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PLAYING BYNUM'S GAME CAUTIOUSLY

L.R. Haff
Department of Mathematics, University of California, San Diego, California
`lhaff@ucsd.edu`

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Abstract

Several sequences of infinitesimals are introduced for the purpose of analyzing a restricted form of Bynum's game or "Eatcake". Two of these have terms with uptimal values (à la Conway and Ryba, 1970s). All others (eight) are specified by "uptimal+ forms," i.e., standard uptimals plus a fractional uptimal. The game itself is played on an $n \times m$ grid of unit squares, and here we describe all followers (submatrices) of the 12×12 grid. Positional values of larger grids become intractable. However, an examination of $n \times n$ squares, $2 \leq n \leq 21$, reveals that all but three of them are equal to $*$, the exceptions being the 10×10, 14×14, and 18×18 cases. Nonetheless, the exceptional cases have "star-like" characteristics: they are of the form $\pm(G)$, confused with both zero and up, and less than double-up.

1. Introduction and Summary

A version of Bynum's game, also known as "Eatcake," is examined. The notation, terminology, and basic concepts of Combinatorial Game Theory are assumed mostly without further comment. From time-to-time, however, certain basics are underscored for the sake of exposition. *Winning Ways for your Mathematical Plays* is highly recommended as an introduction to Combinatorial Game Theory. Other pertinent books include *Lessons in Play, An Introduction to Combinatorial Game Theory* [1] and *An Introduction to Combinatorial Game Theory* [6]. At the graduate level, *Combinatorial Game Theory* [11] is a standard text and reference.

Combinatorial Game Theory embodies two person games of perfect information. Well known games of this kind include Chess, Checkers, and Go. Our players are named *Left* and *Right*, and it is customary to think of *Left* as feminine and positive while *Right* is masculine and negative. The Fundamental Theorem of Combinatorial Game Theory asserts that "Either *Left* can force a win by playing first (on a game G) or else *Right* can force a win by playing second, but not both" ([11], p.9). Accordingly, the following questions are of interest: "Which player can force a win?" and "What is the winning move?" We assume the normal play convention according to which the player who makes the last legal move wins the game.

Eatcake is described and analyzed in *On Numbers and Games,* [3], p. 199-202; also, in [2], pp. 233 - 235. It is an example of a *dicotic game* – this means that both players can move from every nonempty sub-position of the game. (See [11], p. 60.) Every sub-position of a dicotic game is necessarily infinitesimal. In general, a game G is infinitesimal if $-\varepsilon < G < \varepsilon$ for every positive number ε.

The starting position for Bynum's game is an $n \times m$ grid of unit squares where $n \geq 1$ and $m \geq 1$. Any such grid is called a "cake" (with 1×1 as a special case). The rules are now illustrated by referring to the 6×8 starting position shown in Figure 1.1.

If *Left* is *First*, then she is required to completely remove any *column* from the starting position. Looking at Figure 1.1, she has removed the third column – thus splitting it into separate cakes, A and B. For the second move, *Right* must now choose either A or B and remove any *row*. (*Left* always takes *columns*; *Right* always takes *rows.*) In this case, *Right* chooses B and removes the fourth row. Now *Left* moves in either A, C, or D, etc. The game ends when no cakes remain, and the winner is the player who eats the last cake.

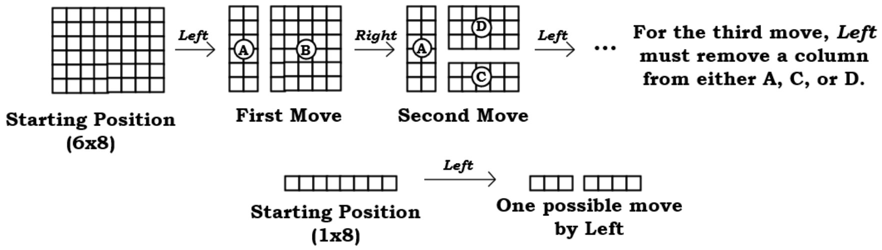


Figure 1.1 Top: Possible opening moves from the 6×8 starting position.
Bottom: A possible move by *Left* from the 1×8 position.

Any single row (or column) can appear as a position in Bynum's game. For example, at the bottom of Figure 1.1 we see that *Left* has moved in the 1×8 cake by taking the fourth square. However, a move by *Right* (from the starting position) would eliminate the entire row. This rather obvious position is pointed out only because individual rows or columns are eliminated as soon as they appear in our version of the game. (They become zero positions.)

In the above, it is clear that *Left* had 4 distinct options to start with. She started by eliminating the third column, but the same result would appear had she eaten the sixth column instead. Similarly, had *Right* been *First*, his removal of the second row (for example) would have the same meaning as the removal of the fifth row. Consequently, the expression "third column" will refer to the third column from either the left edge or the right edge. Similarly, "second row" will refer either to the second row from the top or the second from the bottom, etc.

Our present version of Bynum's game is played on an $n \times m$ grid with $n \geq 2$ and $m \geq 2$. Moves are made exactly as in Bynum's (original) game along with the following modification: if a single row or column becomes isolated by a move, then it is treated as a zero position (such leftovers become tainted!) In the above example, suppose *Left* had used the first move to take the second column instead of the third. In this case, the first column becomes a zero-position due to its isolation. Likewise, in Figure 1.1, consider *Left's* options for the third move. If she chooses A, then both columns are effectively removed from play. Finally, consider the $n \times 3$ (or $3 \times m$) case. At some point in the game, if *Right* opts to play in D, for example, then he can move in exactly two different ways. He can eat the first row (as in Bynum's game) or he can eat the second row in which case D completely disappears because the two remaining rows become isolated.

We denote an $n \times m$ starting position by $[n \times m]$. A *follower* of $[n \times m]$ is any game position that can appear after a number of moves have been made; and such positions consist of a number of cakes or *components*. Again, see Figure 1, for example. After two moves, the follower consists of components A, C, and D.

First, all values of $[2 \times m]$ for $m \geq 2$ are *uptimals*. Values of this kind were introduced by Conway and Ryba in the 1980's. This initial work was unpublished, but is often cited in books and research articles. In this regard, see McKay [8], p.210-214; and [11], p.95 ff. In particular, McKay [8] extended the theory of uptimals and provided computer code for computing their canonical forms.

In this paper, all values of $[n \times m]$ for $2 < n \leq 12$ and $2 < m \leq 12$ are expressed as "uptimal+ forms"; that is, ordinary uptimals increased (or decreased) by multiples of $\uparrow^{3/2}$. Ten sequences of infinitesimals are defined for this purpose, and considerable effort is devoted to establishing their properties. Notably, CGSuite output is translated into relatively tractable terms via these sequences.

The followers of $[n \times n]$, $12 < n \leq 21$, become prohibitively complex. (The $[18 \times 18]$ case, for example, requires 1,832 pages to print out.) Nevertheless, we are able to describe the values of all *square* positions in this range. In this regard, we encounter a peculiarity that we have been unable to explain: all $[n \times n]$ positions, $2 \leq n \leq 21$, are equal to $*$ *except* for $n=$ 10, 14, and 18. But, these exceptional cases have "star-like" characteristics. These are of the form $\pm(G)$, confused with both zero and up, and less than double-up.

2. Game-theoretic Values

The *left* and *right options* of $[n \times m]$, a position from Bynum's game (modified) are given by

$$\text{LeftOptions}\ ([n \times m]) = \{[n \times (m-i)] + [n \times (i-1)]\ \text{ for } 1 \leq i \leq a\} \tag{2.1}$$

and

$$\text{RightOptions } ([n \times m]) = \{[(n-i) \ \times m] + [(i-1) \times m] \ \text{ for } 1 \le i \le b\}$$

where a and b are the number of left and right options, respectively,

$$a = \ \lfloor (m+1)/2 \rfloor \ \text{ and b } = \lfloor (n+1)/2 \rfloor.$$

Here $[n \times 0] = [n \times 1] = [0 \times m] = \ [1 \times m] = \ 0$ (and $\lfloor x \rfloor$ is the greatest integer less than or equal to x). Since $[n \times m] = -[m \times n]$, the values are given only for $n \le m$.

Since the values of $[2 \times m]$, $m \ge 2$, are uptimals, we begin with a brief review of such terms. In addition, we briefly review fractional uptimals and atomic weight.

3. Uptimals, Fractional Uptimals, and Atomic Weight

3.1. The First Instances

The uptimals are defined in terms of the basic units *zero*, $0 = \{ \ \ | \ \ \}$; *star*, $* = \{0 \mid 0\}$; and *up*, $\uparrow = \{0 \mid *\}$. Specifically, they are given by

$$\uparrow^{[0]} = 0, \uparrow^{[1]} = \uparrow, \ldots, \uparrow^{[n]} = \{\uparrow^{[n-1]} \mid *\},\ n \ge 1;$$

$$\uparrow^{0} = 0, \uparrow^{1} = \uparrow, \ldots, \uparrow^{n} = \{\ 0 \mid \downarrow^{[n-1]} * \ \},\ n \ge 1.$$

The game $\uparrow^n$ is called "up-nth." And the negatives of these games are "down,"

$$\downarrow = -\uparrow = \{* \mid 0\};\ \ \downarrow^{[n]} = -\uparrow^{[n]} = \{* \mid \downarrow^{[n-1]}\};\ \text{and}\ \ \downarrow^{n} = -\uparrow^{n} = \{\uparrow^{[n-1]} * \mid 0\}.$$

The following (traditional) notation appears throughout: for any game G, we write $G* \ = G + *$ and $G \uparrow = G + \uparrow$, *etc.*

Uptimals are the first of the *dicotic games.* (Recall that such games have an option for both players at every position during the game.) An example of a game that is infinitesimal but *not* dicotic is $\{0|\{0|-2\}\}$ In this case, *Right* has an option in every non-zero sub-position, but *Left* has no option in $-2 = \{ \ |-1\}$.

A number of properties are now stated for future reference. Again, see [8] for a tidy presentation of proofs.

Lemma 3.1. *The sequences* $\uparrow^{[n]}$ *and* $\uparrow^{n}$ *are positive; also, they are increasing and decreasing, respectively.*

Lemma 3.2. *As canonical forms, we have*

$$\uparrow^{[n]} * = \{0, \uparrow^{[n-1]} * \mid 0\} \ \textit{and} \ \uparrow^{n} * = \{0, * \mid \downarrow^{[n-1]}\}.$$

Lemma 3.3. *The games* $\uparrow^{[n]} *$ *and* $\uparrow^{n} *$ *are confused with 0 (i.e., they are* fuzzy*).*

Lemma 3.4. *The games* $\uparrow^{[p]}$ *and* $\uparrow^{[q]} *$ *are confused for all nonnegative integers* p *and* q*.*

Theorem 3.5. *For each* $n \geq 2$, *we have* $\uparrow^{[n]} > m \cdot \uparrow^{n}$ *for all positive integers* m*. (Thus we say that "* $\uparrow^{n}$ *is* infinitesimal *with respect to* $\uparrow^{[n]}$*".)*

Lemma 3.6. *For* $n \geq 0$, *we have* $\uparrow^{[n]} - \uparrow^{[n-1]} = \uparrow^{n}$*, or, equivalently,*

$$\uparrow^{[n]} = \uparrow + \uparrow^{2} + \cdots + \uparrow^{n} .$$

3.2. Certain Generalizations

More generally, uptimals are given as follows.

Definition 3.7. An *uptimal* u is a game of the form

$$u = d_0 \cdot * + d_1 \cdot \uparrow + d_2 \cdot \uparrow^{2} + \ldots + d_n \cdot \uparrow^{n}$$

where d_0 is either 0 or 1 and $d_{\mathrm{i}} \in \mathbb{Z}$ for $i > 0$.

Following Conway and Ryba, the above expression is sometimes written as

$$u = .d_1 d_2 \ldots d_n *$$

where the additive $*$ may or may not be present.

An example follows. Let $G * +2 \cdot \uparrow +3 \cdot \uparrow^{2} + \uparrow^{3}$ and $H = - \uparrow -4 \cdot \uparrow^{2} +3 \cdot \uparrow^{3}$. These are represented by $G = .231*$ *and* $H = .\overline{1}\overline{4}3$ where accents on the digits of H indicate negation. As a sum of games, we merely add coefficients; e.g., $G + H = .231* + .\overline{1}\overline{4}3 = .1\overline{1}4*$. Naturally it follows that $-(G + H) = -G - H = .\overline{1}1\overline{4}*$, etc.

Now *fractional uptimals* are briefly described. For $G = \{G^L | G^R\}$ and $H = \{H^L | H^R\}$, their *ordinal sum* is defined by

$$G : H = \{G^L,\ G : H^L | G^R,\ G : H^R\}.$$

Among the first instances, we have

$$* : \mathrm{n} = \uparrow^{[n]} *, \quad n = 0,\ 1,\ 2,\ \ldots$$

Correspondingly, for dyadic rational numbers $\mathrm{x} \geq 0$, the *fractional uptimals* are defined by $\uparrow^{[x]} * = (*\!: x)$ and $\uparrow^{x+1} = \{\ 0 \mid \downarrow^{[x]} *\}$ along with $\downarrow^{[x]} = - \uparrow^{[x]}$ and $\downarrow^{x+1} = - \uparrow^{x+1}$.

The special cases $\uparrow^{[1/2]} = \{0 | *,\ \uparrow\}$ and $\uparrow^{3/2} = \{0 || 0,\ \downarrow * | 0,\ *\}$ are of particular importance in what follows. From Siegel [11], Exercise 4.24, p.99, it is seen that

$$\uparrow^{[1/2]} + \uparrow^{3/2} = \uparrow . \tag{3.8}$$

Equation (3.8) motivates most of our present work. Indeed, $\downarrow + \uparrow^{[1/2]} + \uparrow^{3/2}$ $(= 0)$ is the first instance of an uptimal+ form. In general, these forms are given by

$$d_0 \cdot * + d_1 \cdot \uparrow + d_2 \cdot \uparrow^2 + \ldots + d_n \cdot \uparrow^n + d_{n+1} \cdot \uparrow^{3/2} \tag{3.9}$$

(where the d_i are integers). All positional values in our present game are written in these terms.

3.3. Atomic Weight

The atomic weight of a game, $aw(\mathrm{G})$, is equal to the number of "ups" most closely approximating G; so, in particular, $aw(\uparrow) = 1$. See [2], p. 321, for the traditional presentation of atomic weight. In what follows, we simply state these values as provided by CGSuite.

To begin with, it is the case that $aw(*) = aw(\uparrow^r) = 0$ for all dyadic rational $r > 1$. Since $aw(\uparrow^{3/2}) = 0$, it follows from Equation (3.8) and the linearity of $aw(\cdot)$ that

$$aw(\uparrow^{[1/2]}) = aw(\uparrow + \uparrow^{3/2}) = 1.$$

Likewise, for any uptimal+ form, we have

$$aw(d_0 \cdot * + \ d_1 \cdot \uparrow + d_2 \cdot \uparrow^2 + \ldots \ + d_n \cdot \uparrow^n + \ d_{n+1} \cdot \uparrow^{3/2}) = d_1.$$

Atomic weight theory provides the following key result: for any infinitesimal G, it follows that

a. $aw(G) \geq 2$ implies $G > 0$,

b. $aw(G) \leq -2$ implies $G < 0$, otherwise,

c. if $-2 \leq aw(G) \leq 2$, then the unconditional winner is undetermined.

Experts have noted, however, that sharper tools than atomic weight are needed for extending the theory of all-small games. Here is a simple example: the distinction between $\uparrow^{[p]}$ and $\uparrow^{[q]}, p \neq q$, can be important. But that distinction is lost if we consider atomic weights only. In particular, we have $aw(\uparrow^{[p]}) = aw(\uparrow^{[q]}) = 1$ for $p, q > 0$.

3.4. Basic Properties of $\uparrow^{[1/2]}$ and $\uparrow^{3/2}$

Several inequalities involving $\uparrow^{[1/2]}$ and $\uparrow^{3/2}$ will be needed in what follows. While the proofs are routine, they are included for completeness; also, the author has not encountered some of these. In what follows, the expression $\mathrm{G} \xrightarrow{L} \mathrm{A} \xrightarrow{R} \mathrm{B}$ means "*Left* plays in G and chooses option A"; "*Right* plays in A and chooses B", etc.

Lemma 3.10. *It follows that*

(a) $\uparrow^{[1/2]} > 0$ *and* $\uparrow^{3/2} > 0$, *also,*

(b) $\uparrow^{[1/2]} \| *$ *and* $\uparrow^{3/2} \| *$.

Proof. (a) Clearly both $\uparrow^{[1/2]} \mathrel{|\!\triangleright} 0$ and $\uparrow^{[1/2]} \geq 0$ and, hence, $\uparrow^{[1/2]} > 0$. Likewise, it is clear (for the same reasons) that $\uparrow^{3/2} > 0$. The proof of (b) is also immediate and thus omitted. □

The next two theorems are easily verified by CGSuite, and we omit their proofs.

Theorem 3.11. *We have a.* $\uparrow^{[1/2]} * = \{0, * | 0, \uparrow *\}$ *and b.* $\uparrow^{3/2} * = \left\{0, * \middle| \downarrow^{[1/2]}\right\}$.

Theorem 3.12. *The following inequalities hold:* $\uparrow > \uparrow^{[1/2]} > \uparrow^{3/2} > \uparrow^{2} > 0$.

Theorem 3.13. *We have*

(a) $\uparrow^{[1/2]} > n \cdot \uparrow^{3/2}$ *for all* $n > 1$ *(i.e.,* $\uparrow^{3/2}$ *is infinitesimal with respect to* $\uparrow^{[1/2]}$*),*

(b) $\uparrow^{3/2} > n \cdot \uparrow^{2}$ *for all* $n > 1$ *(*$\uparrow^{2}$ *is infinitesimal with respect to* $\uparrow^{3/2}$*).*

Proof. We prove only (a), as the proof of (b) is similar. The case $n = 1$ was proven above. Hence, we set $\mathcal{D}_n = \uparrow^{[1/2]} + n \cdot \downarrow^{3/2}$. Suppose that *Left* is *First* in $\mathcal{D}_n$. Then she will choose $\uparrow^{[1/2]} + (n-1) \cdot \downarrow^{3/2} + \uparrow^{[1/2]} + *$, a game of atomic weight 2, and thus $\mathcal{D}_n \mathrel{|\!\triangleright} 0$.

If *Right* is *First* in $\mathcal{D}_n$, then his three options are (i) $* + n \cdot \downarrow^{3/2}$, (ii) $\uparrow + n \cdot \downarrow^{3/2}$, and (iii) $\uparrow^{[1/2]} + (n-1) \cdot \downarrow^{3/2}$. It follows that $(i) \| (ii)$ and $(ii) = (iii)$. Accordingly, we will have $(i) \xrightarrow{L} * + (n-1) \cdot \downarrow^{3/2} + \uparrow^{[1/2]} * = (n-1) \cdot \downarrow^{3/2} + \uparrow^{[1/2]} > 0$ (by induction) and $(ii) \xrightarrow{L} \uparrow + (n-1) \cdot \downarrow^{3/2} + \uparrow^{[1/2]} * > 0$ (a game of atomic weight 2). We conclude that $\mathcal{D}_n \geq 0$ and hence $\uparrow^{[1/2]} > n \cdot \uparrow^{3/2}$ for all $n > 1$. □

4. Values of $[n \times m]$ for $2 \leq n \leq 12$ and $2 \leq m \leq 12$

4.1. The $[2 \times m]$ Positions, $m \geq 2$

Now we simplify the notation by setting $T_m = [2 \times m]$ for $m \geq 2$ and $T_0 = T_1 = 0$. (Recall that the single right option of T_m is 0.) From Equation (2.1), the first six values of T_m are as follows:

- $T_2 = *$ since $\{T_1 | \, 0\} = \{0|0\}$,
- $T_3 = \uparrow^{[1]} *$ since $\{T_2, T_1 + T_1 | 0\} = \{*, 0 | 0\} = \uparrow * = \uparrow^{[1]} *$.

- $T_4 = \downarrow^2$ since $\{T_3, T_2 |\ 0\} = \{\uparrow^{[1]} *,\ *|0\} = \{\uparrow^{[1]} *|0\}$. Here $\uparrow^{[1]} *$ dominates $*$ as a left option. While $T_4 < 0$, all other T_n are seen to be *fuzzy.*
- $T_5 = \uparrow^{[2]} *$ since $\{T_4\ , T_3{+}T_1, T_2{+}T_2\} = \{\downarrow^2, \uparrow^{[1]} *, 0|0\} = \{\uparrow^{[1]} *, 0|0\} = \uparrow^{[2]} *$. (In this case, 0 dominates $\downarrow^2$ as a left option.)
- $T_6 = \uparrow^{[3]} *$; i.e., $\{T_5, T_4 + T_1, T_3 + T_2|0\} = \{\uparrow^{[2]} *, \downarrow^2, \uparrow |0\} = \{\uparrow^{[2]} *, \uparrow |0\}$.

($\uparrow$ dominates $\downarrow^2$ as a left option while $\uparrow^{[2]}\ *$ and $\uparrow$ are confused – see Lemma 3.4.) The last equation follows since $\uparrow$ reverses to 0 and thus $\{0,\ \uparrow^{[2]} *|0\} = \uparrow^{[3]}$.

One can likewise verify that $T_m = \uparrow^{[m-3]} *$ for $7 \le m \le 10$. While the details are muddied a bit by a quirky term, $T_4 = \downarrow^2$, they pose no difficulties. For $m > 10$, the terms "smooth out" and we get the following result.

Theorem 4.1. *For $m > 10$, we have $T_m = [2 \times m] = \uparrow^{[m-3]} *$.*

Proof. We proceed by induction: for fixed $m > 10$ and all n such that $10 < n < m$, we assume that $T_n = \uparrow^{\mathbf{[n-3]}} \boldsymbol{*}$. The proof that follows is for the case $m = 2k + 1$, $k \ge 5$. That for the even case is entirely similar (hence omitted). There are $k{+}1$ left options of T_m. And these are given as follows:

$$
\begin{aligned}
(1)&\ \ T_{m-1} = \uparrow^{[m-4]} *\\
(2)&\ \ T_{m-2} = \uparrow^{[m-5]} *\\
(3)&\ \ T_{m-3} + T_2 = \uparrow^{[m-6]}\\
(4)&\ \ T_{m-4} + T_3 = \uparrow^{[m-7]} + \uparrow\\
(5)&\ \ T_{m-5} + T_4 = \uparrow^{[m-8]} * + \downarrow^2\\
(6)&\ \ T_{m-6} + T_5 = \uparrow^{[m-9]} + \uparrow^{[2]}\\
(7)&\ \ T_{m-7} + T_6 = \uparrow^{[m-10]} + \uparrow^{[3]}\\
&\ \ \vdots\\
(k+1)&\ \ T_{m-k-1} + T_k = \uparrow^{[m-k-4]} + \uparrow^{[k-3]}\ .
\end{aligned}
$$

First, options (2), (3), (4), and (5) are dominated as left options. In particular, the following are true.

- Option (2) is dominated by (1) (see Lemma 3.1).
- Option (3) is dominated by (4), i.e., we have $\uparrow^{[m-7]} + \uparrow\ > \uparrow^{[m-6]}$ or, equivalently, $\uparrow^1 > \uparrow^{m-6}$ (recall that $\uparrow^n$ is a decreasing sequence).
- Both (4) and (5) are dominated by (6). This follows since

$$(4)\ \uparrow^{[m-7]} + \uparrow\ = .211\ldots1_{\text{m-7}},\quad (5)\ \uparrow^{[m-8]} + \downarrow^{2} = .101\ldots1_{\text{m-8}},\text{ and}$$
$$(6)\ \uparrow^{[m-9]} + \uparrow^{[2]} = .221\ldots1_{\text{m-9}}.$$

Now we dispatch with the remaining dominated options. The values of (6) through $(k+1)$ (inclusive) are all of the form $\uparrow^{[m-p]} + \uparrow^{[q]}$ where $p-q=7$ and $p=9,\ldots,k+1$. Now we claim that option $(k+1)$ dominates (6) thru (k), inclusive. In this regard, we rewrite option $(k+1)$ as

$$T_{m-k-1} + T_k = T_k + T_k = \uparrow^{[k-3]} + \uparrow^{[k-3]} = .22\cdots2_{k-3},$$

and the j^{th} predecessor of option $(k+1)$ as

$$T_{k+j} + T_{k-j} = \uparrow^{[(k-3)+j]} + \uparrow^{[(k-3)-j]} = .22\cdots2_{(k-3)-j}11\cdots1_{(k-j)+3},$$

from which the dominance follows. So far we have established that

$$T_{2k+1} = \{\ \uparrow^{[2k-3]} *, \uparrow^{[k-3]} + \uparrow^{[k-3]}\ | 0\ \}.$$

Now we claim that option $(k+1)$ reverses to 0. First, options (1) and $(k+1)$ are confused is equivalent to saying that

$$\text{“}\uparrow^{[2k-3]} + \downarrow^{[k-3]} + \downarrow^{[k-3]} \text{ is confused with } *\text{”}.$$

This follows since $\uparrow^{[2k-3]} + \downarrow^{[k-3]} + \downarrow^{[k-3]} = \overline{11}\cdots\overline{1}_{k-3}11\cdots1_{2k-3}$, and a result from Siegel [11], p. 96, confirms that the sum is confused with $*$.

Finally, we show that $\uparrow^{[k-3]} + \uparrow^{[k-3]}$ reverses to 0. If *Left* chooses this option, then *Right* can only respond with $\uparrow^{[k-3]} * = \{0, \uparrow^{[k-4]} * |\ 0\}$. Hence we must show that $\mathcal{D} = T_{2k+1} + \downarrow^{[k-3]} * \geq 0$. If *Right* is *First* in $\mathcal{D}$, then he has 3 options, namely,

$$(i)\ 0 + \downarrow^{[k-3]} *,\ (ii)\ T_{2k+1} + 0,\ \text{and}\ (iii)\ T_{2k+1} + \downarrow^{[k-4]} *.$$

But *Left* has winning moves in each case. In (i), she plays in $\downarrow^{[k-3]} *$ and chooses 0; in (ii) she chooses $\uparrow^{[k-3]} + \uparrow^{[k-3]} > 0$; and in (iii) she plays in T_{2k+1}, chooses $\uparrow^{[2k-3]} *$ and gets the overall result $\uparrow^{[2k-3]} * + \downarrow^{[k-4]} * = \uparrow^{[2k-3]} + \downarrow^{[k-4]} > 0$. This completes the proof. □

Proceeding with $3 \leq n \leq 12$ and $3 \leq m \leq 12$, several sequences of infinitesimals are defined. Their properties are described; positional values are written in terms of the sequences; and a couple of examples are given. First, two preliminaries:

Definition 4.2. A sequence of games G_n is *linear* if $G_{n+1} - G_n$ is constant; that is, the terms are independent of n for $n \geq 1$. Otherwise such a sequence is *nonlinear.*

Definition 4.3. A sequence of infinitesimals G_n is *virtually increasing* if $G_n < G_{n+1}*$ or, equivalently, $G_n* < G_{n+1}$, for $n \geq 1$. (*Virtually decreasing* is similarly defined.)

4.2. Linear Sequences of *Extended Uptimal Forms*

Four sequences follow ($\alpha,\ \beta,\ \gamma^a,\ and\ \gamma^b$), all of which are virtually increasing. In each case, the first differences are equal to $\uparrow *$.

Definition 4.4. The *alpha-sequence* is given by

$$\alpha_1 = \uparrow^{1/2} \text{ and } \alpha_n = \{0 \mid \alpha_{n-1}\} \text{ for } n \geq 2.$$

Definition 4.5. The *beta-sequence* is given by

$$\beta_1 = \uparrow^{3/2} \text{ and } \beta_n = \{0 \mid \beta_{n-1}\} \text{ for } n \geq 2.$$

Let us rewrite Equation (3.8) as $\alpha_1 = \uparrow - \uparrow^{3/2}$. (As mentioned earlier, this is the first example of an uptimal+ form.) Recall that $\uparrow^{3/2}$ is infinitesimal with respect to α_1.

Lemma 4.6. *We have $\alpha_n > 0$ and $\beta_n > 0$ for all $n \geq 1$.*

Proof. The case $n = 1$ appeared in Lemma 3.10. For $\alpha_n = \{0 \,|\alpha_{n-1}\}$, again $\alpha_n \mathrel{|\!\triangleright} 0$. We have $\alpha_{n-1} > 0$ by induction; hence $\alpha_n \geq 0$ and, consequently, $\alpha_n > 0$. The details for the beta sequence are similar (hence omitted). □

Lemma 4.7. *Further comparisons follow for the alpha and beta sequences:*

(a) *for $n \geq 1,$ we have $\alpha_n > \beta_n$;*

(b) *$\alpha_1 \| *$ and $\alpha_n > *$ for $n > 1$;*

(c) *$\alpha_1 < \uparrow$, $\alpha_2 \| \uparrow$, and $\alpha_n > \uparrow$ for $n \geq 3$;*

(d) *$\alpha_{n+1} \| \alpha_n$ for $n \geq 1$.*

Moreover, statements (b), (c), and (d) hold for the beta sequence as well.

Proof. (a) For $n = 1$, the inequality follows from Equation 3.8 and Lemma 4.6. Suppose $n > 1$ and set $\mathcal{D} = \alpha_n - \beta_n = \{0 \,|\alpha_{n-1}\} + \{-\beta_{n-1}|0\}$. If *Left* is *First* in $\mathcal{D}$, then $\mathcal{D} \xrightarrow{L} \alpha_n - \beta_{n-1}$ and *Right* can reply with either $\alpha_{n-1} - \beta_{n-1} > 0$ (by induction) or $\alpha_n - 0 > 0$. Thus $\mathcal{D} \mathrel{|\!\triangleright} 0$. Otherwise, if *Right* starts, then either

$$\mathcal{D} \xrightarrow{R} \alpha_{n-1} - \beta_n \xrightarrow{L} \alpha_{n-1} - \beta_{n-1} > 0 \text{ or } \mathcal{D} \xrightarrow{R} \alpha_n + 0 > 0$$

and hence $\mathcal{D} \geq 0$. This proves that $\alpha_n > \beta_n$ for $n \geq 1$.

(b) Recall that $\alpha_1 \parallel *$ (from Lemma 3.10). Now consider $\alpha_n > *$ for $n \geq 2$. Here, we need to show that $\mathcal{D}_n = \alpha_n + * = \{0 \mid \alpha_{n-1}\} + \{0 \mid 0\} > 0$. It is clear that $\mathcal{D}_n \mathrel{|\!\triangleright} 0$. Also, $\mathcal{D}_n \geq 0$ since either $\mathcal{D}_n \xrightarrow{R} \alpha_{n-1} + * > 0$ (by induction) or $\mathcal{D}_n \xrightarrow{R} \alpha_n > 0$. This proves the inequality.

(c) Again, the first inequality follows from Equation (3.8). For the second one, set $\mathcal{D}_2 = \alpha_2 + \Downarrow$. *Left's* winning move in $\mathcal{D}_2$ is $\alpha_2 + * > 0$. Otherwise, if *Right* is *First*, his winning move is to $\alpha_1 + \Downarrow < 0$ (from Lemma 3.12); hence $\alpha_2 || \uparrow$. The inequality $\alpha_n > \uparrow$ for $n \geq 3$ follows by a simple induction.

(d) Set $\mathcal{D}_n = \alpha_n - \alpha_{n+1} = \{0|\alpha_{n-1}\} + \{-\alpha_n\,|0\}$. In particular, for $n = 2$, $\mathcal{D}_2 = \alpha_1 - \alpha_2 = \{0|*, \uparrow\} + \{*, \downarrow|0||0\}$. We have $\mathcal{D}_2 \mathrel{|\!\triangleright} 0$ because $\mathcal{D}_2 \xrightarrow{L} \alpha_1 - \alpha_1 = 0$. Also, $\mathcal{D}_2 \mathrel{\triangleleft\!|} 0$ since $\mathrm{D}_2 \xrightarrow{R} * - \alpha_2 < 0$. Thus we have $\alpha_2 || \alpha_1$.

In the general case: again, we have $\mathcal{D}_n \mathrel{|\!\triangleright} 0$. Now, $\mathcal{D}_n \mathrel{\triangleleft\!|} 0$ since $\mathcal{D}_n \xrightarrow{R} \alpha_{n-1} - \alpha_{n+1}$ from which *Left* can respond with $0 - \alpha_{n+1} < 0$ or $\alpha_{n-1} - \alpha_n$. The latter is fuzzy by induction. In the above proof, $\alpha_{n-1} - \alpha_n \mathrel{\triangleleft\!|} 0$, and so $\alpha_n \,||\, \alpha_{n+1}$. ☐

The proofs of (b), (c), and (d) for the beta sequence are omitted.

Theorem 4.8. *The sequences α_n and β_n are virtually increasing.*

Proof. Set $n = 1$ and we show that $\alpha_2 * > \alpha_1$ or $\mathcal{D}_1 = \alpha_2 - \alpha_1 + * > 0$. We have $\mathcal{D}_1 \mathrel{|\!\triangleright} 0$ because $\mathcal{D}_1 \xrightarrow{L} \alpha_2 + * + * > 0$ (Lemma 4.6). Also, we have $\mathcal{D}_1 \geq 0$ because either

$$\mathcal{D}_1 \xrightarrow{R} \alpha_1 - \alpha_1 + * \xrightarrow{L} \alpha_1 - \alpha_1 = 0 \text{ or } \mathcal{D}_1 \xrightarrow{R} \alpha_2 + 0 + * > 0 \text{ (by Lemma 4.7b.)}$$

$$\text{or } \mathcal{D}_1 \xrightarrow{R} \alpha_2 - \alpha_1 + 0 \xrightarrow{L} \alpha_2 + * > 0.$$

Thus $\alpha_2 * > \alpha_1$. For $n \geq 2$, set $\mathcal{D}_n = \alpha_{n+1} * - \alpha_n = \{0|\alpha_n\} + \{-\alpha_{n-1}|0\} + *$. *Left* wins $\mathcal{D}_n$ by choosing $\alpha_{n+1} * - \alpha_{n-1}$ because *Right's* two options from the latter are

$$\alpha_n * - \alpha_{n-1} > 0 \text{ (by induction) or } \alpha_{n+1} * > 0.$$

Consequently, $\mathcal{D}_n \mathrel{|\!\triangleright} 0$. Also, we claim that $\mathcal{D}_n \geq 0$ because either

$$\mathcal{D}_n \xrightarrow{R} (\alpha_n *) - \alpha_n \xrightarrow{L} 0 \text{ or } \mathcal{D}_n \xrightarrow{R} \alpha_{n+1} * > 0.$$

Thus we have $\alpha_{n+1} * > \alpha_n$. ☐

Lemma 4.9. *For $n \geq 2$, $\alpha_n * = \{0 \mid \alpha_{n-1} *\}$ and $\beta_n * = \{0 \mid \beta_{n-1} *\}$.*

Proof. For $n = 1$, see Theorem 3.11. From definition, $\alpha_n + * = \{*, \alpha_n | \alpha_{n-1} *, \alpha_n\}$. *Left's* option $*$ is dominated by α_n (from Lemma 4.7(b)). Also, α_n reverses to 0 since

$$\alpha_n * \xrightarrow{L} \alpha_n \xrightarrow{R} \alpha_{n-1} \text{ and } \alpha_{n-1} < \alpha_n * . \text{ (from Theorem 4.8).}$$

Lastly, α_n is dominated by $\alpha_{n-1} *$ as a right option. The proof for $\beta_n *$ is omitted. ☐

Theorem 4.10. *The successive differences follow. We have $\alpha_n - \alpha_{n-1} = \uparrow *$ and $\beta_n - \beta_{n-1} = \uparrow *$, $n \geq 2$.*

Proof. We will consider the alpha case only. For $n = 2$, set

$$\mathcal{D}_2 = \ \alpha_2 - \alpha_1 + \ \downarrow * \ = \{0\ |\alpha_1\ \} + \{*,\ \downarrow |0\} + \{0|0,\ *\}.$$

Suppose *Left* is *First* in $\mathcal{D}_2$. Then she has four options:

$$(i)\ 0 - \alpha_1\ + \downarrow *,\ (ii)\ \alpha_2 + * + \downarrow *,\ (iii)\ \alpha_2 + \ \downarrow + \downarrow *, \text{ and } (iv)\ \ \alpha_2 - \alpha_1\ + 0.$$

One readily shows that (i), (ii), and (iii) are all dominated by (iv) as a left option. Moreover, *Right's* response to (iv) is immediate, that is, $(iv) \xrightarrow{R} \alpha_1 - \alpha_1 + 0 = 0$. We conclude that $\mathcal{D}_2 \leq 0$.

If *Right* is *First* in $\mathcal{D}_2$, then his options are (i) $\alpha_1 - \alpha_1 + \ \downarrow *$, (ii) $\alpha_2 + 0 + \downarrow *$, (iii) $\alpha_2 - \alpha_1 + 0$, and (iv) $\alpha_2 - \alpha_1\ + *$. In this case, (ii), (iii) *and* (iv) are all dominated as right options by (i), so we need only consider $(i) \xrightarrow{L} 0$ (where *Left* plays in $\downarrow *$). Consequently, $\mathcal{D}_2 \geq 0$, and we conclude that $\mathcal{D}_2 = 0$.

For $n > 2$, we set $\mathcal{D}_n = \ \alpha_n\ - \alpha_{n-1} + \ \downarrow * = \{0\ |\alpha_{n-1}\ \} + \{-\alpha_{n-2}\ |0\} + \{0|0,\ *\}$. Here, *Left's* options are

$$(i)\ 0 - \alpha_{n-1} + \ \downarrow *,\ (ii)\ \alpha_n - \alpha_{n-2} + \ \downarrow *, \ \text{ and } (iii)\ \alpha_n - \alpha_{n-1}\ + \ 0.$$

Now it happens that (iii) dominates both (i) and (ii) as left options. In the first case, $(iii) - (i) = \alpha_n + \ \uparrow *$ has two *Right* options, but *Left* wins both, that is,

$$\alpha_{n-1} + \ \uparrow * \xrightarrow{R} \alpha_{n-1} + 0\ > 0 \text{ and } \ \alpha_{n-1} + \ \uparrow *\ \ \alpha_n + 0\ > 0.$$

Additionally, (iii) is equal to (ii) by induction. Thus, we need only consider (iii). Since $(iii) \xrightarrow{R} \alpha_{n-1} - \alpha_{n-1}\ =\ 0$, it follows that $\mathcal{D}_n \leq 0$.

If *Right* is *First in* $\mathcal{D}_n$, then his options are given by

(i) $\downarrow *$ (a play in α_n), (ii) $\alpha_n\ + \ \downarrow *$ (a play in $-\alpha_{n-1}$),
(iii) $\alpha_n - \alpha_{n-1}$ (a play in $\downarrow *$), and (iv) $\alpha_n - \alpha_{n-1} + *$ (a play in $\downarrow *$).

It is left for the reader to show that option (i) dominates each of the other three as a right option. That done, we will have $(i) \xrightarrow{L} 0$ and, consequently, $\mathcal{D}_n \geq 0$. We have shown that $\mathcal{D}_n = 0$. □

Theorem 4.11. *For integers $n > m > 0$,*

$$\alpha_n - \ \alpha_m = \begin{cases} (n-m)\cdot \uparrow + * & \text{if } n \text{ and } m \text{ are of opposite parity} \\ (n-m)\cdot \uparrow & \text{if } n \text{ and } m \text{ are of same parity} \end{cases}.$$

Proof. From Theorem 4.10,

$$\sum_{i=2}^{n} (\alpha_i - \alpha_{i-1}) = (n-1) \cdot (\uparrow *)$$

$$\text{and } \sum_{i=2}^{n}(\alpha_i - \alpha_{i-1}) = \sum_{i=2}^{m}(\alpha_i - \alpha_{i-1}) + \sum_{i=m+1}^{n}(\alpha_i - \alpha_{i-1})$$
$$= (m-1) \cdot (\uparrow *) + (\alpha_n - \alpha_m)\,,$$

from which the result follows. □

Neil A. McKay responded to an earlier version of this work by suggesting the following equation.

Corollary 4.12. *It follows that* $\alpha_n = \begin{cases} (n-1)\cdot \uparrow + \, \alpha_1 & \text{if } n \text{ is odd} \\ (n-1)\cdot \uparrow + \, \alpha_1 * & \text{if } n \text{ is even.} \end{cases}$

Proof. In Theorem 4.11, set m = 1. □

When $\alpha_1 = \uparrow - \uparrow^{3/2}$ is applied to Corollary 4.12 we obtain the following.

Corollary 4.13. *The Uptimal+ Form is given by*

$$\alpha_n = \begin{cases} n\cdot \uparrow - \uparrow^{3/2} & \text{if } n \text{ is odd} \\ * + \; n\cdot \uparrow - \uparrow^{3/2} & \text{if } n \text{ is even.} \end{cases}$$

We mentioned that $aw(\uparrow^{3/2}) = 0$, so it follows that $aw\,(\alpha_n) = n$.

Corollary 4.14. *It follows that* $\alpha_1 < \uparrow < \; \alpha_2 * < 2\cdot \uparrow < \alpha_3 < 3\cdot \uparrow \cdots$.

Proof. Let n be an odd positive integer. From Corollary 4.13, we have $\alpha_n \; = \; n\cdot \uparrow - \; \uparrow^{3/2} \; < \; n\cdot \uparrow$ (since $\uparrow^{3/2} > 0$). If n is even, then

$$\alpha_n = \; *\!+ n\cdot \uparrow - \; \uparrow^{3/2} = \; *\!+ (n-1)\cdot \uparrow + \left(\uparrow - \; \uparrow^{3/2}\right) > \; *\!+ (n-1)\cdot \uparrow$$

since $\uparrow - \; \uparrow^{3/2} > 0$. □

Exactly the same inequalities (as in Corollary 4.14) hold for the β-sequence. Moreover, the same reasoning that led to Corollary 4.13 can be used to provide a similar statement for the β-sequence (the details are omitted).

Proposition 4.15. *The Uptimal+ Form is given by*

$$\beta_n = \begin{cases} (n-1)\cdot \uparrow + \uparrow^{3/2} & \text{if } n \text{ is odd} \\ * + \; (n-1)\cdot \uparrow + \uparrow^{3/2} & \text{if } n \text{ is even.} \end{cases}$$

Here we have $aw(\beta_n) = n - 1$.

Definition 4.16. The *gamma-a* sequence is given by

$$\gamma_1^a = \{0 \mid \alpha_1, \alpha_2\} \text{ and } \gamma_n^a = \{0 \mid \gamma_{n-1}^a\} \text{ for } n \geq 2.$$

Lemma 4.17. *It follows that*

(a) $\gamma_n^a > 0$ *and* $\gamma_n^a > *$ *for* $n \geq 1$,

(b) $\gamma_1^a \parallel \uparrow$ *while* $\gamma_n^a > \uparrow$ *for* $n \geq 2$,

(c) $\gamma_n^a * > \alpha_n$, $n \geq 1$.

Proof. (a) The first inequality is obvious. For the latter, set $\mathcal{D}_n = \gamma_n^a + *$. For $n = 1$, we have $\mathcal{D}_1 \xrightarrow{L} \gamma_1^a + 0 > 0$. But if Right starts, then

$$[\mathcal{D}_1 \xrightarrow{R} \alpha_1 + * \xrightarrow{L} \alpha_1 > 0] \text{ or } [\mathcal{D}_1 \xrightarrow{R} \alpha_2 + * \xrightarrow{L} \alpha_2 > 0] \text{ or } [\mathcal{D}_1 \xrightarrow{R} \gamma_1^a > 0];$$

consequently, $\gamma_1^a > *$. For $n > 1$, $\mathcal{D}_n \xrightarrow{R} \gamma_n^a > 0$. Otherwise, we have

$$\mathcal{D}_n \xrightarrow{R} \gamma_{n-1}^a + * = 0 \text{ (by induction) or } \mathcal{D}_n \xrightarrow{R} \gamma_n^a > 0.$$

This shows that $\gamma_n^a > *$ for all $n \geq 1$.

(b) Set $\mathcal{D}_n = \gamma_n^a + \downarrow$. For the first relation, we have $\mathcal{D}_1 \xrightarrow{L} \gamma_1^a + * > 0$ (by part (a)) and $\mathcal{D}_1 \xrightarrow{R} \alpha_1 + \downarrow < 0$ (since $\alpha_1 + \uparrow^{3/2} = \uparrow$); consequently, $\gamma_1^a \parallel \uparrow$. For the second inequality, we first consider $\mathcal{D}_2 = \gamma_2^a + \downarrow$. In this case, *Left's* winning move is $\mathcal{D}_2 \xrightarrow{L} \gamma_2^a + * > 0$. If *Right* is *First* in $\mathcal{D}_2$, then either $\mathcal{D}_2 \xrightarrow{R} \gamma_1^a + \downarrow \xrightarrow{L} \gamma_1^a + * > 0$ or $\mathcal{D}_2 \xrightarrow{R} \gamma_2^a + 0 > 0$. This shows that $\gamma_2^a > \uparrow$. Finally, we show that $\gamma_n^a > \uparrow$ for all $n \geq 2$. Similarly, *Left's* winning move is $\mathcal{D}_n \xrightarrow{L} \gamma_n^a + * > 0$. Otherwise,

$$\mathcal{D}_n \xrightarrow{R} \gamma_{n-1}^a + \downarrow > 0 \text{ (by induction) or } \mathcal{D}_n \xrightarrow{R} \gamma_n^a + 0 > 0.$$

We conclude that $\gamma_n^a > \uparrow$ for all $n \geq 2$.

(c) For $n \geq 1$, we prove that $\mathcal{D}_n > 0$ where

$$\mathcal{D}_n = \gamma_n^a - \alpha_n + * = \{0 | \gamma_{n-1}^a\} + \{-\alpha_{n-1} \mid 0\} + \{0|0\}.$$

First, we claim that $\mathcal{D}_1 \,|\!\triangleright\, 0$ where $\mathcal{D}_1 = \{0 \mid \alpha_1, \alpha_2\} + \{*, \downarrow | 0\} + \{0 \mid 0\}$. If *Left* is *First*, then she will choose $*$ in $-\alpha_1$ and the result is $\gamma_1^a > 0$ (from part (a)). Now suppose *Right* is *First.* Then his options are (i) $\alpha_1 - \alpha_1 + * \xrightarrow{L} 0$, (ii) $\alpha_2 - \alpha_1 + * > 0$ (from $Theorem$ 4.8), (iii) $\gamma_n^a + 0 + * > 0$ (from part (a)), and (iv) $\gamma_1^a - \alpha_1 + 0 \xrightarrow{L} \gamma_1^a + * > 0$. Hence $\mathcal{D}_1 \geq 0$ and the foregoing shows that $\gamma_1^a * > \alpha_1$.

If *Left* is *First* in $\mathcal{D}_n$ $(n \geq 2)$, then she will choose $\gamma_n^a - \alpha_{n-1} + *$ from which *Right* has three options:

(i) $\gamma_{n-1}^a - \alpha_{n-1} + * > 0$ (by induction), (ii) $\gamma_n^a + 0 + * > 0$ (from part (a)), and

$$(iii)\ \gamma_n^a - \alpha_n.$$

Left will respond to (iii) with $\gamma_n^a - \alpha_{n-1}$. By induction, it follows that

$$\gamma_n^a - \alpha_{n-1} \xrightarrow{R} \gamma_{n-1}^a - \alpha_{n-1} = (\gamma_{n-1}^a *) \ - \alpha_{n-1} + * \xrightarrow{L} (\gamma_{n-1}^a *) \ - \alpha_{n-1} > 0$$

$$\text{or } \gamma_n^a - \alpha_{n-1} \xrightarrow{R} \gamma_n^a + 0 > 0.$$

This shows that $\mathcal{D}_n \mathrel{|\!\rhd} 0$. Now let *Right* be *First*. In this case,

$$\mathcal{D}_n \xrightarrow{R} \gamma_{n-1}^a - \alpha_n + \ * \xrightarrow{L} \gamma_{n-1}^a - \alpha_{n-1} \ + \ * \ > 0 \text{ (by induction)}$$

$$\text{or } \mathcal{D}_n \ \xrightarrow{R} \gamma_n^a + 0 + \ * > 0 \text{ (by part (a))}$$

$$\text{or } \mathcal{D}_n \xrightarrow{R} \gamma_n^a \ - \alpha_n + \ 0 \xrightarrow{L} \gamma_n^a \ - \alpha_{n-1} \ .$$

From the last position, we have either $\gamma_n^a - \alpha_{n-1} \xrightarrow{R} \gamma_{n-1} - \alpha_{n-1} = (\gamma_{n-1}^a *) - \alpha_{n-1} + *$ (from which *Left* plays in $*$ and wins by induction) or $\gamma_n^a - \alpha_{n-1} \xrightarrow{R} \gamma_n^a + 0 > 0$. We now have $\mathcal{D}_n \geq 0$ and thus $\mathcal{D}_n > 0$. We conclude that $\gamma_n^a * > \alpha_n$. ☐

Theorem 4.18. *The sequence* γ_n^a *is virtually increasing.*

Proof. Set $\mathcal{D}_n = \gamma_n^a - \gamma_{n-1}^a + * = \{0 \mid \gamma_{n-1}^a\} + \{-\gamma_{n-2}^a \mid 0\} + \{0 \mid 0\}$. For $\gamma_2^a > \gamma_1^a *$, we have $\mathcal{D}_2 = \gamma_2^a - \gamma_1^a + * = \{0 \mid \gamma_1^a\} + \{-\alpha_1, -\alpha_2 \mid 0\} + \{0 \mid 0\}$. If *Left* starts in $\mathcal{D}_2$, then her winning move is to $\gamma_2^a - \alpha_1 + *$ because *Right* can only respond to the latter with

$$(i)\ \gamma_1^a - \alpha_1 + * \ \text{ or } \ (ii)\ \gamma_2^a + 0 + * \ \text{ or } \ (iii)\ \gamma_2^a - \alpha_1 + 0.$$

But (i) and (ii) are positive (Lemma 4.17 (a), (c)). Also, (iii) is positive since $\gamma_2^a - \alpha_1 > \ (\alpha_2 *) - \alpha_1 > 0$ by Lemma 4.17 (c) and the fact that α_n is virtually increasing. It follows that $\mathcal{D}_2 \mathrel{|\!\rhd} 0$. Now suppose *Right* is *First* in $\mathcal{D}_2$. Then his options are

$$(i)\ \gamma_1^a - \gamma_1^a + \ *,\ \ (ii) \quad \gamma_2^a + \ 0 + \ *, \ \text{ and } \ (iii) \quad \gamma_2^a - \gamma_1^a + \ 0.$$

Left clearly wins (i). And from from Lemma 4.17 (a), it follows that (iii) dominates (ii) as a *Right* option. It is left for the reader to show that *Left* can win (iii). This done we will have $\mathcal{D}_2 \geq 0$ and, consequently, $\gamma_2^a > \ \gamma_1^a *$.

Now we claim that $\gamma_n^a > \gamma_{n-1}^a *$ for $n \geq 3$. If *Left* is *First* in $\mathcal{D}_n$, then her winning move is to $\gamma_n^a - \gamma_{n-2}^a + *$ since *Right*'s options from the latter are

$$(i) \quad \gamma_{n-1}^a - \gamma_{n-2}^a + \ * \ , \ \ (ii) \quad \gamma_n^a + \ 0 + \ * \ \ \textit{and}\ (iii)\ \ \gamma_n^a - \gamma_{n-2}^a + 0,$$

none of which will provide him with a winning move. Again, (iii) dominates (ii) as a right option. *Left* wins (i) by induction; and following (iii), *Left* will respond with

$$\gamma_{n-1}^a - \gamma_{n-3}^a = (\gamma_{n-1}^a + \gamma_{n-2}^a *) + (\gamma_{n-2}^a * - \gamma_{n-3}^a) > 0 \ \text{ (again by induction)}\,.$$

Thus we have $\mathcal{D}_n |\!\!\triangleright 0$. Now let *Right* move first in $\mathcal{D}_n$. Then he has three options, namely,

$$(i)\ \mathcal{D}_n \xrightarrow{R} \gamma^a_{n-1} + \gamma^a_{n-1} + *,\ (ii)\ \mathcal{D}_n \xrightarrow{R} \gamma^a_n + 0 + * \text{ and } (iii)\ \mathcal{D}_n \xrightarrow{R} \gamma^a_n + \gamma^a_{n-1} + 0.$$

Here (ii) dominates (iii) as a right option. And the results of (i) and (ii) are positive from Lemma 4.17 (a). Hence $\mathcal{D}_n \geq 0$ and we conclude that $\gamma^a_n > \gamma^a_{n-1}*$ $(n \geq 3)$. □

Lemma 4.19. *We have* $\gamma^a_1 * = \{0 \mid \alpha_1 *, \alpha_2 *\}$; *and, for* $n \geq 2$, $\gamma^a_n * = \{0 \mid \gamma^a_{n-1} *\}$.

Proof. By definition, $\gamma^a_1 + * = \{*, \gamma^a_1 \mid \gamma^a_1, \alpha_1 *, \alpha_2 *\}$. First, γ^a_1 is dominated by $\alpha_1 *$ as a right option (by Lemma 4.17 c). It is also the case that the left options $*$ and γ^a_1 reverse to 0 (the details are omitted).

Now let $n \geq 2$, so that $\gamma^a_n + * = \{*, \gamma^a_n \mid \gamma^a_{n-1} *,\ \gamma^a_n\}$. In this case, $\gamma^a_{n-1} *$ dominates γ^a_n as a right option (since γ^a_n is virtually increasing). Also, γ^a_n dominates $*$ as a left option. Finally, γ^a_n reverses to 0 since $(\gamma^a_n + *) \xrightarrow{L} \gamma^a_n \xrightarrow{R} \gamma^a_{n-1} \xrightarrow{L} 0$ and $\gamma^a_n * > \gamma^a_{n-1}$. □

The lemma that follows is stated without proof.

Lemma 4.20. *The first term of* γ^a_n *is given by*

$$\gamma^a_1 = \ 2 \cdot \alpha_1 + \ * \ = 2 \cdot \uparrow + 2 \cdot \downarrow^{3/2} + \ *.$$

Theorem 4.21. *For* $n \geq 2$, *we have* $\gamma^a_n - \gamma^a_{n-1} = \uparrow *$.

Proof. Set $\mathcal{D}_n = \gamma^a_n - \gamma^a_{n-1} + \downarrow * = \{0 \mid \gamma^a_{n-1}\} + \{-\gamma^a_{n-2} \mid 0\} + \{0 \mid 0, *\}$.
Suppose *Left* moves first in

$$\mathcal{D}_2 = \gamma^a_2 - \gamma^a_1 + \downarrow * = \{0 \mid \gamma^a_1\} + \{-\alpha_1, -\alpha_2 \mid 0\} + \{0 \mid 0\}.$$

Then she has four options:

$$(i)\ 0 - \gamma^a_1 + \downarrow *,\ \ (ii)\ \gamma^a_2 - \alpha_1 + \downarrow *,\ \ (iii)\ \gamma^a_2 - \alpha_2 + \downarrow *,\ \text{and}\ (iv)\ \gamma^a_2 - \gamma^a_1 + 0.$$

However, option (iv) dominates the first three of these as a left option.

To begin with, $(iv) - (i) = \gamma^a_2 + \uparrow * \geq 0$ because

$$\begin{aligned}
(\gamma^a_2 + \uparrow *) \xrightarrow{R} (\gamma^a_1 + \uparrow *) \xrightarrow{L} (\gamma^a_1 + 0) > 0 \text{ or } (\gamma^a_2 + \uparrow *) &\xrightarrow{R} (\gamma^a_2 + 0) > 0,\\
(iv) - (ii) &= -\gamma^a_1 + \alpha_1 + \uparrow = -\alpha_1 + \uparrow > 0 \text{ (by Theorem 3.12), or}\\
(iv) - (iii) &= -\gamma^a_1 + \alpha_2 + \uparrow * = \alpha_2 - 2 \cdot \alpha_1 + \uparrow\\
&= (\alpha_2 - \alpha_1) - (\alpha_1 + \downarrow) = \uparrow * + \beta_1 \text{ (from Theorem 4.10)}\\
&= (\beta_2 - \beta_1) + \beta_1 = \beta_2 > 0.
\end{aligned}$$

Thus we need only consider (iv) $\gamma_2^a - \gamma_1^a \xrightarrow{R} \gamma_1^a - \gamma_1^a = 0$. This proves that $\mathcal{D}_2 \leq 0$. If *Right* is *First* in $\mathcal{D}_2$, then he also has four options:

$$(i)\ \gamma_1^a - \gamma_1^a + \downarrow *, \quad (ii)\ \gamma_2^a + 0 + \downarrow *, \quad (iii)\ \gamma_2^a - \gamma_1^a + 0, \text{ and } (iv)\ \gamma_2^a - \gamma_1^a + *.$$

In this case, it is left for the reader to show that option (i) is the single, dominant right option. This done, we will have (i) $\downarrow * \xrightarrow{L} 0$ so that $\mathcal{D}_1 \geq 0$. This establishes that $\mathcal{D}_1 = 0$.

In general, *Left's* options for $n \geq 3$ are

$$(i)\ -\gamma_n^a - \gamma_{n-2}^a + \downarrow *, \quad (ii)\ \gamma_n^a - \gamma_{n-2}^a + \downarrow *, \text{ and } (iii)\ \gamma_n^a - \gamma_{n-1}^a.$$

Here we claim that (ii) dominates both (i) and (iii) as a left option since

$$\begin{aligned}(ii) - (i) = \gamma_n^a + (\gamma_{n-1}^a - \gamma_{n-2}^a) &= \gamma_n^a + \uparrow * \quad \text{(by induction)}\\ &\geq \alpha_n * + \uparrow \quad \text{(by Lemma 4.17(c))}\\ &> 0,\\ (ii) - (iii) = \gamma_{n-1}^a - \gamma_{n-2}^a + \downarrow * &= 0 \quad \text{(by induction)}.\end{aligned}$$

Finally, (ii) $\gamma_n^a - \gamma_{n-2}^a + \downarrow * \xrightarrow{R} \gamma_{n-1}^a - \gamma_{n-2}^a + \downarrow * = 0$, Thus $\mathcal{D}_n \leq 0$. In the proof that $\mathcal{D}_n \geq 0$, there are four right options among which $\downarrow *$ is dominant (the details are omitted). Thus $\downarrow * \xrightarrow{L} 0$, so we declare that $\gamma_n^a - \gamma_{n-1}^a = \ \uparrow *$ for $n \geq 2$. □

A proof of the following is omitted since it is similar to that of Theorem 4.11.

Corollary 4.22. *The Uptimal+ Form is given by*

$$\gamma_n^a = \begin{cases} * + \ (n+1)\cdot \uparrow - \ 2\cdot \uparrow^{3/2} & \text{if } n \text{ is odd}\\ (n+1)\cdot \uparrow - \ 2\cdot \uparrow^{3/2} & \text{if } n \text{ is even.}\end{cases}$$

Thus we see that $aw\,(\gamma_n^a) = n+1$. Now, stated without proof, we have the following.

Corollary 4.23. *Upper and lower bounds on* γ_n^a *are given by*

(a) $(n-1)\cdot \uparrow < \gamma_n^a < (n+1)\cdot \uparrow$ *if n is even,*

(b) $(n-1)\cdot \uparrow < \gamma_n^a < (n+1)\cdot \uparrow + *$ *if n is odd.*

The three sequences that we have seen thus far are related by the following corollary. (Again, the details are omitted.)

Corollary 4.24. *For* $n \geq 1$ *and* $m \geq 1$, *we have*

(a) $\beta_n + \gamma_m^a \ = \alpha_{n+m}$,

(b) $\alpha_n + \alpha_m = \gamma_{n+m-1}^a + *$.

Definition 4.25. The *gamma-b* sequence is given by

$$\gamma_1^b = \{0 \mid \gamma_1^a, \gamma_2^a\} \text{ and } \gamma_n^b = \{0 \mid \gamma_{n-1}^b\} \text{ for } n \geq 2.$$

Lemma 4.26. *For all $n \geq 1$, we have*

(a) $\gamma_n^b > 0$ *and* $\gamma_n^b > *$,

(b) $\gamma_n^b > \uparrow$,

(c) $\gamma_n^b > \gamma_n^a * > 0$.

Proof. We submit the proof of part (c) only. Accordingly, set

$$\mathcal{D}_n = \gamma_n^b - \gamma_n^a * = \{0 \mid \gamma_{n-1}^b\} + \{-\gamma_{n-1}^a * \mid 0\};$$

so, in particular, $\mathcal{D}_1 = \{0 \mid \gamma_1^a, \gamma_2^a\} + \{-\alpha_1 *, -\alpha_2 * \mid 0\}$. For $n = 1$, *Left's* winning is

$$\mathcal{D}_1 \xrightarrow{L} \{0 \mid \gamma_1^a, \gamma_2^a\} + \{0, \downarrow * \mid 0, *\}. \text{ (Recall that } -\alpha_1 * = \{0, \downarrow * \mid 0, *\}.)$$

This follows since *Right*'s options in the latter are

$$(i)\ \gamma_1^a - \alpha_1 *, \quad (ii)\ \gamma_2^a - \alpha_1 *, \quad (iii)\ \gamma_1^b + 0, \text{ and } (iv)\ \gamma_1^b + *.$$

Left wins the first two of these by choosing 0 in $-\alpha_1 *$, and (iii) is positive from part (a). Additionally, *Left* wins (iv) by playing in $*$. Hence $\mathcal{D}_1 \mathrel{|\!\triangleright} 0$.

If *Right* is *First* in $\mathcal{D}_1$, then his options are (i) $\gamma_1^a - \gamma_1^a *$, (ii) $\gamma_2^a - \gamma_1^a *$, and (iii) $\gamma_1^b + 0$. *Left* obviously wins (i) and (iii). And he wins (ii) since γ_n^a is virtually increasing. Thus we have $\mathcal{D}_1 \geq 0$ and hence $\gamma_1^b > \gamma_1^a *$.

If *Left* moves first in $\mathcal{D}_n$, then we will have $\mathcal{D}_n \xrightarrow{L} \gamma_n^b - \gamma_{n-1}^a *$. Now *Right* can reply with $\gamma_{n-1}^b - \gamma_{n-1}^a * > 0$ (by induction) or $\gamma_n^b > 0$. Hence $\mathcal{D}_n \mathrel{|\!\triangleright} 0$. Otherwise, if *Right* is *First*, then we can have

$$\mathcal{D}_n \xrightarrow{R} \gamma_{n-1}^b - \gamma_n^a * \xrightarrow{L} \gamma_{n-1}^b - \gamma_{n-1}^a * = 0 \text{ (by induction) or } \mathcal{D}_n \xrightarrow{R} \gamma_n^b > 0.$$

This shows that $\mathcal{D}_n \geq 0$ and, finally, $\gamma_n^b > \gamma_n^a *$. □

Theorem 4.27. *The sequence γ_n^b is virtually increasing.*

Proof. We claim that $\mathcal{D}_n = \gamma_n^b - \gamma_{n-1}^b + * > 0$. For $n = 2$, we have

$$\mathcal{D}_2 = \gamma_2^b - \gamma_1^b + * = \{0 \mid \gamma_1^b\} + \{-\gamma_1^a, -\gamma_2^a \mid 0\} + *.$$

First, $\mathcal{D}_2 \mathrel{|\!\triangleright} 0$ because *Left* will choose $\gamma_2^b - \gamma_2^a + * > 0$ (where the inequality follows from Lemma 4.26 c.) Otherwise, if *Right* is *First*, then his options are

$$(i)\ \gamma_1^b - \gamma_1^b + * \xrightarrow{L} 0, \ (ii)\ \gamma_2^b - 0 + * > 0 \text{ and } (iii)\ \gamma_2^b - \gamma_1^b + 0 \xrightarrow{L} 0.$$

Thus $\mathcal{D}_2 \geq 0$ and, consequently, $\gamma_2^b > \gamma_1^b *$. Suppose that $n \geq 3$. In this case, *Left's* winning move is given by

$$\mathcal{D}_n \xrightarrow{L} \gamma_n^b - \gamma_{n-2}^b + * = (\gamma_n^b - \gamma_{n-1}^b) + (\gamma_{n-1}^b - \gamma_{n-2}^b *)$$

in which $\gamma_n^b - \gamma_{n-1}^b \geq 0$ and $\gamma_{n-1}^b - \gamma_{n-2}^b * > 0$. (The first inequality is immediate, and the second one follows by induction.) Therefore $\mathcal{D}_n \mathrel{|\!\triangleright} 0$. It is easily shown that *Right* has no winning moves in $\mathcal{D}_n$. Thus $\mathcal{D}_n \geq 0$ and the proof is done. □

The following lemma is easily verified by CGSuite, so its proof is omitted.

Lemma 4.28. *We have* $\gamma_1^b * = \{0 \mid \gamma_1^a *, \gamma_1^a *\}$. *And, for* $n \geq 2$, $\gamma_n^b * = \{0 \mid \gamma_{n-1}^b *\}$.

Lemma 4.29. *The first term of* γ_n^b *is given by*

$$\gamma_1^b = \gamma_1^a + \alpha_1 * = 3\cdot\uparrow + 3\cdot\downarrow^{3/2}.$$

Proof. Now we set

$$\mathcal{D} = \gamma_1^b - \gamma_1^a - \alpha_1 * = \{0 \mid \gamma_1^a, \gamma_2^a\} + \{-\alpha_1, -\alpha_2 \mid 0\} + \{0, \downarrow * \mid 0, *\}$$

and show that $\mathcal{D} = 0$. If *Left* is *First*, then she will not play in γ_1^b because, if she does, then *Right* will choose 0 in $-\alpha_1 *$. Thus, we consider *Left's* other options:

$$(i)\ \gamma_1^b - \alpha_1 - \alpha_1 *,\ (ii)\ \gamma_1^b - \alpha_2 - \alpha_1 *,\ (iii)\ \gamma_1^b - \gamma_1^a + 0, \text{ and } (iv)\ \gamma_1^b - \gamma_1^a + \downarrow *.$$

Here, (i) is confused with (ii) from Lemma 4.7 (d). Also, (i) is equal to (iii) from Lemma 4.20; and (ii) is equal to (iv) due to Corollary 4.13 and Lemma 4.20. Thus we need only consider

$$(i) \xrightarrow{R} \gamma_1^a - \alpha_1 - \alpha_1 * = 0 \text{ (by Lemma 4.20) and}$$

$$(ii) \xrightarrow{R} \gamma_2^a - \alpha_2 - \alpha_1 * = 0 \text{ (by Corollary 4.13 and Lemma 4.20).}$$

Thus $\mathcal{D} \leq 0$. Now suppose *Right* is *First*. Then his options in $\mathcal{D}$ are as follows:

$$(i)\ -\alpha_1 *,\ (ii)\ \gamma_2^a - \gamma_1^a - \alpha_1 *,\ (iii)\ \gamma_1^b + 0 - \alpha_1 *,$$

$$(iv)\ \gamma_1^b - \gamma_1^a + 0, \text{ and } (v)\ \gamma_1^b - \gamma_1^a + *.$$

In this case, (i) clearly dominates (iii) as a right option. Additionally, (i) dominates both (iv) and (v) as right options. In particular,

$$\begin{aligned} (i) - (iv) &= \alpha_1 - \gamma_1^b \text{ (by Lemma 4.20)} \\ &< \alpha_1 - \gamma_1^a * \text{ (by Lemma 4.26 (c))} \\ &< 0 \text{ (by Lemma 4.17 (c))} \\ \text{and } (i) - (v) &= -3\cdot\alpha_1 - \gamma_1^b * \text{ (by Lemma 4.20)} \\ &< -3\cdot\alpha_1 - \gamma_1^a < 0. \end{aligned}$$

However, (i) and (ii) are confused so we need to observe that

$$(i) \xrightarrow{L} 0 \text{ and } (ii) \xrightarrow{L} \gamma_2^a - \alpha_2 \ - \alpha_1 * \ = 0 \text{ (by using Corollaries 4.12 and 4.22).}$$

Now we have $\mathcal{D} \geq 0$ and, hence, $\gamma_1^b = \gamma_1^a + \alpha_1 *$. The second equation in the lemma follows from substitution by using previously established uptimal+ forms. □

It is fairly obvious that the method we used to obtain the first differences, uptimal+ forms, and so on, for all previous sequences can be used here as well. Consequently, further properties of the γ^b- sequence and of the six sequences defined in the sequel are all stated without proof.

Theorem 4.30. *It follows that* $\gamma_n^b \ - \gamma_{n-1}^b = \uparrow *$ *for all* $n \geq 2$.

Corollary 4.31. *The Uptimal+ Form is given by*

$$\gamma_n^b = \begin{cases} (n+2)\cdot \uparrow -3\cdot \uparrow^{3/2} & \text{if } n \text{ is odd} \\ * + \ (n+2)\cdot \uparrow - \ 3\cdot \uparrow^{3/2} & \text{if } n \text{ is even.} \end{cases}$$

Hence $aw(\gamma_n^b) = \ n+2$.

4.3. Nonlinear Sequences of Extended Uptimals

The sequences defined above and those which follow were stated (are stated) mostly in the order in which they occurred while translating CGSuite output. While the sequences α_n, β_n, γ_n^a, and γ_n^b were seen to be linear, the next four are nonlinear. Also, by comparison, the next four are strictly monotonic.

Definition 4.32. The *gamma-c* sequence is given by

$$\gamma_1^c = \{\gamma_3^a * \mid \gamma_1^a *, \gamma_2^a *\} \text{ and } \gamma_n^c = \{\gamma_3^a * \mid \gamma_1^a *, \gamma_{n-1}^c *\} \text{ for } n \geq 2.$$

Lemma 4.33. *For* $n \geq 1$, *we have*

(a) $\gamma_n^c > 0$ *and* $\gamma_n^c > *$,

(b) $\uparrow < \gamma_n^c < 2\cdot \uparrow$ *and* $2\cdot \uparrow < \gamma_n^c * < 3\cdot \uparrow$,

(c) $\gamma_n^c \parallel \gamma_1^b$ *but* $\gamma_n^c < \gamma_2^b$.

Lemma 4.34. *The first term of* γ_n^c *is given by*

$$\gamma_1^c = \gamma_2^a + \downarrow^2 *.$$

Theorem 4.35. *For* $n \geq 2$, *we have* $\gamma_n^c - \gamma_{n-1}^c = \downarrow^{n+1}$.

Corollary 4.36. *The* γ_n^c *sequence is* strictly decreasing.

Corollary 4.37. *The Uptimal+ Form is given by*

$$\gamma_n^c \;=\; * + \; 4\cdot \uparrow \; + \; \downarrow^{[n+1]} -2\cdot \uparrow^{3/2} .$$

In this case, $aw(\gamma_n^c) = \; 3$ for all n. Next, we have another strictly decreasing sequence.

Definition 4.38. The *delta* sequence is given by

$$\delta_1 = \{\alpha_2 \mid *, \alpha_1\} \text{ and } \delta_n = \{\alpha_2 \mid *, \delta_{n-1}\} \text{ for } n \geq 2.$$

Lemma 4.39. *For all $n \geq 1$, we have*

(a) $\delta_n > 0$,

(b) $\delta_n \| *$ *but* $2 \cdot \delta_n \; > \; *$,

(c) $\uparrow > \; \delta_n > \uparrow^2$ *for all* $n \geq 1$,

(d) $\delta_1 < \gamma_n^c$.

Lemma 4.40. *The first term of δ_n is given by*

$$\delta_1 = \; \alpha_1 + \; \downarrow^2 = \uparrow + \; \downarrow^2 - \uparrow^{3/2} .$$

Lemma 4.41. *We have* $\delta_n - \delta_{n-1} = \uparrow^{n+2}$ *for* $n \geq$ *1.*

Corollary 4.42. *The delta sequence is strictly increasing.*

Corollary 4.43. *The Uptimal+ Form is given by*

$$\delta_n = \; 2\cdot \uparrow \; + \; \downarrow^{[n+1]} - \; \uparrow^{3/2} \; \textit{for all } n \geq 1.$$

Hence $aw\,(\delta_n) = 1, \;\; n \geq 1$.

Definition 4.44. The *epsilon-a* sequence is given by

$$\varepsilon_1^a = \{0, * \mid -\alpha_1\} \text{ and } \varepsilon_n^a = \{0, \varepsilon_{n-1}^a \mid -\alpha_1\} \text{ for } n \geq 2.$$

The first term, ε_1^a, was given earlier by Theorem 3.11 (b). It is restated below (in Lemma 4.48) because it appears in the uptimal+ form.

Lemma 4.45. *We have*

(a) $\varepsilon_n^a \; || 0$, $\varepsilon_n^a \; || \uparrow$, *and* $\varepsilon_n^a < \Uparrow$, *and* $2 \cdot \varepsilon_n^a < \uparrow$, *for all* $n \geq 1$,

(b) $\varepsilon_n^a \; || \uparrow^m$ *for all* $m \neq n$,

(c) $\varepsilon_n^a > *$ *for all* $n \geq 1$,

(d) $\varepsilon_n^a || \delta_m$ *for all* $m \neq n$.

Lemma 4.46. *The first term of* ε_n^a *is given by*

$$\varepsilon_1^a = \beta_1 * \ = \ \uparrow^{3/2} * \, .$$

Theorem 4.47. *For* $n \geq 2$, *we have* $\varepsilon_n^a - \varepsilon_{n-1}^a = \uparrow^n$.

Corollary 4.48. *The* epsilon-*a sequence is strictly increasing.*

Corollary 4.49. *The Uptimal+ Form is given by*

$$\varepsilon_n^a = *+ \downarrow + \uparrow^{[n]} + \uparrow^{3/2} .$$

Hence $aw(\varepsilon_n^a) = 0$ for all n.

Definition 4.50. The *theta-a* sequence is given by

$$\theta_1^a = \{\gamma_1^a * \mid 0, \alpha_1 *\} \text{ and } \theta_n^a = \{\gamma_1^a * \mid 0, \theta_{n-1}^a\} \text{ for } n \geq 2.$$

Lemma 4.51. *The inequalities from Lemma 4.47 a, b, and c are also valid for* θ_n^a *with the exception that* $2 \cdot \theta_n^a > \uparrow$. *In addition,* $\theta_n^a > \varepsilon_m^a$ *for all* $n \neq m$.

Lemma 4.52. *The first term of* θ_n^a *is given by*

$$\theta_1^a \ = \ * + \ \alpha_1 - \ \uparrow^{3/2} .$$

Lemma 4.53. *For* $n \geq 2$, *we have* $\theta_n^a - \ \theta_{n-1}^a = \downarrow^n$.

Lemma 4.54. *The* θ_n^a*-sequence is strictly decreasing.*

Theorem 4.55. *The Uptimal+ Form is given by*

$$\theta_n^a = \ * + \ 2 \cdot \uparrow \ + \downarrow^{[n]} - \ 2 \cdot \uparrow^{3/2}$$

Hence, $aw\,(\theta_n^a) = 1$, for $n \geq 1$.

4.4. Two sequences of uptimals (in the usual sense)

Finally, two additional sequences follow in which the terms are not appended by $\uparrow^{3/2}$.

Definition 4.56. The *epsilon-b* sequence is given by

$$\varepsilon_1^b = \{0, \uparrow^2 * \mid \downarrow\} \text{ and } \varepsilon_n^b = \{0, \varepsilon_{n-1}^b \mid \downarrow\} \text{ for } n \geq 2.$$

The terms of ε^a and ε^b both have "star-like" characteristics (vis-à-vis comparisons with 0 and $\uparrow$) while each term of ε_n^a dominates every term of ε_n^a.

Lemma 4.57. *The inequalities from Lemma 4.45 a, b, and c are also valid for the* ε_n^b *sequence. In addition,* $\varepsilon_n^b < \varepsilon_m^a$ *for all* $n \neq m$ *.*

Lemma 4.58. *The first term of* ε_n^b *is given by*

$$\varepsilon_1^b = \varepsilon_3^a - \beta_1 \quad = \quad * + \downarrow + \uparrow^{[3]} .$$

Theorem 4.59. *For* $n \geq 2$*, we have* $\varepsilon_n^b - \varepsilon_{n-1}^b = \uparrow^{n+2}$ *.*

Corollary 4.60. *The Uptimal+ Form is given by*

$$\varepsilon_n^b = *+ \downarrow + \uparrow^{[n+2]} .$$

Hence $aw(\varepsilon_n^b) = 0$ for all n.

Definition 4.61. The *theta-b* sequence is given by

$$\theta_1^b = \{\Uparrow \mid 0, \uparrow *\} \text{ and } \theta_n^b = \{\Uparrow \mid 0, \theta_{n-1}^b\} \text{ for } n \geq 2.$$

Lemma 4.62. *The inequalities from Lemma 4.45 a, b, and c are also valid for* θ_n^b *with the exception that* $2 \cdot \theta_n^b > \uparrow$ *. In addition,* $\theta_n^b > \theta_n^a$ *for all* $n \neq m$ *.*

Lemma 4.63. *The first term of* θ_n^b *is given by*

$$\theta_1^b = * + \delta_1 + \varepsilon = * + \uparrow + \downarrow^2 .$$

Theorem 4.64. *The Uptimal+ Form is given by*

$$\theta_n^b = * + 2 \cdot \uparrow + \downarrow^{[n+1]}.$$

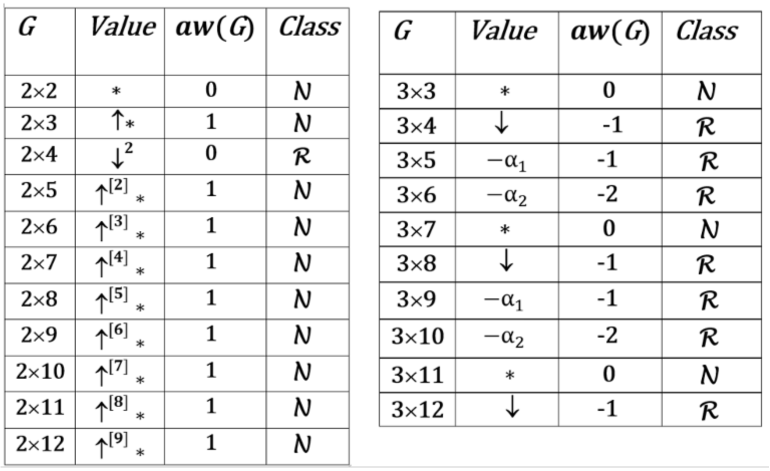
Hence, $aw\left(\theta_n^b\right) = 1$ for $n \geq 1$.

4.5. Table of values for $2 \leq n \leq 6$ and $2 \leq m \leq 12$ and two examples

The values of $[n \times m]$ (from (2.1)) for $2 \leq n \leq 6$ and $2 \leq m \leq 12$ are given in Table 4.66 below. This table includes translations of CGSuite values into the terms of the sequences previously discussed. The translations were all done by hand; and, as yet, no computer code exists for providing them.

Along with each value, the atomic weight and outcome class is stated. Recall that every combinatorial game G is found in exactly one of four outcome classes:

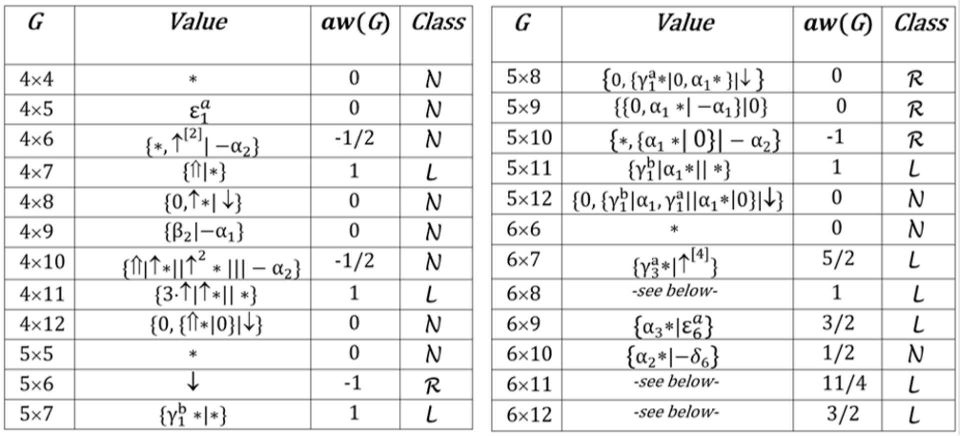
$G \in \mathcal{N}$ if *First* can win; $G \in \mathcal{L}$ if *Left* can win;
$G \in \mathcal{P}$ if *Second* can win; $G \in \mathcal{R}$ if *Right* can win.



G	*Value*	$aw(G)$	*Class*
2×2	$*$	0	$\mathcal{N}$
2×3	$\uparrow *$	1	$\mathcal{N}$
2×4	$\downarrow^{2}$	0	$\mathcal{R}$
2×5	$\uparrow^{[2]} *$	1	$\mathcal{N}$
2×6	$\uparrow^{[3]} *$	1	$\mathcal{N}$
2×7	$\uparrow^{[4]} *$	1	$\mathcal{N}$
2×8	$\uparrow^{[5]} *$	1	$\mathcal{N}$
2×9	$\uparrow^{[6]} *$	1	$\mathcal{N}$
2×10	$\uparrow^{[7]} *$	1	$\mathcal{N}$
2×11	$\uparrow^{[8]} *$	1	$\mathcal{N}$
2×12	$\uparrow^{[9]} *$	1	$\mathcal{N}$

G	*Value*	$aw(G)$	*Class*
3×3	$*$	0	$\mathcal{N}$
3×4	$\downarrow$	-1	$\mathcal{R}$
3×5	$-\alpha_1$	-1	$\mathcal{R}$
3×6	$-\alpha_2$	-2	$\mathcal{R}$
3×7	$*$	0	$\mathcal{N}$
3×8	$\downarrow$	-1	$\mathcal{R}$
3×9	$-\alpha_1$	-1	$\mathcal{R}$
3×10	$-\alpha_2$	-2	$\mathcal{R}$
3×11	$*$	0	$\mathcal{N}$
3×12	$\downarrow$	-1	$\mathcal{R}$

Table 4.66. Values of $[n \times m]$ for $n = 2, 3$; and $2 \leq m \leq 12$.



G	*Value*	$aw(G)$	*Class*
4×4	$*$	0	$\mathcal{N}$
4×5	ε_1^a	0	$\mathcal{N}$
4×6	$\{*, \uparrow^{[2]} \mid -\alpha_2\}$	-1/2	$\mathcal{N}$
4×7	$\{\Uparrow \mid *\}$	1	$\mathcal{L}$
4×8	$\{0, \uparrow * \mid \downarrow\}$	0	$\mathcal{N}$
4×9	$\{\beta_2 \mid -\alpha_1\}$	0	$\mathcal{N}$
4×10	$\{\Uparrow \mid \uparrow * \mid\mid \uparrow^2 * \mid\mid\mid -\alpha_2\}$	-1/2	$\mathcal{N}$
4×11	$\{3{\cdot}\uparrow \mid \uparrow * \mid\mid *\}$	1	$\mathcal{L}$
4×12	$\{0, \{\Uparrow * \mid 0\} \mid \downarrow\}$	0	$\mathcal{N}$
5×5	$*$	0	$\mathcal{N}$
5×6	$\downarrow$	-1	$\mathcal{R}$
5×7	$\{\gamma_1^b * \mid *\}$	1	$\mathcal{L}$

G	*Value*	$aw(G)$	*Class*
5×8	$\{0, \{\gamma_1^a * \mid 0, \alpha_1 *\} \mid \downarrow\}$	0	$\mathcal{R}$
5×9	$\{\{0, \alpha_1 * \mid -\alpha_1\} \mid 0\}$	0	$\mathcal{R}$
5×10	$\{*, \{\alpha_1 * \mid 0\} \mid -\alpha_2\}$	-1	$\mathcal{R}$
5×11	$\{\gamma_1^b \mid \alpha_1 * \mid\mid *\}$	1	$\mathcal{L}$
5×12	$\{0, \{\gamma_1^b \mid \alpha_1, \gamma_1^a \mid\mid \alpha_1 * \mid 0\} \mid \downarrow\}$	0	$\mathcal{N}$
6×6	$*$	0	$\mathcal{N}$
6×7	$\{\gamma_3^a * \mid \uparrow^{[4]}\}$	5/2	$\mathcal{L}$
6×8	*-see below-*	1	$\mathcal{L}$
6×9	$\{\alpha_3 * \mid \varepsilon_6^a\}$	3/2	$\mathcal{L}$
6×10	$\{\alpha_2 * \mid -\delta_6\}$	1/2	$\mathcal{N}$
6×11	*-see below-*	11/4	$\mathcal{L}$
6×12	*-see below-*	3/2	$\mathcal{L}$

Table 4.67. Values of $[n \times m]$ for $n = 4, 5, 6$; and $2 \leq m \leq 2$

The following entries are a part of Table 4.67:

$[6 \times 8] = \{G^L \mid \varepsilon_3^b\}$ where $G^L = \{\gamma_3^a * | \gamma_1^a *,\ \{\alpha_2 * \mid 0, \alpha_1 *\}\}$,

$[6 \times 11] = \{\gamma_4^b,\ H | \uparrow^{[8]}\}$ where $H = \ \{\alpha_3 * | \alpha_2 * || \alpha_2 * \ ||| \ \alpha_2 *\}$,

$[6 \times 12] = \{G^L \mid \varepsilon_7^b\}$ where $G^L = \ \{\gamma_4^b | \gamma_2^b,\ \gamma_1^c * |||\{0 \mid \alpha_1 * || \alpha_1 * ||| \alpha_1 *\} | \alpha_1 * || \alpha_1 *\}$.

Example 4.67. Figure 4.69 a. shows the board position following 10 moves from a

13×16 starting position. Five components remain, and their corresponding atomic weights are given. *Left* moved first so, again, its *Left's* turn.

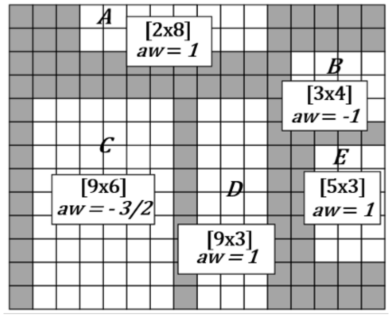


Figure 4.69a. The first 10 moves (indicated by shading)

The components shown in Figure 4.69a. are

$$A = \uparrow^{[5]} *, \; B = \downarrow, \; C = \{-\varepsilon_6^a \mid -\alpha_3 *\}, \; D = \alpha_1, \text{ and } E = \alpha_1;$$

so the present position is a sum $S = A + B + \cdots + E$.

Since $aw(S) = aw(A) + aw(B) + \ldots + aw(E) = 1/2$, it follows that *Left* has a winning move – see [2], p. 236. In this case, she can win by choosing the canonical left option of C, namely, $[9 \times 3] + [9 \times 2] = -\varepsilon_6^a$, as depicted in Figure 4.69b.

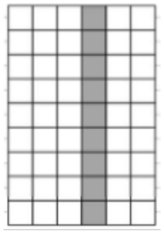


Figure 4.69b. The Canonical Left Option of $C = [9 \times 6]$:
$[9 \times 6]^L = [9 \times 3] + [9 \times 2] = \alpha_1 + \downarrow^{[6]} * = -\varepsilon_6^a$.

The standard output for $[9 \times 6]^L$ is given by

```
{{0|*,↑||0,{0|*,↑||0,{0|*,↑||0,{0|*,↑||0,{0|*,↑||0,{0|*,↑||0,*}}}}}}}
```

.

By comparison with the previous expression, the information provided by its translation, $[9 \times 6]^L = \alpha_1 + \ \downarrow^{[6]} *$, is immediately available. Indeed, such translations are generally more tractable. For large positions, however, they are not readily obtained by hand.

The option $[9 \times 6]^L$ specifies an overall position S^L that is given by the sum of the following:

$$\begin{array}{lllllll}
A & = & * + & & & & \uparrow^{[5]} \\
B & = & & \downarrow & & & \\
C^L & = & & \uparrow & + & \downarrow^{[6]} & - \uparrow^{3/2} \\
D & = & & \uparrow & - & & \uparrow^{3/2} \\
E & = & & \uparrow & - & & \uparrow^{3/2}
\end{array}.$$

We obtain $S^L = * + \ 2 \cdot \uparrow + \ \downarrow^6 \ - \ 3 \cdot \varepsilon$, a position of atomic weight 2. Hence, S^L is a winning position for *Left.*

Example 4.69. This example shows a portion of Table 4.67 for $n = 5$. Here the term $\varphi = \{\alpha_1 \mid -\alpha_1 *\}$ is used to achieve further simplification. In regards to φ, one finds that $aw(\varphi) = 0$, $\varphi > 0$, $\varphi \mid\mid *$, and $\varphi < \uparrow$.

5×7	$\{\gamma_1^b * \mid *\}$	1	$\mathcal{L}$
5×8	$\{0, \{\gamma_1^a * \mid 0, \gamma_1^a * \ \}$	0	$\mathcal{R}$
5×9	$\{\{0, \ \alpha_1 * \mid \ -\alpha_1\} \mid 0\}$	0	$\mathcal{R}$
5×10	$\{*, \{\alpha_1 * \mid \ 0\} \mid -\alpha_2\}$	-1	$\mathcal{L}$

Table 4.71. Certain Entries from Table 4.67 for $n = 5$.

Upon translation of the entries in Table 4.71, we obtain certain "extensions of extended uptimals", namely, $[5 \times 7] = (\uparrow - \uparrow^{3/2}\ ) + \varphi$, $[5 \times 8] = (* - \uparrow^{3/2}\ ) + \varphi$, $[5 \times 9] = * + \varphi$, and $[5 \times 10] = \uparrow + \varphi$.

5. Values of Larger Positions

Values of $[n \times m]$ follow for $7 \leq n \leq 12$ and $2 \leq m \leq 12$. All of these values have been translated (likewise, by hand) into the sequential terms that were featured in Section 4. These particular expressions become exceedingly complex, however, so they are not included here. Nevertheless, they are posted on [7].

A peculiarity was introduced by the square positions that we have been unable to explain. Notably, the values of $[n \times n]$, $2 \leq n \leq 12$, are all equal to $* = \{0|0\}$ *with one exception.* Specifically, in standard terms, we find that

$[10 \times 10] =$

```
±({{0|||0||0|{0,∗|0,↑∗},{0||0,∗|0,↑∗}||||⇑∗,{0|{0,∗|0,↑∗},{0||0,∗|0,↑∗}}}|||{0|||0||0,∗|0,↑∗||||{0,∗|0,↑∗},{0||0,
∗|0,↑∗}}|0,{0|||0,∗||∗,↓|0||||{∗,↓|0},{0,∗||∗,↓|0}}||{0|∗,↑||0,{0|∗,↑||0,∗}||||0,↓∗|0,∗|||0,↓∗|0,∗||0}},
{{0|||0||0|{0,∗|0,↑∗},{0||0,∗|0,↑∗}||||{0|{0,∗|0,↑∗},{0||0,∗|0,↑∗}},{0|||0||0|{0,∗|0,↑∗},{0||0,∗|0,↑∗}||||{0|{0,∗|0,
↑∗},{0||0,∗|0,↑∗}},{0|||0||0|{0,∗|0,↑∗},{0||0,∗|0,↑∗}||||{0|{0,∗|0,↑∗},{0||0,∗|0,↑∗}},{0|||0||0|{0,∗|0,↑∗},{0||0,
∗|0,↑∗}||||{0|{0,∗|0,↑∗},{0||0,∗|0,↑∗}},{0|||0||0|{0,∗|0,↑∗},{0||0,∗|0,↑∗}||||{0|{0,∗|0,↑∗},{0||0,∗|0,↑∗}},
{0|||0||0|{0,∗|0,↑∗},{0||0,∗|0,↑∗}||||{0|{0,∗|0,↑∗},{0||0,∗|0,↑∗}},{0||0|{0,∗|0,↑∗},{0||0,∗|0,↑∗}}}}}}}}|∗})
```

which translates reasonably well as

$$[10 \times 10] = \pm\ \{H, \{-\gamma_6^c * |*\}\} \text{ in which } H = \{J|||K|0, L||M\},$$

$$\text{where } J = \{-\gamma_3^a * |\Uparrow *, \quad -\gamma_1^a *\}, K = \{\alpha_3 * \mid \alpha_1 *, \quad \alpha_2 *\}, L = \{0 \mid \varepsilon_1^a|| - \alpha_1, \varepsilon_1^a\}$$

$$\text{and } M = \{-\varepsilon_2^a||- \quad \alpha_1 * \mid - \quad \alpha_2 * \}.$$

It does happen that $[10 \times 10]$ has "star-like" characteristics; namely, it is confused with 0, $*$, and $\uparrow$; and less than $2\cdot \uparrow$. We mention in passing that this position is conspicuous in at least one other way. Namely, $[10 \times 10]$ has *four* winning moves (so two of them are necessarily dominated options). Only the fifth column (row) happens to be a losing move for *Left* (*Right*). The winning moves for *Left* are illustrated in Figure 5.1.

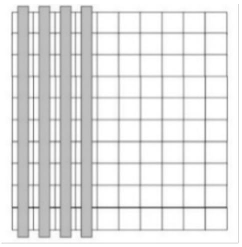


Figure 5.1. *Left's* (4) winning moves from $[10 \times 10]$
(depicted by grey bars)

On the other hand, all other $[n \times n]$ positions for $2 \leq n \leq 12$ admit exactly *one* winning move.

For square positions $[n{\times}n]$, $n \geq 12$, our purpose was to simply evaluate the square positions. With the $[10 \times\ 10]$ case in mind, we were looking for further instances in which such positions were unequal to star. No further attempt was made to analyze the followers of these positions. (The $[18 \times 18]$ case alone would require 1,832 pages of MSWord to print out!)

Similar to the $[10 \times 10]$ case, the values of $[14 \times 14]$ and $[18 \times 18]$ are also unequal to star; moreover, they have the same star-like characteristics. Is there a pattern here? It is yet to be determined whether or not $[22 \times 22]$, $[26 \times 26]$, $\ldots$ are also unequal to star.

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