing k pieces anywhere along the bottom row, effectively claiming k 2×2 boards, and then playing directly above those k pieces. Left cannot prevent Right from making these stacked moves. With each of the first k Right-Left moves, three bottom-row spaces are taken, so that after Left's k^{th} move, exactly k of the $4k$ columns remain empty. Left will be forced to take all k of these spaces as Right plays his k stacked moves in the top row, and since Right went first, Right will run out of moves first.

Right playing second is not as straightforward; Right should not play as if the board were cut into $(2k)G_{2\times 2}$ because that is a next-win position. Right must change the parity using a staggered move. To set himself up for a staggered move at the end of the game, Right will place two pieces adjacent, which we know he can do by Lemma 2 (ii). Here is Right's strategy: place two adjacent pieces in the bottom row and then place $k-2$ more bottom pieces, for a total of k bottom pieces, as before. Since Left went first, after Right's k^{th} move there are $4k - 3k = k$ empty columns. Now Left has to begin taking those empty columns. Right plays $k - 2$ stacked moves above all but his first two pieces, and after that there are two empty columns remaining, as well as an empty 1×4 section above Right's first two pieces. It is Left's turn: she takes one column, leaving exactly one 1×4 and one 2×1, possibly connected. By Lemma 3, Right wins from here with a staggered move. □

We see for $n \equiv 0 \pmod 4$ that Right playing first is 'easy' and involves only stacked moves for Right, while Right playing second requires Right to break parity using a staggered move. We will see the same situation (but vice versa) for $n \equiv 2 \pmod 4$. The hardest case is $n \equiv 3 \pmod 4$, where staggered moves are required for Right going first and second. It turns out that $n \equiv 1 \pmod 4$ is the simplest case: Right only ever needs to place stacked pieces, going first or second.

Theorem 2. *If $n \equiv 1 \pmod 4$ then a $2 \times n$ domineering board is Right-win under misère play. That is,*

$$o^-(G_{2\times 4k+1}) = \mathcal{R}, \text{ for } n = 4k+1.$$

Proof. The case $n = 1$ is clear. For larger $n = 4k+1$, Right playing first or second should follow the 'cut up' strategy from the $4k$ case: that is, Right should place his first k pieces in the bottom row and then place k stacked pieces. After each player has made $2k$ moves, Right has occupied $2k$ columns and Left has occupied $2k$ columns, leaving exactly one column empty. If it is Right's turn next, he has no move and wins; if it is Left's turn next, she takes the last empty column and then Right wins. □

Theorem 3. *If $n \equiv 2 \pmod 4$ and $n \geqslant 22$, then a $2 \times n$ domineering board is Right-win under misère play. That is,*

$$o^-(G_{2\times 4k+2}) = \mathcal{R}, \text{ for } n = 4k+2 \geqslant 22.$$

Proof. For $n = 4k+2 \geqslant 22$, Right playing second should place k pieces in the bottom row followed by k stacked moves in the top row. After Right's $(2k)^{\text{th}}$ move, each player has taken $2k$ columns, so that only 2 columns remain, with Left to move. The columns could be adjacent, forming a 2×2 square, or not; either way, Left moving next loses.

Recall that the first player in an odd sum of 2×2 boards not only loses, but loses with another move to spare; e.g., if Right playing first here places only stacked pieces, then Left will run out of moves and there will still be another 1×2 position remaining. So to prevent Left from winning, it will not be enough to make a single staggered move as in the $4k$ case; Right will have to arrange to make two staggered moves to force Left into the last move. Right should use his first four moves to place two pairs of adjacent pieces in the bottom row, not all adjacent, as per Lemma 2 (iii). Right should then place another $k-3$ pieces in the bottom row, for a total of $k+1$ moves (across $2k+2$ columns), and then place $k-3$ stacked pieces above the latter bottom moves. In this time, Left has taken $2k-2$ columns, so that two empty columns remain, along with two empty 1×4 sections above Right's first four moves. By Lemma 3, Right wins playing first from here with two staggered moves. □

Theorem 4. *If $n \equiv 3 \pmod 4$ and $n \geqslant 27$, then a $2 \times n$ domineering board is Right-win under misère play. That is,*

$$o^-(G_{2\times 4k+3}) = \mathcal{R}, \text{ for } n = 4k+3 \geqslant 27.$$

Proof. Assume $n = 4k + 3 \geqslant 27$. Right playing first will aim to set himself up to make a staggered move at the end of the game. Since $n > 6$, Lemma 2 (i) tells us that Right can place his first two pieces adjacent in the bottom row. Right should then place another $k - 1$ pieces in the bottom row, for a total of $k + 1$ moves; after these $k + 1$ Right–Left moves, $3(k + 1)$ spaces have been taken in the bottom row, leaving $(4k + 3) - (3k + 3) = k$ empty columns. Next, Right makes $k - 1$ stacked moves above his latter $k-1$ bottom moves, while Left places in $k-1$ columns, leaving exactly one empty column, along with one empty 1×4 section above Right's first two pieces. By Lemma 3, Right wins playing first from here with a staggered move.

Right playing second will set himself up to make two staggered moves at the end of the game. By Lemma 2 (iv), Right playing second with $n \geqslant 27$ can use his first four moves to place two pairs of adjacent pieces, not all adjacent, in the bottom row. Right should then make another $k - 3$ moves in the bottom row, for a total of $k + 1$; after these $k + 1$ Left–Right moves, there are $(4k + 3) - 3(k + 1) = k$ empty columns. Now Right makes $k - 3$ stacked moves while Left takes $k - 3$ columns, leaving three empty columns, along with two empty 1×4 sections above Right's first four moves, with left to move. Left has to take one of the empty columns, which leaves two empty columns and two empty 1×4s. By Lemma 3, Right wins from here with two consecutive staggered moves. □

With Theorems 1–4 and the base cases ($n = 14, 15, 18, 19, 23$) obtained computationally, we have the following main result.

Corollary 1. *If $n \geqslant 12$ then a $2 \times n$ domineering board is Right-win under misère play:*

$$o^-(G_{2\times n}) = \mathcal{R}, \text{ for } n \geqslant 12.$$

3. The Algebra of Misère Domineering

3.1. Sums

In normal play, outcomes of sums are somewhat dictated by the outcomes of the summands: for example, the sum of two Left-win games is always Left-win, the sum of a next-win and a Left-win is either Left-win or next-win, and more. An interesting and unfortunate fact about misère play, first proven by [7], is that the outcome of a sum of two games is completely independent from the outcomes of each game: for any outcomes $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3 \in \{\mathcal{P}, \mathcal{N}, \mathcal{L}, \mathcal{R}\}$, there are games G, H such that

$$o^-(G) = \mathcal{O}_1, o^-(H) = \mathcal{O}_2, \text{ and } o^-(G + H) = \mathcal{O}_3.$$

We have found that this property of misère games holds even if restricted to domineering; in fact, our examples (given in Table 1) are restricted to domineering positions that fit within $2 \times n$ and $n \times 2$ boards.

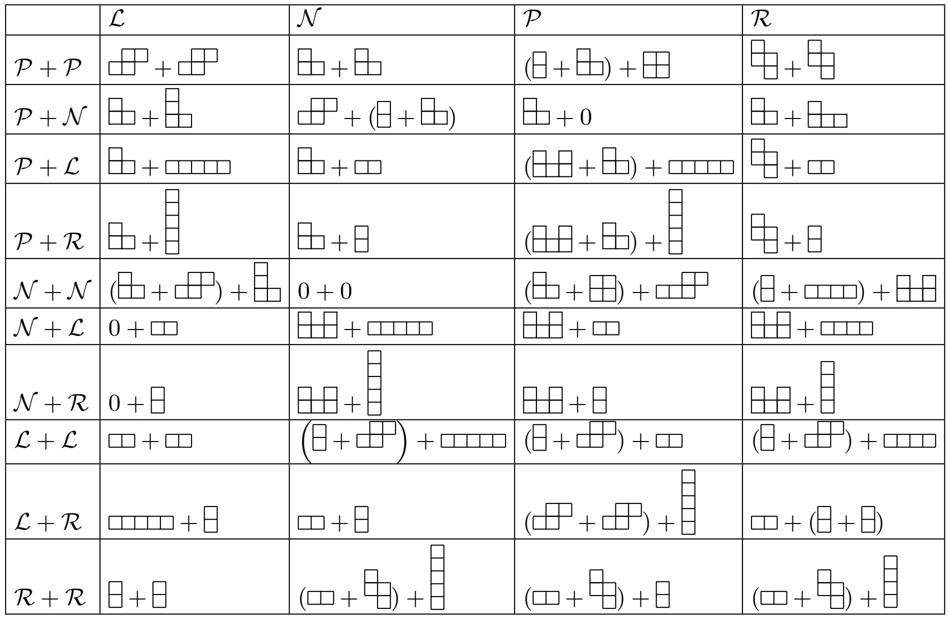


Table 1: Outcomes of sums of domineering positions, demonstrating the lawless addition of misère play.

3.2. Invertibility

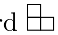
For this and the next subsection, we need to define equivalence and inequality in *restricted* game play. Two games are *equivalent* modulo $\mathcal{U}$ for a set (*universe*) of games $\mathcal{U}$ if they can be interchanged in any sum of games from $\mathcal{U}$ without affecting the outcome:

$$G \equiv_{\mathcal{U}} H \text{ if for all } X \in \mathcal{U},\ \mathrm{o}(G+X) = \mathrm{o}(H+X)[9].$$

Note that this equivalence relation is weaker than the usual equality of games, for which $\mathcal{U}$ is taken to be any sum of game positions.

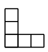
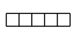
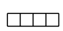
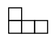
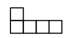
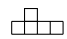
In general misère play (i.e., when $\mathcal{U}$ is the set of all games), $G+(-G)$ is not equal to zero for any nonzero G [7], but games may be invertible modulo restricted universes. Let $\mathcal{E}$ be the universe of *dead-ending* games, defined by the following property: if a player is currently unable to move in a position, then they are never subsequently able to move in that position, even after play by the opponent. For example, John Conway's game *Hackenbush* is dead-ending, while Richard Guy's *Toads and Frogs* is not[6]. Domineering is dead-ending. From [8], we know all *ends*

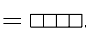
[6]The position [ ] [T] [F] [F] has no available move for Left (Toad), but if Right (Frog) jumps, Left will have a move.

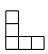
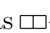
(games in which at least one player has no move) are invertible modulo $\mathcal{E}$; therefore, all $1 \times n$ domineering boards are invertible. However, most positions are not invertible, even modulo $\mathcal{E}$. For example, the game $*$, which occurs as the board  in domineering, is not invertible modulo $\mathcal{E}$ [7].

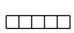
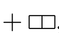
It is an open question to classify all invertible dead-ending positions. We have found the positions given in Theorem 5 to be the only modulo-$\mathcal{E}$ invertible domineering boards with game trees of depth 2 (i.e., games of *rank* 2). This was determined computationally using a recursive test from [6] to check $G + (-G)$ for equivalence to zero modulo $\mathcal{E}$, but we prove the invertibility here directly, with the definition of equivalence.

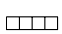
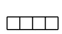
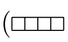
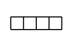
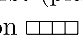
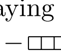
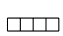
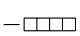
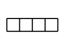
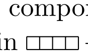
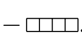
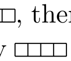
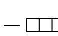
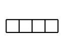
Theorem 5. *If G is a domineering position of rank 2 and $G + (-G) \equiv_{\mathcal{E}} 0$, then G is one of the following boards or their negatives:*

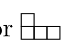
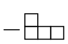
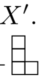
(1)  (2)  (3)  (4)  (5)  (6) 

Proof. We will show each of these positions satisfies $G + (-G) \equiv_{\mathcal{E}} 0$; i.e., that $o(G + -G + X) = o(X)$ for all $X \in \mathcal{E}$. For all other rank-2 domineering boards G, the position $G + (-G)$ can be distinguished from 0 with $X =$  or $X =$ .

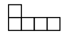
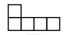
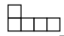
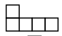
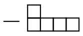
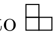
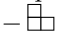
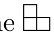
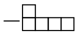
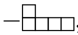
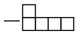
(1) The position  has the same game tree as +, and so is actually equivalent to zero modulo $\mathcal{E}$.

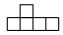
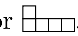
(2) The position  has the same game tree as  + , which we know to be invertible as it is an end.

(3) To show  −  $\equiv_{\mathcal{E}} 0$, we will show o( −  + X) = o(X) for all dead-ending games X. Suppose Left wins X playing first (playing second follows analgously). Left should follow the same strategy on  −  + X; if Right plays in the  −  component, Left can reply with the inverse, as all options of  are invertible, bringing that component to zero. Left then resumes winning on X. If Right does not play in  − , then when Left runs out of moves in X, say at a left end X', she should play  −  + X' to  + X', leaving a position with no Left moves and at least one Right move. By the definition of dead-ending games, Left has no further moves, and so wins.

(4) Because all options of  are invertible, the proof for  −  $\equiv_{\mathcal{E}} 0$ is almost identical to the proof for (3). The only additional consideration is when Left runs out of moves in X, say at X'. At that point, Left should play in the , bringing the position to  +  + X'. From here, Right has at least two moves and Left has only one, so Left will win.

[7] The game $*$ is not invertible in any universe containing the game '1' ( in domineering).[1]

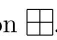
(5) The proof for  −  $\equiv_{\mathcal{E}} 0$ is similar, except that Right could play in the  −  $+ X$ to  −  $+ X$. Left cannot just play to  −  $+ X$, because  −  is not equivalent to zero. Instead, Left should take the  and leave − $+ X$. If Right plays in −, Left can can bring that position to zero and resume winning on X; otherwise, Left runs out of moves in X and plays − to , leaving a left end with at least one move for Right.

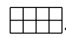
(6) The proof for  is nearly identical to the proof for . □

3.3. Comparisons

Inequality modulo $\mathcal{U}$ is defined by

$$G >_{\mathcal{U}} H \text{ if for all } X \in \mathcal{U}, \text{o}(G + X) > \text{o}(H + X).$$

Comparability is much less common and much harder to prove in misère play than in normal play. Even among just domineering positions, we cannot say that Left would always prefer the zero game to the position  — there are situations in which Left would rather have an extra move than not, including when playing first on .

In Section 2 we claimed that  +  $\geqslant_{\mathcal{E}}$ . We give the justification here. It requires a series of inequalities that build upon each other and the *hand-tying principle*. The hand-tying principle says that if G and H have identical Right options, and the Left options of H are a nonempty[8] subset of the Left options of G, then $G \geqslant H$. This is true because if Left has a good move in H, then that same move is available in G. Similarly, if G and H have identical Left options and the Right options of G are a nonempty subset of the Right options of H, then $G \geqslant H$, because Right prefers H.

The inequalities in Proposition 1 follow from Theorem 6, a weaker version of the 3-part comparison test for $\mathcal{E}$ proven in [6]. Let G^L (G^R) denote a single Left (Right) option of G.

Theorem 6. *If $G, H \in \mathcal{E}$ and*

(1) for every G^R there is an H^R such that $H^R \leqslant_{\mathcal{E}} G^R$, and

(2) for every H^L there is a G^L such that $G^L \geqslant_{\mathcal{E}} H^L$,

then $G \geqslant_{\mathcal{E}} H$.

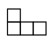
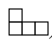
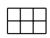
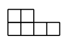
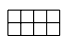
Proof. Let $X \in \mathcal{E}$ and assume Right wins $G+X$. We must show Right wins $H+X$. Right should follow his strategy for $G + X$. If at some follower $H + X'$ the good Right move in $G+X'$ would be G^R+X', then there is an $H^R \leqslant_{\mathcal{E}} G^R$, so Right will do just as well playing to $H^R + X'$. Otherwise, if Right does not move in the H

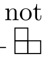
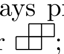
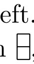
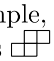
[8]This is required in misère play; in normal play, inequality holds even if this set is empty.

component first, then at some point Left moves to $H^L + X'$. But for this H^L there is a $G^L \geqslant_{\mathcal{E}} H^L$, so $H^L + X'$ is better for Right than $G^L + X'$, and Right would have a winning reply to any such $G^L + X'$. So Right wins from $H^L + X'$. □

The following inequalities can now be verified using the hand-tying principle, Theorem 6, and earlier inequalities.

Proposition 1. *The following comparisons are true modulo $\mathcal{E}$, the set of dead-ending games.*

1.  +  $\geqslant_{\mathcal{E}}$  *(i.e.,* $0 \geqslant_{\mathcal{E}}$ *)*
2.  +  $\geqslant_{\mathcal{E}}$ 
3.  +  $\geqslant_{\mathcal{E}}$ 
4.  +  $\geqslant_{\mathcal{E}}$ 

Note that 'splitting' a board does not always produce a better game for Left. For example, Left does not prefer  +  over ; playing each in a sum with , Left likes  better.

4. Summary and Discussion

In this paper we have shown that all $2 \times n$ domineering boards are Right-win under misère play, after $n \geqslant 12$. We have also found some interesting properties for misère domineering more generally: e.g., even among only $2 \times n$ and $n \times 2$ positions, there is no predictability about the outcome of $G + H$ based on the outcomes of G and H. We have identified the invertible rank-2 domineering boards and have proven some inequalities among $2 \times n$ positions, all modulo dead-ending games.

The strategy described for $2 \times n$ boards has Right placing two adjacent pieces at the start of the game, in order to guarantee that he can make a staggered move at the end of the game. We suspect it is also possible (and more efficient) for Right to make the staggered move(s) right away. This would reduce the lower bounds for n, which come directly from Right needing room to place one or two pairs of adjacent pieces (Lemma 2). This strategy seems to work for small boards that we have tried by hand: Right can either box in the 2×3 section containing his staggered moves or can force Left to do so, which makes the board effectively one column shorter, or if Left prevents this, then we can still find a way for Right to force the win. However, we could not find a general argument to show Right can always win with this strategy.

For larger rectangular $m \times n$ boards with $n > m$, our intuition is that the outcome will always skew in favour of Right, if n is sufficiently larger than m. Interestingly, the computational results for $m \times n$ boards in Appendix A, Table 2, suggest that $3 \times n$ boards become Right-win even sooner than $2 \times n$ boards; however, there may be other outcomes for $m = 3, n \geqslant 12$ that we have not observed.

The next steps for this work would be to determine outcomes for $3 \times n$ and larger boards, and to consider other interactions, such as comparisons and outcomes of sums, among small (say rank 2 or 3) domineering positions. For the latter, continued advancements in the larger universe of dead-ending games — e.g., determining which games are invertible — may provide interesting insights into the algebra of misère domineering.

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[4] N. Bullock, Domineering: Solving large combinatorial search spaces, *ICGA J.* **25** (2002), 67–84.

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[6] U. Larsson, R. Milley, R.J. Nowakowski, G. Renault, and C.P. Santos, Recursive comparison tests for dicots and dead-ending games under misère play, *Integers*, to appear.

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[10] J.W.H.M. Uiterwirjk, 11×11 domineering is solved: the first player wins, in A. Plaat, W. Kosters, J. van den Herik (Eds.) *Computers and Games*, Springer, Leiden (2016), 129–136.

Appendix

Using our (modest) domineering program, we have the following outcomes for $m \times n$ domineering boards under misère play.

$m \backslash n$	2	3	4	5	6	7	8	9	10	11	$\geqslant 12$
2	$\mathcal{P}$	$\mathcal{L}$	$\mathcal{N}$	$\mathcal{R}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{N}$	$\mathcal{R}$
3	$\mathcal{R}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{L}$	$\mathcal{N}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	?
4	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{R}$	?	?	?
5	$\mathcal{L}$	$\mathcal{R}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{R}$	$\mathcal{N}$	?	?	?	?	?
6	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{L}$	$\mathcal{N}$	?	?	?	?	?	?

Table 2: Misère outcomes for $m \times n$ domineering.