RELATOR GAMES ON GROUPS

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Received: 12/24/20, Revised: 2/26/21, Accepted: 6/1/21, Published: 12/20/21

Abstract

We define two impartial games, the *Relator Achievement Game* `REL` and the *Relator Avoidance Game* `RAV`. Given a finite group G and generating set S, both games begin with the empty word. Two players form a word in S by alternately appending an element from $S \cup S^{-1}$ at each turn. The first player to form a word equivalent in G to a previous word wins the game `REL` but loses the game `RAV`. Alternatively, one can think of `REL` and `RAV` as *make a cycle* and *avoid a cycle* games on the Cayley graph $\Gamma(G, S)$. We determine winning strategies for several families of finite groups including dihedral, dicyclic, and products of cyclic groups.

1. Introduction

In this paper we define two 2-player combinatorial games: the *Relator Achievement Game* `REL` and the *Relator Avoidance Game* `RAV`. Given a finite group G and generating set S, two players take turns choosing s or s^{-1}, where s is a generator from S. The only stipulation is that, if the previous player chose s, the next player cannot choose s^{-1} and vice versa. The players' choice of group elements builds a word in S. The goal of `REL` is to be the first player to achieve a subword equivalent to the identity in G. The game of `RAV` is the misère version of `REL`, meaning the first player to achieve a subword equivalent to the identity loses the game. One can play these games on the Cayley graph of G formed by using the generating set S. Since paths in a Cayley graph correspond to words in S, the players' choices of generators form a path in the Cayley graph without backtracking. Hence, when viewed graphically, the goal of `REL` is to be the first player to make a cycle, whereas for `RAV` the goal is to avoid cycles.

One motivation for the development of the games REL and RAV originated from recent results by Ernst and Sieben in [8] and also with Benesh in [3, 4, 5, 6] for the combinatorial games GEN and DNG, which were first defined by Anderson and Harary in [2]. In these games, two players alternate choosing distinct elements from a finite group G until G is generated by the chosen elements. The first player to generate the group on their turn wins the game GEN but loses DNG. Taking inspiration from this work, our goal was to create a pair of games that incorporates the geometry of a group G through its Cayley graph.

We have found in the current literature several combinatorial games on graphs, including some on Cayley graphs. However, REL and RAV are distinct from these combinatorial games. For example, *Cops and Robbers* (see [13], [14]), a popular pursuit-evasion game, has been studied specifically on Cayley graphs (see [9]), and firefighting games have been studied on Cayley graphs as well (see [10]). More recently, the *Game of Cycles* was introduced by Su in [16] and expounded in [1] by Alvarado, et al. This game involves planar graphs and two players taking turns marking previously unmarked edges with a chosen direction. The *Game of Cycles* is the closest of these combinatorial games to REL and RAV, since the goal of the game is to create a cycle. However, the parameters for doing so are very different than in our game of REL.

This paper is structured as follows. In Section 2 we give a precise definition (Definition 2.1) of the games REL and RAV along with some examples and initial results concerning complete bipartite and complete Cayley graphs (see Theorem 2.5 and Theorem 2.6). In Section 3, we explore the family of dihedral groups D_n, $n \geq 3$, with its canonical generating sets. We show winning strategies for the game REL in Theorem 3.2 and for the game RAV in Corollary 3.10. Corollary 3.10 follows from the more general result in Theorem 3.9, which applies to any group with a generating set including an element of order 2. In Section 4, we explore the family of dicyclic groups with two common generating sets. These are results Theorem 4.1, Theorem 4.2, Theorem 4.3, and Theorem 4.4. In Section 5, we examine REL for products of cyclic groups $\mathbb{Z}_n \times \mathbb{Z}_m$, where the results depend on n modulo m (Theorem 5.1). In Section 6 we discuss two different n player versions of REL and prove winning strategies for three-player REL on the dihedral groups D_n in Theorem 6.1 and Theorem 6.3. Lastly, we conclude with some open questions in Section 7.

2. Two-Player Relator Games REL and RAV

Let G be a finite group and let S be a generating set for G (with $e \notin S$). We define two two-player impartial combinatorial games, called the *Relator Achievement Game* REL(G, S), and the *Relator Avoidance Game* RAV(G, S), as follows.

Definition 2.1. On turn 1, Player 1 begins with the empty word w_0. Player 1 chooses an element $s_1 \in S \cup S^{-1}$ to create the word $w_1 = w_0 s_1 = s_1$. The players then alternate choosing elements of $S \cup S^{-1}$. On turn n, with $n > 1$, the current player begins with a word

$$w_{n-1} = s_1 s_2 \dots s_{n-1}.$$

They then select a generator $s_n \in S \cup S^{-1}$ such that $s_n \neq s_{n-1}^{-1}$ and form the word w_n:

$$w_n = w_{n-1} s_n.$$

If a player forms w_n such that $w_n \equiv_G w_k$, that is, w_n and w_k represent the same element of G, for some k, $0 \leq k < n$, then that player wins REL(G, S) and loses RAV(G, S), respectively. If from any position there are no legal moves, then the next player loses. Otherwise, play passes to the next player and continues as described above.

Remark 2.2. When the group and generating set are clear from context, we will use the shorthand REL or RAV to refer to the Relator Achievement Game or the Relator Avoidance Game, respectively, for a group G and generating set S.

We forbid the trivial relator ss^{-1} in our games since every group contains these relators, and we are seeking non-trivial relators. We also assume in our definition that a generating set S does not contain the identity for similar reasons.

For the trivial group and the cyclic group of order 2, with their canonical generating sets, both games end due to the eventual absence of a legal move. These are, in fact, the only groups where this occurs.

Recall that the Cayley graph $\Gamma(G, S)$ for a group G and generating set S is a graph with vertices the elements of G and a directed edge from vertex g to vertex h if $h = gs$ for some $s \in S$. Such an edge would be labeled by s.

If we consider a path of edges in the Cayley graph $\Gamma(G, S)$, this will correspond to a word $w = s_1 s_2 \dots s_{n-1} s_n$ in G with letters in $S \cup S^{-1}$. Therefore, one can visually play the games of REL and RAV on a Cayley graph: a players' choice of element $s_n \in S \cup S^{-1}$ will correspond to traversing an undirected edge in the Cayley graph. A player wins REL if they are the first to form a cycle (a relator) in the Cayley graph. Likewise, a player loses RAV if they are the first to form a cycle in the Cayley graph. The rule stating that a player may not choose the inverse of the last generator chosen translates to disallowing backtracking in the Cayley graph.

We mention this visual Cayley graph correspondence as a useful way to analyze the games REL and RAV. It can be helpful to play these games on a Cayley graph to understand the winning strategies for different groups and generating sets. Note that, due to how players choose elements from $S \cup S^{-1}$, whenever we discuss Cayley graphs, we refer to the undirected Cayley graph.

Example 2.3. Consider REL$(\mathbb{Z}_n, \{1\})$, where $\mathbb{Z}_n$ denotes the additive group of integers modulo n with $n > 2$. The corresponding Cayley graph for $(\mathbb{Z}_n, \{1\})$ is an

n-sided polygon. Hence, the games REL($\mathbb{Z}_n, \{1\}$) and RAV($\mathbb{Z}_n, \{1\}$) are completely determined by the parity of n. If n is even, then Player 2 will win REL and Player 1 will win RAV. If n is odd, then Player 1 will win REL and Player 2 will win RAV.

Example 2.4. Consider the quaternion group Q_8 with generating set $S = \{i, j\}$. We can investigate the game REL(Q_8, S) by means of the Cayley graph $\Gamma(Q_8, S)$. See Figure 1, where the labels i and j are denoted by blue and red, respectively. Note that $\Gamma(Q_8, S)$ is a complete, bipartite graph. That is, the vertices can be partitioned into two sets A and B such that for any two vertices $a \in A$ and $b \in B$, there is an edge joining a and b, and for any two elements from the same set, there is no edge between them. In this case, we have $A = \{\pm 1, \pm k\}$ and $B = \{\pm i, \pm j\}$. The set B is shaded in Figure 1.

To determine a winning strategy for REL(Q_8, S), note that Player 1 must choose from the set B on their first turn. Player 2 cannot backtrack to 1 and so must move to a vertex from $A - \{1\}$. Next, Player 1 moves to another vertex from B distinct from their previous choice and thus cannot win on this turn. Finally, Player 2 wins on their second turn by moving back to 1.

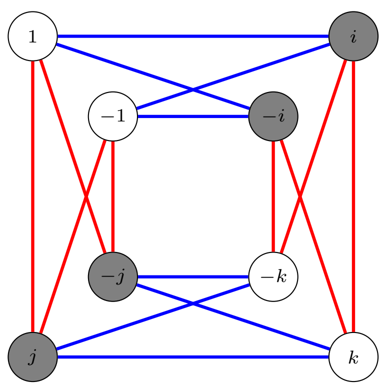

Figure 1: Cayley Graph for Q_8 with generating set $\{i, j\}$.

In general, if a group G contains an index 2 subgroup H and we let $S = G - H$, then $\Gamma(G, S)$ is complete bipartite. In this case, we have the following theorem:

Theorem 2.5. *If G is a finite group of order $2n$, with $n \geq 2$, and S is a generating set such that $\Gamma(G, S)$ is complete bipartite, then Player 2 wins REL(G, S) and Player 1 wins RAV(G, S).*

Proof. The proof that Player 2 wins REL(G, S) follows the same argument as in Example 2.4.

For RAV(G, S), the game is one of exhaustion. If $|G| = 2n$, then Player 1 has n possible vertices to move to on their first turn, while Player 2 has $n - 1$ options due to the game starting at the identity. In general, Player 1 has $n - k$ vertex options after their kth turn, while Player 2 has $n-k-1$ vertex options after their kth turn. These options always exist because Γ is complete bipartite. Hence, Player 2 will exhaust their options before Player 1, and thus Player 1 wins $\texttt{RAV}(G, S)$. □

For any non-trivial finite group G, if we let $S = G - \{e\}$, then $\Gamma(G, S)$ is a complete graph. Such a case is also easy to analyze.

Theorem 2.6. *If G is a finite group of order at least 3 and S is a generating set such that $\Gamma(G, S)$ is a complete graph, then Player 1 wins $\texttt{REL}(G, S)$. Player 1 wins $\texttt{RAV}(G, S)$ if $|G|$ is even and Player 2 wins $\texttt{RAV}(G, S)$ if $|G|$ is odd.*

Proof. If $\Gamma(G, S)$ is complete and $|G| \geq 3$, then Player 1 wins $\texttt{REL}(G, S)$ on their second turn by moving back to e since Player 2 may not backtrack to e on their first turn. $\texttt{RAV}(G, S)$ is a game of exhaustion as in the complete bipartite case. If $|G|$ is even, then Player 1 will complete a Hamiltonian path in $\Gamma(G, S)$ on turn $|G|-1$ and thus win $\texttt{RAV}(G, S)$ since Player 2 will have no available moves on the next turn. If $|G|$ is odd, then Player 2 wins by completing a Hamiltonian path for the same reason. □

While generating sets that yield complete bipartite or complete Cayley graphs allow for quick analysis of REL and RAV, they are rarely canonical generating sets for groups. In this sense, Q_8 is an outlier with its canonical generating set yielding a complete bipartite Cayley graph.

We close this section with an answer to a natural question. Suppose two groups G and H have isomorphic, undirected Cayley graphs, $\Gamma(G, S)$ and $\Gamma(H, T)$. Are the games of REL and RAV the same for both groups? The answer is yes. If a winning strategy dictates a player move along the edge from g to gs in $\Gamma(G, S)$, then the same player has a winning strategy on the other group by moving along the corresponding edge in $\Gamma(H, T)$. We state this explicitly as the following theorem.

Theorem 2.7. *Suppose $\Gamma(G, S)$ and $\Gamma(H, T)$ are isomorphic as undirected Cayley graphs. A player has a winning strategy for $\texttt{REL}(G, S)$ (respectively, $\texttt{RAV}(G, S)$), if and only if that player has a winning strategy for $\texttt{REL}(H, T)$ (respectively, $\texttt{RAV}(H, T)$).*

We provide an example of this result in Example 3.1 at the beginning of the next section.

3. Dihedral Groups

For the dihedral groups D_n of order $2n$, with $n \geq 3$, there are two common generating sets: one is the Coxeter generating set composed of two reflections; the other is composed of one reflection and one rotation. First we examine the Coxeter generating set.

Example 3.1. Suppose $S = \{s, t\}$ is a Coxeter generating set for the dihedral group D_n. That is,

$$D_n = \langle s, t \mid s^2 = t^2 = (st)^n = e \rangle.$$

In this case, the games $\texttt{REL}(D_n, S)$ and $\texttt{RAV}(D_n, S)$ have the same outcomes as $\texttt{REL}(\mathbb{Z}_{2n}, \{1\})$ and $\texttt{RAV}(\mathbb{Z}_{2n}, \{1\})$ since the undirected Cayley graphs $\Gamma(\mathbb{Z}_{2n}, \{1\})$ and $\Gamma(D_n, \{s, t\})$ are isomorphic (cf. Theorem 2.7).

Hence, we focus our attention for the rest of this paper on the following presentation for the dihedral groups:

$$D_n = \langle r, s \mid r^n = s^2 = rsrs = e \rangle.$$

3.1. $\texttt{REL}(D_n, \{r, s\})$

In this section, we investigate the Relator Achievement Game on D_n with generating set $\{r, s\}$. Note that each element of D_n can be written uniquely as $r^i s^j$, for some integers i and j with $0 \leq i \leq n - 1$ and $0 \leq j \leq 1$.

Theorem 3.2. *If n is odd, then Player 1 has a winning strategy for $\texttt{REL}(D_n, \{r, s\})$. If n is even, Player 1 has a winning strategy if $n \equiv 2 \mod 6$ while Player 2 has a winning strategy otherwise.*

Before we begin the proof, we provide some remarks and a lemma that will aid in the proof.

Remark 3.3. The Cayley graph $\Gamma(D_n, \{r, s\})$ contains n "squares", each corresponding to the relation $rsrs = e$. Given the normal form $r^i s^j$, where $0 \leq i \leq n-1$ and $0 \leq j \leq 1$, we number the squares in increasing order by i. Square 1 contains $\{e, s, r, rs\}$, Square n contains $\{r^{n-1}, r^{n-1}s, e, s\}$, and, in general, Square i contains $\{r^{i-1}, r^{i-1}s, r^i, r^i s\}$. See Figure 2 and Figure 3.

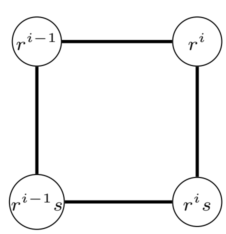

Figure 2: Square i in D_n

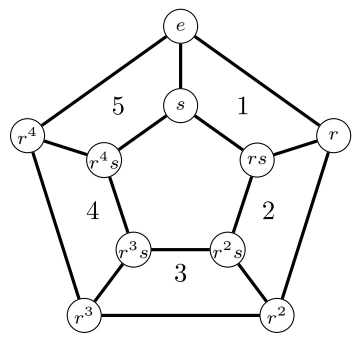

Figure 3: $\Gamma(D_5, \{r, s\})$ with squares 1 through 5

Remark 3.4. If two edges of a square have already been traversed, then neither player will move along a third edge of that square unless it is a winning play since traversing a third edge sets up the opposing player to win on their next turn.

Because of the previous two remarks, once the first r or r^{-1} edge is chosen, the players will move in one direction, clockwise or counter-clockwise, along the Cayley graph until a cycle is completed.

Remark 3.5. If a player chooses s, then the next two moves (if they exist) are both determined by Remark 3.4 and therefore must either both be r or both be r^{-1}.

We now introduce a definition that will be useful in the proof.

Definition 3.6. We say that a player *enters* Square i at vertex $g \in \{r^{i-1}, r^i, r^{i-1}s, r^i s\}$ on turn k if their choice of $s_k \in S \cup S^{-1}$ yields $w_k \equiv_G g$, and if $w_j \equiv_G h$ for some $j < k$, then $h \notin \{r^{i-1}, r^i, r^{i-1}s, r^i s\}$.

When the context is clear, we will state that a player has entered a square without referring to the specific turn.

The following lemma will be used in the proof of Theorem 3.2. Note that the appearance of the modulo 6 condition in Theorem 3.2 is due to this lemma.

Lemma 3.7. *For* REL$(D_n, \{r, s\})$, *suppose all moves have occurred on squares 1 through* $k-3$ *for some* k, *with* $4 \leq k \leq n$. *If* $5 \leq k \leq n$ *and a player enters square* $k-3$ *at the vertex* $r^{k-4}s$, *then that player can guarantee entering square* k *at vertex* r^{k-1}. *If* $4 \leq k \leq n$ *and a player enters square* $k-3$ *at vertex* r^{k-4}, *then that player can guarantee entering square* k *at vertex* $r^{k-1}s$.

Proof. We assume that all prior moves have occurred on squares 1 through $k-3$ and that $5 \leq k \leq n$. Let $A, B \in \{1, 2\}$ with $A \neq B$. Suppose that Player A enters square $k-3$ at the vertex $r^{k-4}s$, which must be done via a choice of r^{-1}. We then have two cases since Player B may either play r^{-1} to move to $r^{k-3}s$ or s to move to r^{k-4}.

If Player B chooses r^{-1}, then Player A will follow by choosing s to move to r^{k-3}. The next two moves are then forced by Remark 3.5 if both players are to avoid making the third edge on a square. Hence Player B will move to r^{k-2}, and Player A will move to r^{k-1}, entering square k at this vertex.

If Player B chooses s, then the next two moves are forced, so Player A moves to r^{k-3}, and Player B moves to r^{k-2}. Player A then has the option to choose r and enter square k at r^{k-1}.

Now suppose $4 \leq k \leq n$ and that Player A enters square $k-3$ at vertex r^{k-4}. For $k=4$, we note that the player enters at $r^0 = e$ on the 0th move of the game. For $k \geq 5$, this must be done by playing r. The rest of the proof is similar to the previous case. □

Remark 3.8. Allowing for repeated use of Lemma 3.7, we can effectively expand our options for moves to the set $\{r, r^{-1}, s, t(\alpha), u(\alpha)\}$ for $\alpha \geq 1$ instead of $\{r, r^{-1}, s\}$, where $r^i s^j t(\alpha) = r^{i+1+3\alpha} s^{j+\alpha}$ and $r^i s^j u(\alpha) = r^{i+3\alpha} s^{j+\alpha}$. Note that $u(\alpha)$ is a move available only to Player 2 from the $k=4$ case in Lemma 3.7 by entering Square 1 at $r^0 = e$ on their 0th move. We further note that by the proof of Lemma 3.7, the position prior to $r^{i+1+3\alpha} s^{j+\alpha}$ is $r^{i+3\alpha} s^{j+\alpha}$.

Proof of Theorem 3.2. First we suppose without loss of generality that r is played before r^{-1}. A consequence of this and Remark 3.4 is that i is nondecreasing in the normal form $r^i s^j$, where $0 \leq i \leq n-1$ and $0 \leq j \leq 1$. Suppose n is odd. Then Player 1's strategy is to play r from vertices r^k where $0 \leq k \leq n-1$ and r^{-1} from vertices $r^k s$ (see Figure 4 for an example). Since elements of $S \cup S^{-1}$ change the parity of exactly one of i and j in the normal form by exactly one, we note that Player 1 moves only to elements $r^a s^b$ where $a+b$ is odd while Player 2 moves only to elements $r^c s^d$ where $c+d$ is even. By Player 1's strategy, the power of r is strictly increasing after every two moves. Hence we will eventually move to $r^n s^0 = e$ or $r^n s^1 = s$. If the game moves to $r^n s^0 = e$, the game is over and Player 1 wins since $n+0$ is odd. If the game moves to $r^n s^1$, then Player 1 moves next since $n+1$ is even. Then Player 1 plays s to win at e.

Now suppose n is even. If Player 1 begins by playing r, then Player 2 wins by the same strategy as the case when n is odd. Hence we may assume that Player 1 begins by playing s. In this case, a player wins by moving to $r^n = e$ or $r^n s = s$. Repeated use of Lemma 3.7 allows us to expand our move set to $\{r, r^{-1}, s, t(\alpha), u(\alpha)\}$, with $t(\alpha)$ and $u(\alpha)$ as defined in Remark 3.8. We recall from Remark 3.8 that the position immediately prior to $r^{i+3\alpha} s^{j+\alpha}$ is $r^{i+3\alpha-1} s^{j+\alpha}$. We now split into three cases:

1. Suppose that $n = 6k+2$ for some k (see Figure 5). Player 1 first moves to s, and Player 2 is forced, without loss of generality, to move to rs. Player 1 can

now play $t(2k)$ to move from rs to $r^{1+1+3(2k)}s^{1+2k} = r^{6k+2}s = r^n s = s$ with the penultimate position being $r^{n-1}s$. Thus Player 1 wins.

2. Suppose that $n = 6k$ for some k. Player 2 effectively moves to $r^0s^0 = e$ on the 0th move, which is the start of Player 2 using $u(2k)$ to move to $r^{3(2k)}s^{2k} = r^{6k}s^0 = r^n = e$, with the penultimate position being r^{n-1}. Thus Player 2 wins.

3. Suppose that $n = 6k + 4$ for some k. We assume that Player 1 moves to s. Then Player 2 uses $t(2k + 1)$ to move from s to $r^{1+3(2k+1)}s^{1+(2k+1)} = r^{6k+4}s^{2k+2} = r^n = e$, with the penultimate position being r^{n-1}. Thus Player 2 wins.

□

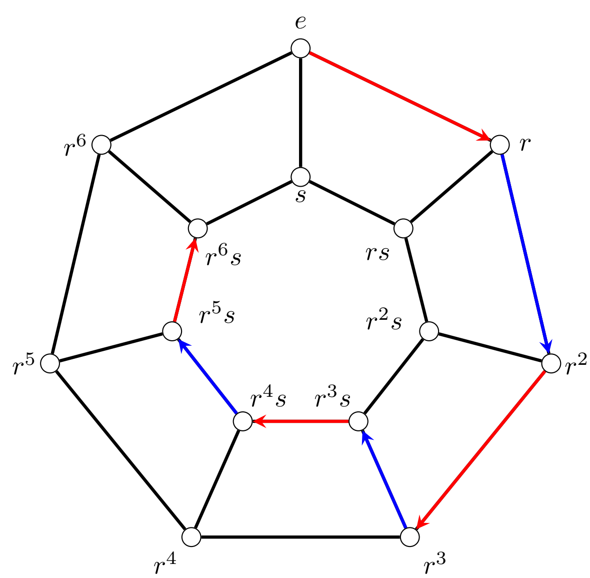

Figure 4: Example of Player 1 strategy for $\texttt{REL}(D_7)$ as described in Theorem 3.2. Player 1 moves are colored 'red' and Player 2 moves are colored 'blue'. Player 1 only needs to choose r or r^{-1} generators to reach a winning position. Note that when Player 2 chooses an s, the next two moves are forced by Remark 3.4. Regardless of Player 2's next move, Player 1 will win the game.

3.2. RAV for Groups with an Order Two Generator

In contrast with the achievement game for dihedral groups (Theorem 3.2), we have that Player 1 has a winning strategy for $\texttt{RAV}(D_n, \{r, s\})$ for any $n \geq 3$. This strategy

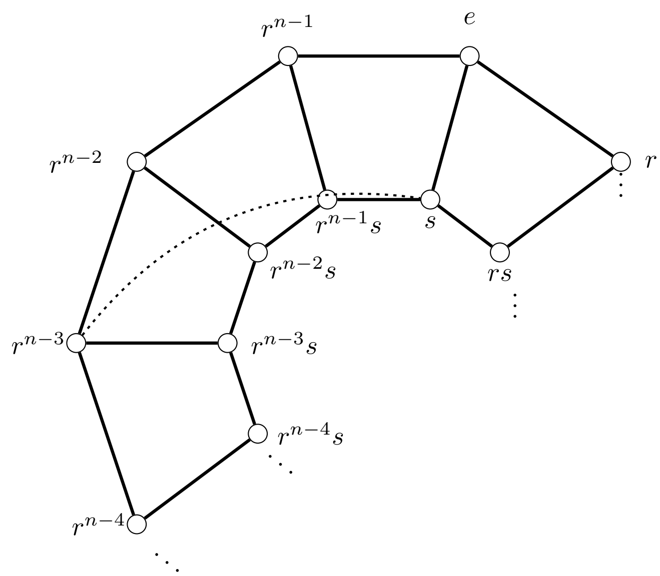

Figure 5: Portion of a general Cayley graph for D_n with moves as in Theorem 3.2, the even case with $n \equiv 2 \mod 6$. By Lemma 3.7, if Player 1 enters square $n-2$ at r^{n-3}, then they can guarantee reaching vertex s.

involves the formation of a Hamiltonian path in the Cayley graph (see Figure 6) by Player 1 always choosing the generator s. In fact, this strategy can be generalized to $\texttt{RAV}(G, S)$ for any group G with generating set S containing an element of order 2.

Theorem 3.9. *Let G be a finite group with generating set S containing an element s of order 2. Then Player 1 has a winning strategy for the game $\texttt{RAV}(G, S)$.*

Proof. The winning strategy of Player 1 is to always choose the order two generator s. Since Player 2 can never choose s due to backtracking, they are forced to choose another element of S.

We first show that a choice of s exists on each turn for Player 1. Indeed, suppose it is Player 1's turn and no such choice is available. Let v denote the vertex in the Cayley graph $\Gamma(G, S)$ representing this point in the game. Because Player 1 has no choice of s available, this means that the edge labeled s from vertex v has been traversed previously. But then the vertex v must have been visited previously, meaning Player 2's last move arriving at v was in fact a losing move for Player 2. Hence, if the choice of generator s is not available for Player 1, then Player 2 already lost the game.

We now show that Player 1's strategy is a winning strategy. Suppose for contradiction that Player 1 choosing s to move from the vertex v to the vertex w is a losing move; that is, this forms the first cycle in the Cayley graph. This means that w has previously been visited. In the case that Player 2 reached w the previous time, then Player 1's strategy implies that they would move to v via choosing s. Hence Player 2 actually formed a cycle by moving to v for the second time, a contradiction.

In the case that Player 1 reached w the previous time, it was from the vertex v, so Player 2 again must have formed a cycle by moving to v for the second time. □

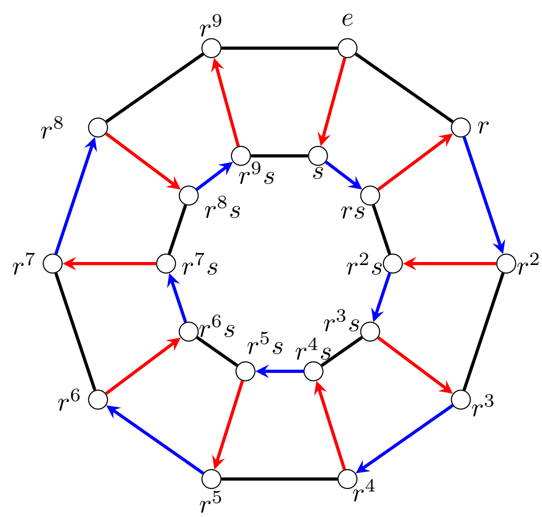

Figure 6: Example of Player 1 winning strategy for $\texttt{RAV}(D_{10})$. Player 1 moves are colored 'red' and Player 2 moves are colored 'blue'. Player 1's strategy is to always choose the generator s

Corollary 3.10. *Player 1 has a winning strategy for* $\texttt{RAV}(D_n, \{r, s\})$ *for any* $n \geq 3$.

Example 3.11. Let H be a finite group with generating set T and let $\{e, s\} = \langle s \rangle \cong \mathbb{Z}_2$ be a cyclic group of order two with generator s. Suppose $G = H \rtimes \mathbb{Z}_2$ with canonical generating set

$$S = (T \times \{e\}) \cup \{(e_H, s)\}.$$

Then Theorem 3.9 implies that Player 1 has a winning strategy for $\texttt{RAV}(G, S)$ by always choosing the generator (e_H, s). This applies in particular to the family of *generalized dihedral groups*, which are defined as the groups $G = H \rtimes \mathbb{Z}_2$ where H is an abelian group and the action of $\mathbb{Z}_2$ on H is that of inversion.

Remark 3.12. Suppose that G is a group of even order. Then G must contain an element of order two. It follows from Theorem 3.9 that there exists a generating set S for which Player 1 has a winning strategy for the game of $\texttt{RAV}(G, S)$.

4. RAV and REL for Dicyclic Groups

4.1. Dicyclic Groups with Generating Set $\{a, x\}$

The dicylic group Dic_n, $n \geq 2$, of order $4n$ is most commonly written via the following presentation:

$$\mathrm{Dic}_n = \langle a, x \mid a^{2n} = x^4 = x^{-1}axa = e \rangle.$$

From the defining relations, one can show that any $g \in \mathrm{Dic}_n$ can be written in a normal form $a^i x^j$, with $0 \leq i < 2n$ and $j \in \{0, 1\}$, and with the following relations:

$$\begin{aligned} a^k a^\ell &= a^{k+\ell} \\ a^k a^\ell x &= a^{k+\ell} x \\ a^k x a^\ell &= a^{k-\ell} x \\ a^k x a^\ell x &= a^{k-\ell+n}. \end{aligned}$$

The winning strategy for the game $\mathtt{RAV}(\mathrm{Dic}_n, \{a, x\})$ is similar to that found in Theorem 3.9. We note here that the generator x is *not* of order two; however, it plays a similar role to that of the order two generator from Theorem 3.9. That is, in the normal form for elements of Dic_n, the possible powers of x are either 0 or 1. Hence, although x has order four, it acts like an element of order two in the normal form.

Theorem 4.1. *Player 1 has a winning strategy for* $\mathtt{RAV}(\mathrm{Dic}_n, \{a, x\})$.

Proof. Note that for $n = 2$, we have $\mathrm{Dic}_2 = Q_8$. Hence, this case is covered by Example 2.4. For the remainder of the proof, suppose $n \geq 3$. See Figure 7 for a Cayley graph of Dic_4.

Using the normal form described above, Player 1 has a winning strategy by moving on each of their turns from $a^k x$ to a^k by choosing x^{-1} or from a^k to $a^k x$ by choosing x, where k is an integer with $0 \leq k < 2n$. Such a move is always available to Player 1, which can be shown via an inductive argument. We will show explicitly the base case and Player 1's second turn.

For the base case, Player 1 starts with a choice of x. That is, they move from $e = a^0$ to $a^0 x = x$.

For Player 1's second turn, we have, based off Player 2's choice of generator, three possible game words:

$$xa \equiv_G a^{2n-1}x, \quad x^2 \equiv_G a^n, \quad xa^{-1} \equiv_G ax.$$

Player 1 can thus choose x^{-1}, x, or x^{-1}, respectively, resulting in:

$$xax^{-1} \equiv_G a^{2n-1}, \quad x^3 \equiv_G a^n x, \quad xa^{-1}x^{-1} \equiv_G a,$$

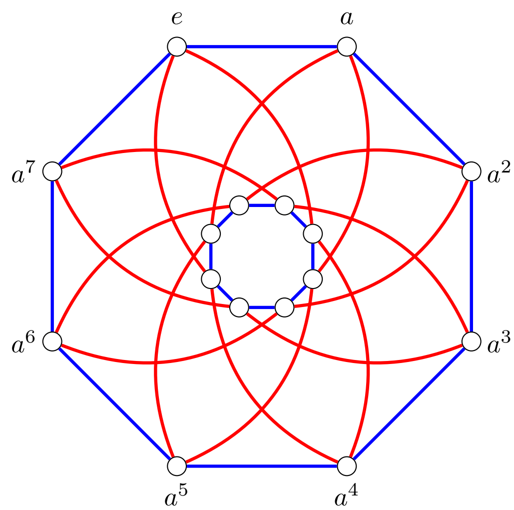

Figure 7: Cayley graph for Dic_4 with generators a and x. The 'blue' edges correspond to the generator a and the 'red' edges to the generator x. On the inner octagon, if one labels the vertices x, ax, a^2x, …, a^7x, in a clockwise order, then a choice of the generator a will move a player counter-clockwise.

and none of these vertices have been previously visited.

Now assume that for the first m turns, Player 1's strategy above has been successfully employed. We want to show that on Player 1's next turn, i.e., on turn $m+2$ of the game, a move from a^kx to a^k via a choice of x^{-1} or a move from a^k to a^kx by choice of x is possible.

Suppose we are in the first case and $w_{m+1} \equiv_G a^kx$. Assume that Player 2 chose x on their last turn. This means Player 2's choice of x moved from a^k to a^kx. Since we are assuming Player 1's strategy has been successfully employed for the first m turns, Player 1 already made the move from a^kx to a^k via choosing x^{-1} on the m-th turn. Hence, Player 2's choice of x is itself an illegal move. Similarly, one can show that Player 1's strategy is possible if $w_{m+1} \equiv_G a^k$.

Knowing that Player 1's strategy is always possible, we now show that this strategy is a winning strategy. Note that Player 1 will always move from a word equivalent to a^ix^ϵ to $a^ix^{1-\epsilon}$, and vice versa, with $\epsilon \in \{0,1\}$. Now suppose that Player 1 loses the game at a word $w_m \equiv_G a^kx^\epsilon$ for some k and m. This means $w_\ell \equiv_G a^kx^\epsilon$ for some $0 \leq \ell < m$. By Player 1's strategy, $w_{m-1} \equiv_G a^kx^{1-\epsilon}$. But then it must also be true that $w_{\ell-1} \equiv_G a^kx^{1-\epsilon}$ if Player 1 moved to w_ℓ or $w_{\ell+1} \equiv_G a^lx^{1-\epsilon}$ if Player 1 moved from w_ℓ previously. In either case, Player 2 must have already lost the game at w_{m-1}. □

In Figure 8 we give a simplified, partial Cayley graph for Dic_n, labeled with respect to the generating set $\{a, x\}$, that may provide a visual aid for the proof of Theorem 4.2 below. In Figure 8, the inner and outer $2n$-gons are given by concentric circles. Instead of drawing all edges, note that a choice of the generator a moves one clockwise on the outer circle but counter-clockwise on the inner circle. A choice of the generator x or x^{-1} will move one from the inner to outer circle or vice versa.

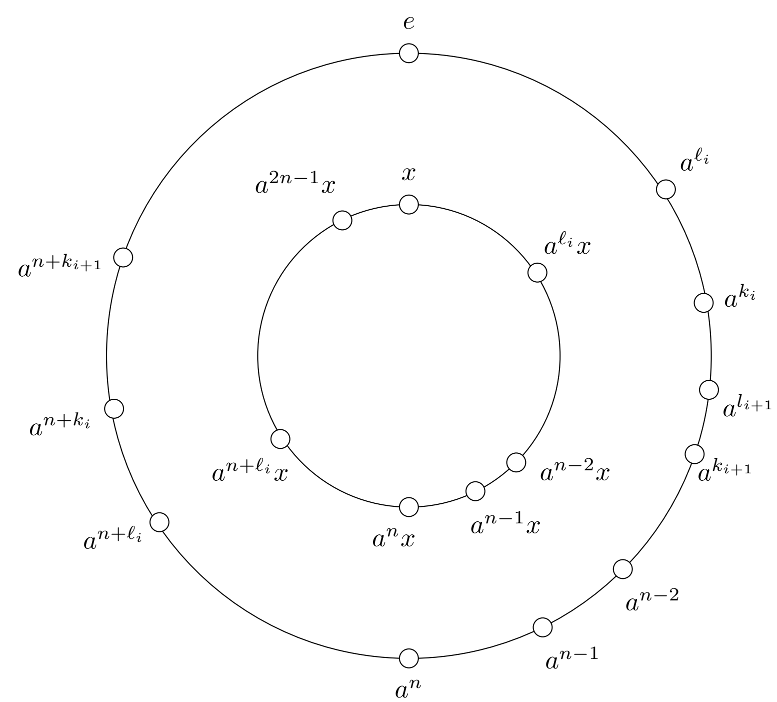

Figure 8: A simplified way to visualize the Cayley graph of Dic_n, which is composed of two $2n$-gons, represented here by concentric circles. The generator a moves one clockwise on the outer circle, but counter-clockwise on the inner circle. The generator x moves one from the inner to outer circle or vice versa.

Theorem 4.2. *Player 1 has a winning strategy for* $\texttt{REL}(\mathrm{Dic}_n, \{a, x\})$ *for odd* n.

Proof. Player 1 begins by choosing a and continues to do so until Player 2 chooses anything other than a. We will show that after Player 2 chooses a move from $\{x, x^{-1}\}$, they will either lose or must choose from $\{x, x^{-1}\}$ again. This allows for an inductive argument showing that Player 1 will necessarily win the game.

First, note that if Player 2 continues to only choose a, then Player 1 lands at a^r, r odd, while Player 2 lands at a^r, with r even. By examining the sequence of moves found in Table 1, we see that Player 1 will win if Player 2 only chooses a. Moreover, any move by Player 2 from a^{n-2} will yield a Player 1 win. Hence, we may assume that Player 2 chooses x to move from a^{ℓ_0} to $a^{\ell_0}x$ or chooses x^{-1} to move from a^{ℓ_0} to $a^{\ell_0}x^{-1} = a^{n+\ell_0}x$ for some odd integer ℓ_0, with $1 \leq \ell_0 \leq n-4$.

We now show that Player 2 must eventually choose again from $\{x, x^{-1}\}$ to arrive at either a^{k_0} or a^{n+k_0}, for some even integer k_0 with $\ell_0 + 1 \leq k_0 \leq n-1$. This is broken into two cases depending on Player 2's move from a^{ℓ_0}.

Player 2 Moves	Player 1 Chooses	If P2 Chooses	Then P1 Wins By
		$\xrightarrow{a^{-1}} a^n x$	$\xrightarrow{x} a^n x^2 = a^{2n} = e$
$a^{n-2} \xrightarrow{a} a^{n-1}$	$\xrightarrow{x} a^{n-1}x$	$\xrightarrow{a} a^{n-2}x$	$\xrightarrow{x^{-1}} a^{n-2}$
		$\xrightarrow{x} a^{2n-1}$	$\xrightarrow{a} a^{2n-1}a = a^{2n} = e$
		$\xrightarrow{a^{-1}} a^n x$	$\xrightarrow{x} a^n x^2 = a^{2n} = e$
$a^{n-2} \xrightarrow{x} a^{n-2}x$	$\xrightarrow{a^{-1}} a^{n-1}x$	$\xrightarrow{x^{-1}} a^{n-1}$	$\xrightarrow{a^{-1}} a^{n-2}$
		$\xrightarrow{x} a^{2n-1}$	$\xrightarrow{a} a^{2n-1}a = a^{2n} = e$
		$\xrightarrow{a^{-1}} a^{2n}x = x$	$\xrightarrow{x^{-1}} e$
$a^{n-2} \xrightarrow{x^{-1}} a^{2n-2}x$	$\xrightarrow{a^{-1}} a^{2n-1}x$	$\xrightarrow{x} a^{n-1}$	$\xrightarrow{a^{-1}} a^{n-2}$
		$\xrightarrow{x^{-1}} a^{2n-1}$	$\xrightarrow{a} a^{2n} = e$

Table 1: The sequence of moves if Player 2 moves to a^{n-3}. Then Player 1 moves to a^{n-2} via a. Regardless of how Player 2 proceeds from a^{n-2}, Player 1 ends up back at e or a^{n-2} to win.

1. Suppose Player 2 chooses x to move from a^{ℓ_0} to $a^{\ell_0}x$. Then Player 1 will choose a^{-1} until Player 2 chooses anything other than a^{-1}.

   Should Player 2 only choose a^{-1}, then Player 2 arrives first to $a^{n-2}x$ since ℓ_0 is odd. Then Player 1 moves to $a^{n-1}x$. By Table 2, we see that Player 2 is forced to choose x^{-1} to arrive at a^{n-1}. Player 1 can then move first from a^{n-1} to a^n, and by Table 2 again, Player 2 must move to a^{n+1} to prevent losing. Player 1 will continue to choose a until Player 2 chooses anything other than a. If play continues in this way, then Player 2 will be the first to reach the vertex $a^{n+\ell_0}$, since ℓ_0 is odd. Then Player 1 would win by playing x^{-1} to move to $a^{\ell_0}x$. Hence, for some even m with $2 \leq m \leq l_0 - 1$, Player 2 must choose x^{ϵ}, with $\epsilon \in \{\pm 1\}$, to move from a^{n+m} to $a^{n+m}x^{\epsilon}$. From here, Player 1 chooses x^{ϵ} to arrive at a^m in either case. Because $2 \leq m \leq \ell_0 - 1$, Player 1 will win.

Hence, Player 2 loses unless for some even k_0, with $\ell_0+1 \leq k_0 \leq n-3 \leq n-1$, they choose either x to move from $a^{k_0}x$ to $a^{k_0}x^2 = a^{n+k_0}$ or x^{-1} to move from $a^{k_0}x$ to $(a^{k_0}x)x^{-1} = a^{k_0}$.

2. Suppose Player 2 chooses x^{-1} to move from a^{ℓ_0} to $a^{n+\ell_0}x$. Then Player 1 will choose a^{-1} until Player 2 chooses anything other than a^{-1}. Should Player 2 only choose a^{-1}, then they will be the first to arrive at $a^{2n}x = x$ because $n + \ell_0$ is even. From here, Player 1 chooses x^{-1} to win at $xx^{-1} = e$.

   Hence, Player 2 loses unless for some even k_0, with $\ell_0 + 1 \leq k_0 \leq n - 1$, they choose from $\{x, x^{-1}\}$ to move from $a^{n+k_0}x$ to $a^{n+k_0}x^2 = a^{2n+k_0} = a^{k_0}$ or to $(a^{n+k_0}x)x^{-1} = a^{n+k_0}$.

Player 1 Moves	Player 2 Chooses	P1 Moves First To
$a^{n-2}x \xrightarrow{a^{-1}} a^{n-1}x$	$\xrightarrow{a^{-1}} (a^{n-1}x)a^{-1} = a^n x$	$\xrightarrow{x} a^n x^2 = a^{2n} = e$
	$\xrightarrow{x} a^{n-1}x^2 = a^{2n-1}$	$\xrightarrow{a} a^{2n} = e$
	$\xrightarrow{x^{-1}} (a^{n-1}x)x^{-1} = a^{n-1}$	$\xrightarrow{a} (a^{n-1})a = a^n$
$a^{n-1} \xrightarrow{a} a^n$	$\xrightarrow{a} a^{n+1}$	$\xrightarrow{a} a^{n+2}$
	$\xrightarrow{x} a^n x$	$\xrightarrow{x} a^n x^2 = e$
	$\xrightarrow{x^{-1}} a^n x^{-1} = x$	$\xrightarrow{x^{-1}} xx^{-1} = e$

Table 2: The first row details the sequence of possible moves should Player 2 move to $a^{n-2}x$. The second row is a continuation of the third move sequence in row one. Note that Player 1 will either win or move to a^{n+2} via choosing a.

We have thus shown above our base case: Player 2 must move to a^{k_0} or a^{n+k_0} for some even k_0 with $\ell_0+1 \leq k_0 \leq n-1$ and where $1 \leq \ell_0 \leq n-4$. Now, let us assume that Player 2 has moved to a^{k_i} or a^{n+k_i} for some even k_i with $\ell_i + 1 \leq k_i \leq n-1$ and where ℓ_i is an odd integer satisfying $1 \leq \ell_i \leq n-4$.

- Suppose Player 2 moves to a^{k_i}, then Player 1 chooses a to move to a^{k_i+1}. Note that if $k_i = n-3$, then Player 1 has a winning sequence of moves by Table 1. If $k_i = n-1$, then by Table 2, Player 2 is forced to move to a^{n+1} and Player 1 will win by the same argument given above in (1). Otherwise, we obtain an odd ℓ_{i+1} such that $k_i+1 \leq \ell_{i+1} \leq n-4$. By the same reasoning as above, we can obtain an even k_{i+1} such that $\ell_{i+1}+1 \leq k_{i+1} \leq n-1$.

- If Player 2 moves instead to a^{n+k_i}, then Player 1 will continue to play a until Player 2 chooses from $\{x, x^{-1}\}$. Because k_i is even and n is odd, Player 1 will be the first to move to $a^{n+n} = e$ if $k_i = n-1$ or Player 2 only chooses a.

Otherwise, Player 2 chooses from $\{x, x^{-1}\}$ to move from $a^{n+\ell_{i+1}}$ to $a^{n+\ell_{i+1}}x$ or $a^{n+\ell_{i+1}}x^{-1} = a^{\ell_{i+1}}x$ for some odd ℓ_{i+1} with $k_i + 1 \leq \ell_{i+1} \leq n-4 \leq n-2$. Similar to the previous cases (1) and (2), we will show that Player 2 must move to $a^{k_{i+1}}$ or $a^{n+k_{i+1}}$ for some even k_{i+1} with $\ell_{i+1} + 1 \leq k_{i+1} \leq n-1$.

- Suppose Player 2 chooses x to move to $a^{n+\ell_{i+1}}x$ for some odd ℓ_{i+1} with $k_i + 1 \leq \ell_{i+1} \leq n-2$. Then Player 1 will continue to play a^{-1} until Player 2 chooses from $\{x, x^{-1}\}$ or Player 2 reaches $a^{n+n}x = x$, in which case Player 1 wins at e by playing x^{-1}. Hence, Player 2 moves to either $(a^{n+k_{i+1}}x)x = a^{k_{i+1}}$ or $(a^{n+k_{i+1}}x)x^{-1} = a^{n+k_{i+1}}$ for some even k_{i+1} where $\ell_{i+1} + 1 \leq k_{i+1} \leq n-1$.
- Suppose Player 2 chooses x^{-1} to move to $a^{\ell_{i+1}}x$ for some odd ℓ_{i+1} with $k_i + 1 \leq \ell_{i+1} \leq n-2$. Then Player 1 continues to play a^{-1} until Player 2 chooses from $\{x, x^{-1}\}$, or until Player 1 reaches $a^{n-1}x$. Player 1 reaches $a^{n-1}x$ first because both $\ell_{i+1} + 1$ and $n-1$ are even. Note that by Table 2, if Player 1 moves to $a^{n-1}x$, then either Player 1 wins or Player 2 moves to a^{n+1}. Then Player 1 can eventually move to either $a^{n+\ell_0}x$ or $a^{n+\ell_0}x^{-1} = a^{\ell_0}$; or Player 1 can move to $a^{n+m}x^{\epsilon}x^{\epsilon} = a^m$ after Player 2 moves to $a^{n+m}x^{\epsilon}$ for some $\epsilon \in \{\pm 1\}$ and $2 \leq m \leq \ell_0 - 1$. In either case, one of $a^{n+\ell_0}x$, a^{ℓ_0} or a^m with $2 \leq m \leq \ell_0 - 1$ has been visited previously, hence Player 1 wins. Hence, as in the previous case, we can assume that Player 2 moves from $a^{k_{i+1}}x$ to either $(a^{k_{i+1}}x)x = a^{n+k_{i+1}}$ or $(a^{k_{i+1}}x)x^{-1} = a^{k_{i+1}}$ for some even k_{i+1} where $\ell_{i+1} + 1 \leq k_{i+1} \leq n-1$.

We have shown that either Player 1 wins or we can generate a strictly increasing sequence of positive integers (k_i) satisfying $k_i \leq n-1$ for all i, since $k_i < \ell_{i+1} < k_{i+1}$. As the set of positive even integers less than or equal to $n-1$ is finite, there must exist some integer j such that $k_j = n-1$. Thus, Player 2 eventually arrives at either a^{2n-1} or a^{n-1}. The choice of a to move from a^{2n-1} to $a^{2n} = e$ is a win for Player 1. By Table 2, Player 1 can choose a to move from a^{n-1} to a^n which, as argued previously, leads to a Player 1 win. Therefore, we conclude that Player 1 wins $\texttt{REL}(\mathrm{Dic}_n, \{a, x\})$ for odd $n \geq 3$. ☐

Theorem 4.3. *Player 2 has a winning strategy for* $\texttt{REL}(\mathrm{Dic}_n, \{a, x\})$ *when* n *is even.*

Proof. Player 2 has a winning strategy via mirroring Player 1. That is, if Player 1 chooses a generator s on their turn, Player 2 will follow with s on their turn. Since $x^2 = (x^{-1})^2 = a^n$ and n is even, we note that this strategy implies Player 1 only lands at a^{ℓ} where ℓ is odd or $a^k x$ where k is even. Meanwhile, Player 2 will only land at a^k where k is even.

To show that Player 2 has a winning strategy, we assume for contradiction that Player 1 has a winning strategy. There are two cases: Player 1 can either win at a^ℓ for ℓ odd or $a^k x$ for k even as stated above.

First suppose that Player 1 wins at a^ℓ, where ℓ is odd. The first time that Player 1 arrived at a^ℓ must have been from either $a^{\ell-1}$ or $a^{\ell+1}$. By Player 2's mirroring strategy, Player 2 would have then moved to the other. Thus, upon reaching a^ℓ for the second time, Player 1 must have moved from $a^{\ell-1}$ or $a^{\ell+1}$, both of which would have been visited for a second time. Hence Player 2 won the previous turn, a contradiction.

Now suppose that Player 1 wins at $a^k x$, where k is even. The first time that Player 1 arrived at $a^k x$ must have been from a^k or a^{n+k}. Player 2 then would have moved to the other. Thus, upon reaching $a^k x$ for the second time, Player 1 would have again moved from a^k or a^{n+k}, both of which would have been visited for a second time. Hence Player 2 won the previous turn, a contradiction. □

4.2. Dicyclic Groups with Triangle Presentation

There is another common presentation for the dicyclic groups, namely, as an instance of a triangle group or binary von Dyck group:

$$\mathrm{Dic}_n = \langle a, b, c \mid a^n = b^2 = c^2 = abc \rangle.$$

Note that the triangle presentation for Dic_n is isomorphic to the one given in Section 4.1 via the mapping

$$a \mapsto a, x \mapsto b^{-1}, ax^{-1} \mapsto c.$$

Hence, we can describe group elements via the normal form $a^i b^j$ with $0 \leq i < 2n$ and $j \in \{0, 1\}$. For the proofs that follow, we will make use of this normal form. See Figure 9 for an example of the Cayley graph of Dic_4 with generating set $\{a, b, c\}$.

For the game of $\texttt{RAV}(\mathrm{Dic}_n, \{a, b, c\})$, we have the same result as Theorem 4.1.

Theorem 4.4. *Player 1 has a winning strategy for* $\texttt{RAV}(\mathrm{Dic}_n, \{a, b, c\})$.

Proof. Player 1 has a winning strategy by choosing b on their first turn, and then moving from $a^k b$ to a^k by choosing b^{-1}, or moving from a^k to $a^k b$ by choosing b on their subsequent turns. The only addition to the previous argument of Theorem 4.1 is accounting for the generator c. Because $c = ab$, we have $bc = bab \equiv_{\mathrm{Dic}_n} a^{n-1}$ and $bc^{-1} = b(b^{-1}a^{-1}) \equiv_{\mathrm{Dic}_n} a^{2n-1}$. Note that, in both cases, Player 1 can choose b on their next turn to move to $a^{n-1}b$ or $a^{2n-1}b$, respectively, thus extending the argument given in Theorem 4.1 for Player 1's second turn. The rest of the proof follows exactly as in Theorem 4.1. □

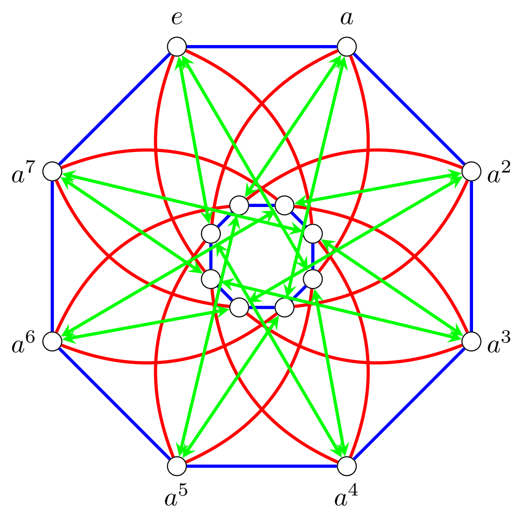

Figure 9: Cayley graph for Dic_4 with generators a, b and c. The 'blue' edges correspond to the generator a, the 'red' to the generator b, and the 'green' edges to c. One has a normal form $a^i b^j$ with $0 \leq i < 2n$ and $j \in \{0, 1\}$, and the inner vertices are labeled b, ab, ab^2, $\ldots$, ab^7, in a clockwise manner.

Despite the addition of the third generator, we can see that Player 2 has the same winning strategy for $\texttt{REL}(\mathrm{Dic}_n, \{a, b, c\})$ as they did for $\texttt{REL}(\mathrm{Dic}_n, \{a, x\})$ when n is even.

Theorem 4.5. *Player 2 has a winning strategy for* $\texttt{REL}(\mathrm{Dic}_n, \{a, b, c\})$ *for* n *even.*

The same argument as described in Theorem 4.3 applies here. Recall we can describe every element of Dic_n via the normal form $a^i b^j$ where $0 \leq i < 2n$ and $j \in \{0, 1\}$. From Theorem 4.3, we know that Player 1 can arrive at words equivalent to $a^k b$ for k even and a^ℓ for ℓ odd. However, with the addition of the generator c, Player 1 can also arrive at words equivalent to $a^\ell b$ for ℓ odd. Player 2 still can only arrive at words equivalent to a^k where k is even. We leave the remaining details to the reader.

As opposed to the game $\texttt{REL}(\mathrm{Dic}_n, \{a, x\})$, Player 2 has a winning strategy for all $n \geq 2$ by Theorem 4.6 below. Note that one can still use Figure 8 as a visual aid for the proof of Theorem 4.6 by replacing x with b and using that $c = ab$.

There are several relators of length three in Dic_n that will be used throughout the proof of Theorem 4.6. These have been collected into Table 3 below for reference.

Triangle Relators			
abc^{-1}	$a^{-1}cb^{-1}$	$ab^{-1}c$	$a^{-1}c^{-1}b$
$ba^{-1}c^{-1}$	$b^{-1}a^{-1}c$	$bc^{-1}a$	$b^{-1}ca$
$cb^{-1}a^{-1}$	$c^{-1}ab$	cab^{-1}	$c^{-1}ba^{-1}$

Table 3: Twelve relators of length three in Dic_n with generating set $\{a,b,c\}$. We refer to these relators as *triangle relators* because in the Cayley graph of Dic_n, they form triangles.

Theorem 4.6. *Player 2 has a winning strategy for* $\texttt{REL}(\mathrm{Dic}_n, \{a,b,c\})$ *when* n *is odd.*

Proof. We will show first that Player 1 cannot win if Player 1 begins the game with $b^{\pm 1}$ or $c^{\pm 1}$.

1. Without loss of generality[1], suppose Player 1 chooses b on their first turn. Then Player 2 will also play b to move to $b^2 = a^n$. Then by the move sequences shown in Table 4, we see that Player 2 wins unless Player 1 chooses a.

   Player 2's strategy now is to continue playing a until Player 1 plays anything other than a, or moves to a^{2n-2}, which must happen because n is odd.

Player 2 Moves	If Player 1 Chooses	Player 2 Moves First To
$b \xrightarrow{b} b^2 = a^n$	$\xrightarrow{a} a^{n+1}$	$\xrightarrow{a} a^{n+2}$
	$\xrightarrow{a^{-1}} a^{n-1} = b(ba^{-1})$	$\xrightarrow{c^{-1}} b(ba^{-1}c^{-1}) = be = b$
	$\xrightarrow{b} a^n b$	$\xrightarrow{b} a^n b^2 = a^{2n} = e$
	$\xrightarrow{c} a^n c$	$\xrightarrow{c} a^n c^2 = a^{2n} = e$
	$\xrightarrow{c^{-1}} a^n c^{-1}$	$\xrightarrow{c^{-1}} a^n c^{-2} = a^{2n} = e$

Table 4: The sequence of moves if Player 1 first plays b. Player 2 will mirror them and move to $b^2 = a^n$. Observe that Player 2 wins unless Player 1 chooses a.

- ⧫ If Player 1 moves to a^{2n-2}, then Player 2 will choose c^{-1} to move to $a^{2n-2}c^{-1} = a^{n-1}b$. By the calculations in Table 5, we may assume that Player 1 moves to a^{n-2} by choosing c^{-1}.

  Since n is odd, Player 2 will continue to choose a^{-1} and be able to move to $a^0 = e$ and win unless Player 1 chooses from $\{b^{\pm 1}, c^{\pm 1}\}$. Thus, let us

[1]For the case where Player 1 chooses c, one simply makes the following changes: b changes to c, c to b, and $a^{\pm 1}$ to $a^{\mp 1}$.

Player 2 Moves	If Player 1 Chooses	Player 2 Moves To
	$\xrightarrow{a} (a^{2n-2}c^{-1})a = a^{2n-2}(c^{-1}a)$	$\xrightarrow{b} a^{2n-2}(c^{-1}ab) = a^{2n-2}e = a^{2n-2}$
	$\xrightarrow{a^{-1}} (a^{n-1}b)a^{-1} = a^n b$	$\xrightarrow{b} a^n b^2 = a^{2n} = e$
$a^{2n-2} \xrightarrow{c^{-1}} a^{n-1}b$	$\xrightarrow{b} a^{n-1}b^2 = a^{2n-1}$	$\xrightarrow{a} a^{2n} = e$
	$\xrightarrow{b^{-1}} (a^{n-1}b)b^{-1} = a^{n-1}$	$\xrightarrow{a} a^n$
	$\xrightarrow{c^{-1}} (a^{2n-2}c^{-1})c^{-1} = a^{n-2}$	$\xrightarrow{a^{-1}} a^{n-2}a^{-1} = a^{n-3}$

Table 5: The sequence of moves if Player 1 moves to a^{2n-2}. Note that all move sequences yield Player 2 winning except the last row.

assume that Player 1 chooses to play $z \in \{b^{\pm1}, c^{\pm1}\}$ to move from a^ℓ to $a^\ell z$, for some even ℓ with $2 \leq \ell \leq n-3$. Then Player 2 will play z to move from $a^\ell z$ to $a^\ell z^2 = a^{n+\ell}$. Since every vertex a^m, $n \leq m \leq 2n-2$ has been visited before, Player 2 wins.

- Suppose that Player 1 plays $z \in \{b^{\pm1}, c^{\pm1}\}$ to move from a^{n+k} to $a^{n+k}z$ for some even k with $2 \leq k \leq n-3$. Because of the triangle relators (Table 3), we can see from the top part of Table 6 that Player 2 will win unless z is c or c^{-1}. Regardless, Player 2 will move the game to a^k. From the bottom part of Table 6, we see that the only possible choice for Player 1 is to choose a^{-1} to move to a^{k-1}, with Player 2 also choosing a^{-1} on the next turn to move to a^{k-2}. Player 2 will continue to choose a^{-1} until Player 1 chooses something other than a^{-1}. Should Player 1 choose only a^{-1}, then Player 2 will first reach $a^{k-k} = a^0 = e$, since k is even. Hence, for some even ℓ, $2 \leq \ell \leq k-2 < n-3$, Player 1 chooses $y \in \{b^{\pm1}, c^{\pm1}\}$ to move from a^ℓ to $a^\ell y$. But since $y^2 = a^n$, Player 2 will also choose y to move from $a^\ell y$ to $a^\ell y^2 = a^{n+\ell}$, which has been previously visited.

2. Having shown that Player 1 loses if they choose b or c to start, Player 1 will choose from $\{a, a^{-1}\}$ on their first turn. Without loss of generality, suppose Player 1 chooses a. Then Player 2 will choose a until Player 1 chooses something other than a. Should Player 1 only choose a, then Player 2 will first reach $a^{2n} = e$ because $2n$ is even. Hence, we assume that Player 1 chooses z in $\{b^{\pm1}, c^{\pm1}\}$ to move from a^ℓ to $a^\ell z$ for some even ℓ with $2 \leq \ell \leq 2n-2$. We have two cases depending on whether $\ell > n$ or $\ell < n$.

   - Suppose $n < \ell \leq 2n-2$. Let $\ell = n+k$ where $1 \leq k \leq n-2$. After Player 1 chooses $z \in \{b^{\pm1}, c^{\pm1}\}$ to move from a^{n+k} to $a^{n+k}z$, Player 2 will choose z to move from $a^{n+k}z$ to $a^{n+k}z^2 = a^{2n+k} = a^k$, since $z^2 = a^n$

	Player 1 Moves	Player 2 Chooses
	$a^{n+k} \xrightarrow{z} a^{n+k}z$	
$z = b$	$a^{n+k}b = a^{n+k-1}(ab)$	$\xrightarrow{c^{-1}} a^{n+k-1}(abc^{-1}) = a^{n+k-1}$
$z = b^{-1}$	$a^{n+k}b^{-1} = a^{n+k-1}(ab^{-1})$	$\xrightarrow{c} a^{n+k-1}(abc) = a^{n+k-1}$
$z = c$	$a^{n+k}c$	$\xrightarrow{c} a^{n+k}c^2 = a^{2n+k} = a^k$
$z = c^{-1}$	$a^{n+k}c^{-1}$	$\xrightarrow{c^{-1}} a^{n+k}c^{-2} = a^{2n+k} = a^k$
	$a^k = a^{n+k}c^\epsilon c^\epsilon \xrightarrow{y} a^k y$	
$y = a$	$(a^{n+k}c^\epsilon c^\epsilon)a$	$\xrightarrow{b^{-\epsilon}} (a^{n+k}c^\alpha)(c^\epsilon ab^{-\epsilon}) = a^{n+k}c^\alpha$
$y = a^{-1}$	$a^k a^{-1} = a^{k-1}$	$\xrightarrow{a^{-1}} a^{k-1}a = a^{k-2}$
$y = b$	$a^k b$	$\xrightarrow{b} (a^k b)b = a^k b^2 = a^{n+k}$
$y = b^{-1}$	$a^k b^{-1}$	$\xrightarrow{b^{-1}} (a^k b^{-1})b^{-1} = a^k b^{-2} = a^{n+k}$
$y = c^\epsilon$	$a^k c^\epsilon$	$\xrightarrow{c^\epsilon} (a^k c^\epsilon)c^\epsilon = a^k c^{2\epsilon} = a^{n+k}$

Table 6: The top part shows the sequence of moves after the game proceeds through a^m for $n \le m \le k$ and Player 1 moves from a^{n+k} to $a^{n+k}z$ for $z \in \{b^{\pm 1}, c^{\pm 1}\}$. Player 2 wins unless Player 1 chooses c or c^{-1}, both of which move the game to a^k. The bottom part shows the sequence of moves following Player 1's choice of c^ϵ for some $\epsilon \in \{\pm 1\}$. Note that the only option that does not immediately yield a Player 2 win is for Player 1 to choose a^{-1}.

for any $z \in \{b^{\pm 1}, c^{\pm 1}\}$. The vertex a^k has previously been visited; hence Player 2 wins.

♦ Suppose $2 \le \ell < n$. By Table 7, we see that only a choice of $z = c^\epsilon$, $\epsilon \in \{\pm 1\}$ is possible for Player 1, hence moving from a^ℓ to $a^\ell c^\epsilon$.

If $\ell = n-1$, Player 2 then wins by choosing b^ϵ to move to $a^{n-1}(c^\epsilon b^\epsilon) = a^{n-1}a^{n+1} = e$. Hence, we assume that $2 \le \ell \le n-3$. In this case, Player 2 will subsequently move to $a^{n+\ell}$. Continuing from the second half of Table 7, we see that Player 1 must choose a^{-1} to move to $a^{n+\ell-1}$. Player 2 will then continue to choose a^{-1} until Player 1 moves to $a^{\ell+2}$, which happens because ℓ is even and n is odd; or until Player 1 plays something other than a^{-1}. We examine each case below in items (a) and (b), respectively.

Player 1 Moves		Player 2 Chooses
$a^\ell \xrightarrow{z} a^\ell z$		
$z = b$	$a^\ell b = a^{\ell-1}(ab)$	$\xrightarrow{c^{-1}} a^{\ell-1}(abc^{-1}) = a^{\ell-1}$
$z = b^{-1}$	$a^\ell b^{-1} = a^{\ell-1}(ab^{-1})$	$\xrightarrow{c} a^{\ell-1}(ab^{-1}c) = a^{\ell-1}$
$z = c^\epsilon$	$a^\ell c^\epsilon$	$\xrightarrow{c^\epsilon} a^\ell c^{2\epsilon} = a^{n+\ell}$
$a^\ell c^\epsilon c^\epsilon = a^{n+\ell} \xrightarrow{z} a^{n+\ell} z$		
$z = c^\epsilon$	$a^{n+\ell} c^\epsilon$	$\xrightarrow{c^\epsilon} a^{n+\ell} c^{2\epsilon} = a^{2n+\ell} = a^\ell$
$z = b^\epsilon$	$a^{n+\ell} b^\epsilon$	$\xrightarrow{b^\epsilon} a^{n+\ell} b^{2\epsilon} = a^{2n+\ell} = a^\ell$
$z = a$	$a^{n+\ell} a$	$\xrightarrow{b^{-\epsilon}} a^{n+\ell}(ab^{-\epsilon}) = a^\ell c^\epsilon (c^\epsilon ab^{-\epsilon}) = a^\ell c^\epsilon$
$z = a^{-1}$	$a^{n+\ell} a^{-1}$	$\xrightarrow{a^{-1}} a^{n+\ell} a^{-2} = a^{n+\ell-2}$

Table 7: The lines of play if Player 1 chooses a on their first turn and Player 2 mirrors them until Player 1 moves from a^ℓ to $a^\ell z$ for some even ℓ, $2 \le \ell \le n-3$, and $z \in \{b^{\pm 1}, c^{\pm 1}\}$. Note that Player 2 wins unless Player 1 chooses either c or c^{-1}. Following this further, the bottom part shows Player 1's options after Player 2 moves to $a^{n+\ell}$ above. The only option that does not result in a Player 1 loss is a^{-1}.

Player 2 Moves	If Player 1 Chooses	Player 2 Moves To
$a^{\ell+2} \xrightarrow{b^{-1}} a^{\ell+2} b^{-1}$	$\xrightarrow{a} (a^{n+\ell+2} b) a = a^{n+\ell+1} b$	$\xrightarrow{c^{-1}} (a^{n+\ell+1} b) c^{-1} = a^{n+\ell}(abc^{-1})$
		$= a^{n+\ell}$
$a^{\ell+2} b^{-1} = a^{n+\ell+2} b$	$\xrightarrow{a^{-1}} (a^{\ell+2} b^{-1}) a^{-1} = a^{\ell+2}(b^{-1}a^{-1})$	$\xrightarrow{c} a^{\ell+2}(b^{-1}a^{-1}c) = a^{\ell+2}$
	$\xrightarrow{b^{-1}} (a^{n+\ell+2} b) b^{-1} = a^{n+\ell+2}$	$\xrightarrow{a} a^{n+\ell+2} a = a^{n+\ell+3}$
	$\xrightarrow{c} (a^{\ell+2} b^{-1}) c = a^{\ell+2}(b^{-1}c)$	$\xrightarrow{a} a^{\ell+2}(b^{-1}ca) = a^{\ell+2}$
	$\xrightarrow{c^{-1}} (a^{n+\ell+2} b) c^{-1} = a^{n+\ell+2}(bb^{-1}a)$	
	$= a^{n+\ell+1}$	$\xrightarrow{a^{-1}} a^{n+\ell}$

Table 8: If play proceeds from $a^{n+\ell-2}$ as in Table 7 with only a^{-1} chosen by both players, then Player 2 will choose b^{-1} from $a^{\ell+2}$. Note that the next move for Player 1 must be b^{-1} to move to $a^{n+\ell+2}$ as all others will be a loss.

(a) Suppose play has continued from $a^{n+\ell-1}$ with both players choosing only a^{-1} until Player 1 moves from $a^{\ell+3}$ to $a^{\ell+3}a^{-1} = a^{\ell+2}$. Then Player 2 will play b^{-1} to move to $a^{\ell+2}b^{-1} = a^{n+\ell+2}b$. From Table 8, we see that any choice other than b^{-1} for Player 1 leads to a loss;

hence Player 1 moves from $a^{n+\ell+2}b$ to $a^{n+\ell+2}$.
In a manner similar to the earlier case (first sub-case of (1) above), Player 2 will continue to choose a until Player 1 chooses something other than a. If $\ell = n-3$, then Player 2 wins immediately by moving from $a^{n+\ell+2} = a^{2n-1}$ to $a^{2n} = e$. Thus we may assume $2 \le \ell \le n-5$. If Player 1 only chooses a, Player 2 will win at $a^{2n} = e$ since $n+\ell+3$ is even. Hence, Player 1 must move from a^{n+m} to $a^{n+m}y$, for some odd m, $\ell+3 \le m \le n-2$ and $y \in \{b^{\pm1}, c^{\pm1}\}$. Then Player 2 will play y to move to $a^{n+m}y^2 = a^{2n+m} = a^m$ since $y^2 = a^n$ for all $y \in \{b^{\pm1}, c^{\pm1}\}$. Since every vertex a^m, $\ell+3 \le m \le n-2$ has been previously visited, Player 2 wins.

	Player 1 Moves	Player 2 Chooses
$a^{k+1}a^{-1} = a^k \xrightarrow{z} a^k z$		
$z = b^\epsilon$	$a^k b^\epsilon$	$\xrightarrow{b^\epsilon} a^k b^{2\epsilon} = a^{n+k}$
$z = c^\epsilon$	$a^k c^\epsilon$	$\xrightarrow{b^{-\epsilon}} (a^k c^\epsilon)b^{-\epsilon} = a^{k+1}(a^{-1}c^\epsilon b^{-\epsilon}) = a^{k+1}$
$a^k b^\epsilon b^\epsilon = a^{n+k} \xrightarrow{y} a^{n+k}y$		
$y = b^\epsilon$	$a^{n+k}b^\epsilon$	$\xrightarrow{b^\epsilon} a^{n+k}b^{2\epsilon} = a^k$
$y = c$	$a^{n+k}c$	$\xrightarrow{c} (a^{n+k}c)c = a^{n+k}c^2 = a^{2n+k} = a^k$
$y = c^{-1}$	$a^{n+k}c^{-1}$	$\xrightarrow{c^{-1}} (a^{n+k}c^{-1})c^{-1} = a^{n+k}c^{-2} = a^{2n+k} = a^k$
$y = a$	$a^{n+k}a = a^{n+k+1}$	$\xrightarrow{a} (a^{n+k+1})a = a^{n+k+2}$
$y = a^{-1}$	$(a^k b^\epsilon b^\epsilon)a^{-1} = a^k(b^\epsilon a^{-1})$	$\xrightarrow{c^{-\epsilon}} (a^k b^\epsilon)(b^\epsilon a^{-1}c^{-\epsilon}) = a^k b^\epsilon$

Table 9: The sequence of moves if play proceeds from $a^{n+\ell-2}$ as in Table 7 until Player 1 chooses $z \in \{b^\epsilon, c^\epsilon\}$, $\epsilon \in \{\pm1\}$ to move from $a^{k+1}a^{-1} = a^k$ to $a^k z$. Note that Player 1 must choose $z = b^\epsilon$ by the relators in Table 3. Following this further, the bottom part shows Player 1's options after Player 2 moves to a^{n+k} above. The only option that does not result in a Player 1 loss is a.

(b) Suppose Player 1 plays $z \in \{b^{\pm1}, c^{\pm1}\}$ to move from a^k to $a^k z$ for some odd k with $\ell+3 \le k \le n+\ell-2$. We see from Table 9 that Player 2 wins if $z \in \{c^{\pm1}\}$. If $k \ge n$, then Player 2 wins if $z \in \{b^{\pm1}\}$ as well since a^{n+k} has already been visited in this case.
Now consider the case where $k \le n-2$ and hence where a^{n+k} has not been visited. By the second part of Table 9, we see that Player 2 wins unless Player 1 chooses a. Player 2 will now choose a until Player 1 chooses otherwise. If $k = n-2$, then Player 2 wins by choosing a to move from a^{2n-1} to $a^{2n} = e$. Otherwise, since $n+k+2$ is

even, Player 2 will win at $a^{2n} = e$ unless Player 1 chooses some $y \in \{b^{\pm 1}, c^{\pm 1}\}$ to move from a^{n+j} to $a^{n+j}y$ where $k+2 \leq j \leq n-2$. In this case, Player 2 will mirror to move to $a^{n+j}y^2 = a^{n+j}a^n = a^j$, which has already been visited. Thus Player 2 wins.

□

5. REL for Products of Cyclic Groups

In this section, we consider the group $\mathbb{Z}_n \times \mathbb{Z}_m$ with the following presentation

$$\langle a, b \mid a^n = b^m = aba^{-1}b^{-1} = e \rangle.$$

By Theorem 3.2, and the fact that the undirected Cayley graphs of $(D_n, \{r, s\})$ and $(\mathbb{Z}_n \times \mathbb{Z}_2, \{(1,0),(0,1)\})$ are isomorphic for $n \geq 3$, we know the winner of REL$(\mathbb{Z}_n \times \mathbb{Z}_2)$ for $n \geq 3$ by Theorem 2.7. Additionally, we know Player 2 has a winning strategy for REL$(\mathbb{Z}_2 \times \mathbb{Z}_2)$ since its undirected Cayley graph is equal to that of $(\mathbb{Z}_4, \{1\})$. Additional cases for REL$(\mathbb{Z}_n \times \mathbb{Z}_m)$ are covered by the following theorem.

Theorem 5.1. *Consider the game REL$(\mathbb{Z}_n \times \mathbb{Z}_m, \{a, b\})$, where $n \geq m-1$ and $n, m \geq 3$.*

1. *If $n \equiv \pm 1 \mod m$, then Player 1 has a winning strategy.*
2. *If $n \equiv 0 \mod m$, then Player 2 has a winning strategy.*
3. *If $m = 4$ and $n \equiv 2 \mod 4$, then Player 2 has a winning strategy.*

Proof. Let $G \cong \mathbb{Z}_n \times \mathbb{Z}_m$ with the presentation above. We can write elements from G in the normal form $a^i b^j$, where $0 \leq i < n$ and $0 \leq j < m$. Note that the Cayley graph for G can be visualized as an n-gon of m-gons (see Figure 10 for a partial Cayley graph example.)

The strategy is similar in all cases, so we will describe the strategy first in terms of the players Winner and Loser as opposed to Player 1 and Player 2. The game begins with Winner completing an initial word $w \equiv_G g$ for some $g \in G$, where a^{-1} and b^{-1} do not appear in w. Without loss of generality, we assume that a is played before a^{-1} and b is played before b^{-1}. After this, Winner's strategy is to play a if Loser plays b and to play b if Loser plays a. Play will continue in this manner unless Loser plays a^{-1} or b^{-1}, which must occur from the vertex $g(ab)^{\ell}$ for some ℓ. Note that, by Winner's strategy, the exponent of a is nondecreasing. We show that this strategy is a winning strategy if Loser plays a^{-1} or b^{-1} and hence that we may assume Loser plays only a and b.

Suppose that Loser plays a^{-1}. Then Winner could not have played a on the previous turn since backtracking is disallowed. Hence Winner played b on the previous turn. By Winner's strategy, this means that Loser must have played a the turn prior. Then the three most recent vertices visited before Loser plays a^{-1} are, in order, $g(ab)^{\ell-1}$, $g(a^{\ell}b^{\ell-1})$, and $g(ab)^{\ell}$. Since the exponent of a is nondecreasing, Loser does not win by playing a^{-1} to move to $g(ab)^{\ell}a^{-1} = g(a^{\ell-1}b^{\ell})$. Then Winner wins by completing a commutation relator and choosing b^{-1} to move to $g(ab)^{\ell-1}$. The case where Loser plays b^{-1} is similar. Thus, we may assume that Loser only plays a and b, and Winner therefore moves to $g(ab)^i$ for all i.

We now consider the following fives cases:

(1) If $n = km + 1$ for some k, then Player 1 initially plays a so that $g = a$ and then execute Winner's strategy for km turns to win at $g(ab)^{km} = a(ab)^{km} = a^{km+1}b^{km} = e$.

(2) If $n = km - 1$ for some k, then Player 1 initially plays b so that $g = b$ and executes Winner's strategy for $km - 1$ turns to win at $g(ab)^{km-1} = b(ab)^{km-1} = a^{km-1}b^{km} = e$.

(3) If $n = km$ for some k, then Player 2 considers $g = e$, which they reach on 0th turn. They then execute Winner's strategy for km turns to win at $g(ab)^{km} = e(ab)^{km} = a^{km}b^{km} = e$.

(4) If $m = 4$ and $n = 4k + 2$ for some k and Player 1 initially plays a, then Player 2 also plays a to complete $g = a^2$ and then executes Winner's strategy for $4k$ turns to win at $g(ab)^{4k} = a^2(ab)^{4k} = a^{4k+2}b^{4k} = e$.

(5) If $m = 4$ and $n = 4k + 2$ for some k and Player 1 initially plays b, then Player 2 plays b to complete $g = b^2$ and executes Winner's strategy for $4k + 2$ turns to win at $g(ab)^{4k+2} = b^2(ab)^{4k+2} = a^{4k+2}b^{4k+4} = e$.

□

6. Three-Player REL for Dihedral Groups

In this section, we examine an extension of the REL game for dihedral groups to three players. With more than two players, one must address the issue of what a player does when they no longer can win. To do so, one may establish a podium rule, or ranking system, to define preferences for each player. This ranking is important because it gives each player a preference for who wins. We will examine REL for dihedral groups with two different podium rules. The first is the podium

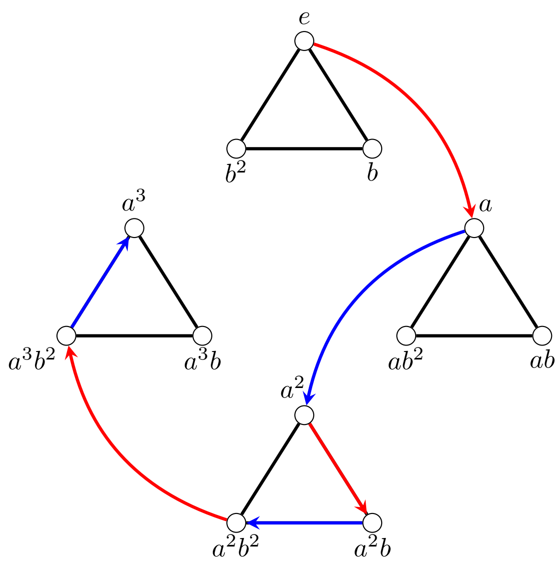

Figure 10: A partial Cayley graph for $\mathbb{Z}_4 \times \mathbb{Z}_3$ - a 4-gon of 3-gons. The colored edges give an example of Player 1's strategy as described in Theorem 5.1. The 'red' edges denote Player 1 moves and the 'blue' edges denote Player 2 moves. By choosing the generator opposite to Player 2's last move, Player 1 will always ensure victory. Note as well that, in this example, the game word is $aabbab \equiv_G a^3b^3 = a^3$, but Player 2 has not achieved a relator.

rule proposed by Li [11] to analyze n-player Nim and studied further in the three-player case by Nowakowski, Santos, and Silva [12]. The second is the podium rule utilized by Benesh and Gaetz [7] to analyze q-player DNG, where q is prime.

First, we define the *standard podium rule* for REL with n players as used by Li and Nowakowski, Santos, and Silva. In this rule, the final player to make a legal move, i.e. the player to complete a relator, is the winner. The penultimate player to move finishes runner-up, the player before that third, and so on. If a player cannot ensure victory for themselves, then they will assist the player who ensures their highest possible ranking to win the game.

For three players, this is a bit simpler. The player to complete a relator is the winner. The previous player is second, and the next player to move is third and last. Because of this podium rule, note that if Player 1 cannot win, then Player 1 will help Player 2 to win. If Player 2 cannot win, Player 2 will help Player 3. Finally, if Player 3 cannot win, Player 3 will help Player 1.

Theorem 6.1. *For three-player* REL$(D_n, \{r, s\})$ *with the standard podium rule, Player 1 has a winning strategy if* $n \equiv 0 \mod 3$ *or* $n \equiv 1 \mod 3$ *and Player 2* *has a winning strategy if* $n \equiv 2 \mod 3$.

Proof. We will show first that, in most cases, a player will choose not to play s. Let $\{A, B, C\} = \{1, 2, 3\}$ such that Player A follows Player C, Player B follows Player A, and Player C follows Player B. Suppose that the game begins with the word w, where w does not end in s, and that, without loss of generality, Player C plays r to move to the word wr. Suppose that Player A does choose s to move to the word wrs. We show that Player A can neither finish first nor second from this point and hence would never have chosen s from wr.

If Player A has a winning strategy from this point, then Player B would finish last. Hence Player B will choose r to move to $wrsr$ from which Player C will choose s to move to $wrsrs \equiv_{D_n} w$ to win, thus securing a second-place finish for Player B. Thus Player A cannot win from this situation. Suppose instead that Player A can finish second and hence that Player B has a winning strategy from the position wrs. Clearly it cannot be by choosing r since this leads to a Player C win. Thus Player B must choose r^{-1} to move to $wrsr^{-1}$. If Player B has a winning strategy from this point, then Player C will play s to move to $wrsr^{-1}s$ from which Player A will play r^{-1} to win at $wrsr^{-1}sr^{-1} \equiv_{D_n} wr$. Thus, Player B has no winning strategy.

Since Player A and Player B cannot have a winning strategy from wrs, it follows that Player C has a winning strategy. However, this leads to a last place finish for Player A, so Player A would never have chosen s.

Now we examine the cases where $n \equiv 0, 1, 2 \mod 3$. First suppose $n \equiv 1 \mod 3$. Player 1 has a winning strategy by always choosing r. After Player 1 chooses r on turn 1, we may consider $w = e$ so that Player 2 is in the situation described above. Hence Player 2 will not choose s and must choose r. This continues until the game reaches $r^n \equiv_{D_n} e$. Since $n \equiv 1 \mod 3$ Player 1 reaches r^n and is thus the winner.

Now suppose $n \equiv 0 \mod 3$. In this case, Player 1 begins by choosing s. Without loss of generality, Player 2 chooses r to move to sr. If Player 3 chooses s, then Player 1 wins by choosing r to move to $srsr \equiv_{D_n} e$. If Player 3 chooses r, then for all following turns we are in the case where the game begins with a word w not ending in s followed by a choice of r. Therefore all players will choose r until the game reaches $sr^n \equiv_{D_n} s$. Since $n \equiv 0 \mod 3$, Player 1 is the player to reach sr^n and hence the winner.

Finally suppose $n \equiv 2 \mod 3$. If Player 1 chooses s, then, without loss of generality, Player 2 will choose r to move to sr. Player 3 must choose r, and each subsequent turn will result in a choice of r until the game reaches $sr^n \equiv_{D_n} s$. Since $n \equiv 2 \mod 3$ and this is a total of $n + 1$ moves, Player 3 reaches sr^n and is the winner. Thus Player 1 finishes last if they choose s on turn 1 and will instead, without loss of generality, choose r. Again by the argument above, we may assume that all players will play r on subsequent turns until the game ends at $r^n \equiv_{D_n} e$.

Since $n \equiv 2 \mod 3$, Player 2 reaches r^n and is the winner. □

We now examine another podium rule for REL with n players as used by Benesh and Gaetz in [7]. The first player to complete a relator still wins the game. The ranking then follows in the opposite manner of the standard podium rule. That is, the following player is runner-up, the next player is third, and so on. We refer to this as the *reverse podium rule*.

In the case of three players, this means that if Player 1 cannot win, then Player 1 will help Player 3 to win. If Player 2 cannot win, Player 2 will help Player 1. Finally, if Player 3 cannot win, Player 3 will help Player 2.

Remark 6.2. Note that Remark 3.4 still holds for dihedral groups. A player will never prefer to be last; if Player m completes the third edge of a square, then Player $m+1 \mod 3$ wins and hence Player m finishes last according to this podium rule. Since finishing last is never preferable, no player will complete a third edge of a square if it can be avoided. Due to this, Remark 3.5 still holds. That is, a choice of the generator s forces the following two moves.

Theorem 6.3. *For three-player REL$(D_n, \{r, s\})$ with the reverse podium rule, Player 1 has a winning strategy if n is odd and Player 3 has a winning strategy if n is even.*

Proof. The key ingredient to this proof is Remark 6.2. As in the proof of Lemma 3.7 we assume, without loss of generality, that players move to words equivalent to $r^i s^j$ with $0 \le i \le n-1$, $0 \le j \le 1$, where i is non-decreasing. Remark 6.2 then implies that a player may play s to guarantee that the game moves from $r^i s^j$ to $r^{i+2} s^{j+1}$ with that same player next to move.

Now suppose $n = 2k+1$ for some k. Then by playing s k consecutive times, Player 1 ensures that the game moves to $r^{2k} s^k$ with Player 1 moving next. Player 1 then moves to $r^{2k+1} s^k \in \{e, s\}$ to win since s has been visited.

Now suppose $n = 2k$ for some k. We note that Player 1 can help Player 3 to win by always playing s. After k times, the game arrives at $r^{2k} s^k = s^k \in \{0, s\}$, where Player 3 makes the last move to win since Player 2 must have moved from $r^{2k-2} s^k$ to $r^{2k-1} s^k$ by Remark 6.2. This implies that Player 2 cannot have a winning strategy since Player 1 will always prefer Player 3 to win instead of Player 2. To conclude that Player 3 must win, we now show that Player 1 does not have a winning strategy when n is even and will therefore help Player 3 to win.

If Player 1 always selects s, we have shown that Player 3 wins. Thus we may assume that Player 1 plays $r^{\pm 1}$ at some point after playing s ℓ consecutive times for some $0 \le \ell < k$. That is, without loss of generality, the game begins with $r^{2\ell+1} s^\ell$ with Player 2 moving next.

Suppose $n \equiv 2 \mod 4$ or $\ell > 0$. We show that Player 2 has a winning strategy. They can play s $k - \ell - 1$ consecutive times to move the game to

$$r^{(2\ell+1)+2(k-\ell-1)} s^{\ell+(k-\ell-1)} = r^{2k-1} s^{k-1},$$

where Player 2 moves next. Player 2 then will move to $r^{2k}s^{k-1} = s^{k-1} \in \{0, s\}$. If $\ell > 0$, then s has been previously visited, so Player 2 wins. If $n \equiv 2 \mod 4$, then $k \equiv 1 \mod 2$, so $k-1$ is even. Hence $s^{k-1} = e$ and Player 2 wins at e.

Now suppose $n = 4m$ for some m and $\ell = 0$. Then, without loss of generality, Player 1 plays r on turn 1. We may assume that Player 2 plays s t consecutive times for some $0 \le t < k$, resulting in the game moving to $r^{2t+1}s^t$ with Player 2 next to move. We look at two cases, namely that Player 2 either always plays s or eventually plays $r^{\pm 1}$. If Player 2 always plays s, then $t = k-1 = 2m-1$, and the game begins with $r^{2(2m-1)+1}s^{2m-1} = r^{4m-1}s^{2m-1} = r^{n-1}s$. From here Player 2 must move to s or to r^{n-1}. Since $\ell = 0$, s has not been visited, so Player 3 then moves to e to win in either case.

Finally, suppose that Player 2 eventually plays $r^{\pm 1}$. That is, suppose that $t < k-1$ and the game begins with $r^{2t+1}s^t$ with Player 2 then moving to $r^{2t+2}s^t$. Player 3 moves next and has a winning strategy by playing s $2m-t-1$ times to move to $r^{(2t+2)+2(2m-t-1)}s^{t+(2m-t-1)} = r^{4m}s^{2m-1} = s$, with Player 3 next to move. Since s has not been visited, Player 3 then moves to e to win. □

7. Open Questions

- When first devising the games REL and RAV, we wanted to create a combinatorial game that utilized the Cayley graph of a group. Although the Cayley graph is not necessary in defining our relator games, we have found it useful when constructing some of our proofs. To that end, one can study the *make a cycle* and *avoid a cycle* games on general graphs. We expect these games to be more challenging to study due to the absence of properties such as graph regularity and symmetry inherent in Cayley graphs.

- A fundamental problem in combinatorial game theory for impartial games is to compute the nim-number of a game (see [15]). These allow one to determine the outcome of the game as well as of game sums. While we have determined the outcome of the games REL and RAV for several families of groups, we leave open the problem of computing their nim-numbers.

- Another goal is to extend results on REL and RAV to n players. For REL on the dihedral groups, this becomes difficult after more than three players are involved since a player can only force moves two ahead and thus loses control over their future moves. Note that the related game DNG for n-players was studied in [7].

- One can of course ask for the outcomes of REL and RAV on other families of finite groups. Of specific interest are the generalized dihedral groups (see

Example 3.11). We have results for RAV for generalized dihedral groups via Theorem 3.9. For a finite generalized dihedral group $G \cong H \rtimes \mathbb{Z}_2$, suppose we have a winning strategy for $\texttt{REL}(H, T)$. Can we then determine a winning strategy for $\texttt{REL}(G, S)$ in a manner similar to that of Theorem 3.9?

- In computing several game trees while working through examples, we observed that most games of REL end after traversing at most half the vertices in the Cayley graph. We have also observed that the game seems to be less complex when more generators are involved. These lead to interesting questions from a computational point of view. Can one find winning strategies utilizing a minimal number of moves or find a correlation between sparseness of the Cayley graph and computational complexity of the game?

- One can certainly explore the games REL and RAV via a computer program. The authors have done some preliminary work on this with College of Wooster undergraduates Minhwa Lee and Pavithra Brahmananda Reddy on this project [2]. One avenue is to apply machine learning techniques such as reinforcement learning to create an A.I. for different families of groups.

- Both authors have incorporated Cayley graphs into their abstract algebra courses. Using the games of REL and RAV for Cayley graphs of dihedral groups and symmetric groups is an alternative way of getting students to practice understanding the structure of these groups. A structured and rigorous implementation of such an approach is a possible direction for pedagogical research.

Acknowledgements. The authors thank Spencer Hamblen, Andrew Kobin, and Mark Schrecengost for providing helpful feedback and comments on an early draft of this paper. The authors also thank the anonymous referee for their immensely helpful suggestions. In particular, their comments greatly improved the clarity of Theorems 3.2, 4.2, and 4.6, and also led to the inclusion of Theorem 6.1.

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