RECURSIVE COMPARISON TESTS FOR DICOT AND DEAD-ENDING GAMES UNDER MISÈRE PLAY

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Abstract

In partizan games, where players Left and Right may have different options, there is a partial order defined as preference by Left: $G \geqslant H$ if Left wins $G+X$ whenever she wins $H+X$, for any game position X. In normal play, there is an easy test for comparison: $G \geqslant H$ if and only if Left wins $G-H$ playing second. In misère play, where the last player to move loses, the same test does not apply — for one thing, there are no additive inverses — and very few games are comparable. If we restrict the arbitrary game X to a subset of games $\mathcal{U}$, we may have $G \geqslant H$ "modulo $\mathcal{U}$"; but without the easy test from normal play, we must give a general argument about the outcomes of $G+X$ and $H+X$ for all $X \in \mathcal{U}$. In this paper, we use the novel theory of *absolute* combinatorial games to develop recursive comparison tests for the well-studied universes of dicots and dead-ending games. This is the first constructive test for comparison of dead-ending games under misère play, using a new family of end-games called *perfect murders*.

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1. Introduction

The purpose of this paper is to develop recursive comparison tests for certain classes of misère-play games. We assume the reader is familiar with the theory of normal-play combinatorial games, including partizan game outcomes, disjunctive sum and negation, and equality and inequality (see Section 2 for a brief review).

In combinatorial games, comparability is a critical relation. Domination and reversibility both rely on comparisons of options to simplify games. In normal play, it is straightforward to check that two games G and H are comparable:

$$G \geqslant H \;\Leftrightarrow\; G - H \geqslant 0 \;\Leftrightarrow\; \text{Left wins } G - H \text{ playing second.}$$

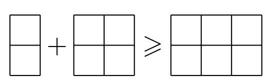
However, in misère play — where the first player unable to move is the winner instead of the loser — this simple test does not apply. Games do not have additive inverses under general misère play, so $H + (-H) \neq 0$. In fact, *no* non-zero game is equal to zero in misère play [11]; in normal play, all previous-win positions are zero. There is a modified *hand-tying principle*[6] for misère play, but nontrivial comparisons are rare in general misère play [16].

If play is restricted to a specific set or *universe* of games $\mathcal{U}$, then G and H may be comparable "modulo $\mathcal{U}$", even if they are not in general. This is *restricted misère play* [14, 15]. However, without the easy test from normal play, finding instances of misère comparison in restricted play requires a proof of a universal statement:

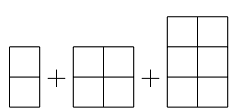
$$G \geqslant_{\mathcal{U}} H \;\text{ if for all } X \in \mathcal{U},\; \mathrm{o}(G + X) \geqslant \mathrm{o}(H + X);$$

i.e., that Left wins $G + X$ whenever she wins $H + X$, for arbitrary $X \in \mathcal{U}$.

As an example, consider the following inequality for the game DOMINEERING in which Left places vertical dominoes and Right places horizontal [3]:

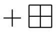


In normal play, it is easy to see that this inequality is true; we simply show Left wins playing second on this sum (where negation is achieved by rotating a board by 90 degrees):



Intuitively, it seems the same should be true in misère play (perhaps modulo a

[6]In misère play, if the Left options of H are a nonempty subset of the Left options of G (or if both are empty), and the Right options of G are a nonempty subset of the Right options of H (or both are empty), then trivially $G \geqslant H$ [16].

suitable restricted universe): Right should be happier to play on the unbroken 2×3 game, since he has more freedom to move and Left's options are the same either way. Without further tools, to prove this in misere play, we would have to prove that for arbitrary X, if Left wins  $+ X$ then she also can win  $+$  $+ X$.

In this paper, we present *recursive* comparison tests for restricted play in two well-studied universes of games. In the summary of our paper in Section 4, we will easily prove the domineering comparison above, and will do so in a way that can be implemented algorithmically. This work is an application of results from *absolute* game theory [8, 9] to the universes of *dicots*, $\mathcal{D}$, and *dead-ending* games, $\mathcal{E}$. The new comparison tests establish $G \geqslant_{\mathcal{D}} H$ or $G \geqslant_{\mathcal{E}} H$ based only on outcomes of G and H and comparisons among their options. The most significant contributions of this paper are the introduction of *strong outcome* as a necessary condition for $G \geqslant_{\mathcal{E}} H$, and the introduction of *perfect murder* games as a way to directly calculate strong outcome.

Section 2 reviews terminology and notation and defines the universes of $\mathcal{D}$ and $\mathcal{E}$. Section 3 proves the main results of the paper. Section 3.1 gives necessary conditions for comparability in $\mathcal{D}$ and $\mathcal{E}$; Section 3.2 proves that with one additional stipulation, these conditions are sufficient for comparability in $\mathcal{D}$; and finally, Section 3.3 proves that a strengthening of that additional stipulation is necessary and sufficient for comparability in $\mathcal{E}$. A moderate amount of new theory is developed in Section 3.3 in order to establish this main result.

2. Definitions

In this section, we review standard definitions from normal-play combinatorial games in the context of misère play, and then define *restricted* misère play and the universes of dicots and dead-ending games.

2.1. Options and Outcomes

We use standard notation to represent a combinatorial game: $G = \left\{G^{\mathcal{L}} \mid G^{\mathcal{R}}\right\}$ is defined by its set of Left and Right options, $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$, respectively, where $G^L \in G^{\mathcal{L}}$ ($G^R \in G^{\mathcal{R}}$) is a typical Left (Right) option of G. The *zero game* is the game with no options for either player: $\mathbf{0} = \{\cdot \mid \cdot\}$. The *followers* of a game are the game itself, its options, the options of its options, and so on; i.e., followers of G are G and all other positions that can be obtained from playing in G.

Because games do not have additive inverses under general misère play, we use $\overline{G}$ instead of $-G$ to denote what we now call the *conjugate* of G (the *negative* in normal play): $\overline{G} = \left\{\overline{G^{\mathcal{R}}} \mid \overline{G^{\mathcal{L}}}\right\}$, where $\overline{G^{\mathcal{L}}} = \left\{\overline{G^L} : G^L \in G^{\mathcal{L}}\right\}$.

In this paper, we often need to discuss who wins "when Left plays first" and who wins "when Right plays first"; we use the terms *left-outcome* and *right-outcome* for this purpose. Under misère play, they are defined as follows, with L > R:

$$\mathrm{o}_L(G) = \begin{cases} \mathrm{L}, & \text{if } G^{\mathcal{L}} = \emptyset; \\ \max \mathrm{o}_R(G^L), & \text{otherwise,} \end{cases}$$

$$\mathrm{o}_R(G) = \begin{cases} \mathrm{R}, & \text{if } G^{\mathcal{R}} = \emptyset; \\ \min \mathrm{o}_L(G^R), & \text{otherwise.} \end{cases}$$

That is, $\mathrm{o}_L(G) = \mathrm{L}$ if and only if Left wins G playing first, and so on. The overall outcome can then be defined by the pair of left-outcome and right-outcome:

$$\mathrm{o}(G) = \begin{cases} \mathscr{L}, & \text{if } (\mathrm{o}_L(G), \mathrm{o}_R(G)) = (\mathrm{L}, \mathrm{L}); \\ \mathscr{N}, & \text{if } (\mathrm{o}_L(G), \mathrm{o}_R(G)) = (\mathrm{L}, \mathrm{R}); \\ \mathscr{P}, & \text{if } (\mathrm{o}_L(G), \mathrm{o}_R(G)) = (\mathrm{R}, \mathrm{L}); \\ \mathscr{R}, & \text{if } (\mathrm{o}_L(G), \mathrm{o}_R(G)) = (\mathrm{R}, \mathrm{R}). \end{cases}$$

The total order L > R induces the standard partial order on the outcomes: $\mathscr{L} > \mathscr{N} > \mathscr{R}$ and $\mathscr{L} > \mathscr{P} > \mathscr{R}$, while $\mathscr{N}$ and $\mathscr{P}$ are incomparable. In this paper, we use $\mathrm{o}(G)$ to mean the outcome under misère play. If necessary, we may use $\mathrm{o}^-(G)$ to distinguish the misère outcome from the normal outcome, $\mathrm{o}^+(G)$.

In a *disjunctive sum* of games, the current player chooses exactly one of the game components and plays in it, while the other components remain the same:

$$G + H = \left\{ G^{\mathcal{L}} + H, G + H^{\mathcal{L}} \mid G^{\mathcal{R}} + H, G + H^{\mathcal{R}} \right\},$$

where $G^{\mathcal{L}} + H = \left\{ G^L + H : G^L \in G^{\mathcal{L}} \right\}$, etc. Comparison is then formally defined by

$$G \geqslant H \;\text{ if for all } X,\; \mathrm{o}(G + X) \geqslant \mathrm{o}(H + X).$$

Two games G, H are *equal* if $G \geqslant H$ and $H \geqslant G$; i.e., if $\mathrm{o}(G + X) = \mathrm{o}(H + X)$ for all games X. That is, games are equal if they can be interchanged in any sum without affecting the outcome. Equality and inequality are dependent upon the ending condition; we assume misère play in this paper, but will use $\geqslant^+$ and $\geqslant^-$ to distinguish normal from misère, respectively, when needed.

2.2. Universes and Restricted Misère Play

Since analysis of games under misère play is difficult, recent work has focused on studying well-suited restrictions of the set of all games (see [13] for a survey). The restrictions relevant for this work are the classes of dicot and dead-ending games.

A game is a *dicot* if, for each follower, either no player can move, or both players can move. These games are called "all-small" in normal play.

Informally, a game is *dead-ending* if, once a player runs out of moves in a game, that player will never again have a move in that game. Many well-studied rulesets, including DOMINEERING and HACKENBUSH (see [3] for descriptions), have the dead-ending property. To define this formally, we need to talk about *ends*: games for which at least one player has no options. In normal-play, ends are simple: every end is an integer. As noted in [16], ends in misère play are very problematic; while in isolation, Left is happy to have no options, when played in a disjunctive sum, she may be unhappy to be forced to play elsewhere.

A game G is a *left-end* if it has no Left option, and a game is a *dead left-end* if each follower is also a left-end. Thus, in a dead left-end, Left has no immediate moves, and there is nothing Right can do to 'open up' moves for Left. Define *right-end* and *dead right-end* analogously. A game is then *dead-ending* if each left-end follower is a dead left-end and each right-end follower is a dead right-end.

The set of all dicot games is denoted by $\mathcal{D}$ and the set of all dead-ending games is denoted by $\mathcal{E}$. Note that $\mathcal{D}$ is the restriction of $\mathcal{E}$, where the only end is the zero game, $\mathbf{0} = \{\cdot \mid \cdot\}$. Both $\mathcal{D}$ and $\mathcal{E}$ satisfy some important closure properties, which classifies each as a *universe* of games.

Definition 1. A *universe*, $\mathcal{U}$, is a non-empty set of games, which satisfies the following properties:

1. option closure: if $G \in \mathcal{U}$ and G' is an option of G then $G' \in \mathcal{U}$;

2. disjunctive sum closure: if $G, H \in \mathcal{U}$ then $G + H \in \mathcal{U}$;

3. conjugate closure: if $G \in \mathcal{U}$ then $\overline{G} \in \mathcal{U}$.

Restricted misère game theory uses weakened definitions of equality and inequality to study games *modulo* a universe $\mathcal{U}$:

$$G \equiv_{\mathcal{U}} H \text{ if for all } X \in \mathcal{U},\ \mathrm{o}(G + X) = \mathrm{o}(H + X)$$
$$G \geqslant_{\mathcal{U}} H \text{ if for all } X \in \mathcal{U},\ \mathrm{o}(G + X) \geqslant \mathrm{o}(H + X).$$

The idea is that we may get equality and comparability, and perhaps reductions and invertibility, 'modulo $\mathcal{U}$', even if the relations do not hold in general, or do not hold in a larger universe.

For example, the game $* = \{\mathbf{0} \mid \mathbf{0}\}$ is invertible modulo $\mathcal{D}$ [1], but not in $\mathcal{E}$ [12]: $* + * \equiv_{\mathcal{D}} 0$, but $* + * \not\equiv_{\mathcal{E}} 0$. The games $\mathbf{1} = \{\mathbf{0} \mid \cdot\}$ and $\overline{\mathbf{1}} = \{\cdot \mid \mathbf{0}\}$ are additive inverses modulo $\mathcal{E}$ (and thus also mod $\mathcal{D}$): $\mathbf{1} + \overline{\mathbf{1}} \equiv_{\mathcal{E}} 0$, but this is not true in general unrestricted misère play.

The universes of dicots and dead-ending games have proven fruitful for misère analysis (see [13]), and we continue the development of that theory by introducing comparison tests for $\mathcal{D}$ and $\mathcal{E}$.

Recently, [8] introduced *absolute combinatorial game theory*, a general theory for combinatorial games under non-specified ending condition. The theory applies to *absolute* universes, which are defined by the *parental* property: for all nonempty sets of games $S, T \subset \mathcal{U}$, if G is a game with $G^{\mathcal{L}} = S$ and $G^{\mathcal{R}} = T$, then G is also in $\mathcal{U}$. Impartial games are not parental; for example, 0 and $*$ are in the impartial universe but $\{0|*\}$ is not. Dicots are parental: if each player has a nonempty set of dicot games as options, the game is a dicot. Likewise, dead-ending games are parental. Our comparison tests for dicots and dead-ending games are specific adaptations of results from absolute game theory.

3. Recursive Comparison

In this section we develop our main results. In Section 3.1 we review a result from [8] that gives necessary and sufficient conditions for $G \geqslant H$ in any absolute universe. In that paper, the conditions are named the *Proviso* and the *Maintenance* property. The Maintenance property is confirmed recursively, but the in general the Proviso is not. In Section 3.2 we show how the Proviso reduces to $\mathrm{o}(G) \geqslant \mathrm{o}(H)$ when the universe is the set of all dicots, $\mathcal{D}$, and this gives an entirely constructive comparison test for dicots. The same idea is implicit in [2], where it is stated in terms of the down-linked relation [16], now generalized by the Maintenance property.

In Section 3.3 we present our main, original results: we show that for the dead-ending universe, $\mathcal{E}$, the Proviso reduces to a consideration of specific end games which we call *perfect murders*. As in $\mathcal{D}$, the result is a completely recursive comparison test for $G \geqslant_{\mathcal{E}} H$.

3.1. Proviso and Maintenance

Theorem 1 states a major result from absolute game theory [8], which gives necessary and sufficient conditions (dependant on the ending condition) for $G \geqslant_{\mathcal{U}} H$ in an absolute universe $\mathcal{U}$. The proof (not included here) is highly non-trivial and uses the adjoint operation, down-linked relation, and other concepts from Siegel, Ettinger et al. [16, 6]. We will apply the result to the universes of dicots and dead-ending games under misère play.

Theorem 1 (Proviso and Maintenance [8])**.** *Let $\mathcal{U}$ be an absolute universe and let $G, H \in \mathcal{U}$. Then $G \geqslant_{\mathcal{U}} H$ if and only if the following hold:*
Proviso:

1. $\mathrm{o}_L(G + X) \geqslant \mathrm{o}_L(H + X)$ *for all Left ends* $X \in \mathcal{U}$*;*

2. $\mathrm{o}_R(G + X) \geqslant \mathrm{o}_R(H + X)$ *for all Right ends* $X \in \mathcal{U}$*;*

Maintenance:

1. $\forall H^L \in H^{\mathcal{L}}, \exists G^L \in G^{\mathcal{L}} : G^L \geqslant_{\mathcal{U}} H^L$ *or* $\exists H^{LR} \in H^{L\mathcal{R}} : G \geqslant_{\mathcal{U}} H^{LR}$*; and*

2. $\forall G^R \in G^{\mathcal{R}}, \exists H^R \in H^{\mathcal{R}} : G^R \geqslant_{\mathcal{U}} H^R$ *or* $\exists G^{RL} \in G^{R\mathcal{L}} : G^{RL} \geqslant_{\mathcal{U}} H$*.*

The idea behind the Maintenance property is that, in an absolute universe, when playing $G + \overline{H}$ with $G \geqslant_{\mathcal{U}} H$, Left can always "maintain" her position: no matter what move Right makes in $G + \overline{H}$, Left can bring the game back to a position that is just as good for Left as before Right moved. In some sense this is a generalization of the fact that in normal play, $G \geqslant H \Rightarrow G + \overline{H} \geqslant \mathbf{0}$ (recall that we do not have this in misère play, because $H + \overline{H}$ is generally not equal to $\mathbf{0}$). Note that a Right move in $\overline{H}$ is a Left move in H, and so the conditions in Theorem 1 are stated without reference to conjugates.

Note that all inequality relations in this section are considered with misère-play ending condition, unless otherwise specified; if necessary, we use $\geqslant^-$ for misère play and $\geqslant^+$ for normal play.

Incidentally, Theorem 1 implies the existence of an order-preserving map of misère-play into normal-play: i.e., if $G \geqslant^-_{\mathcal{U}} H$ then $G \geqslant^+ H$. This result is already known [2, 8], but we give the argument in Corollary 1 below to illustrate how it follows from Theorem 1. We can gain some intuition for the idea by considering the game $\{\boldsymbol{n} \mid -\boldsymbol{n}\}$ for a large integer $\boldsymbol{n}$. Using Chess terminology, this game acts like a "large zugzwang" under misère play: players do not want to be the first to play in this position, and so in both $G + \{\boldsymbol{n} \mid -\boldsymbol{n}\}$ and $H + \{\boldsymbol{n} \mid -\boldsymbol{n}\}$, players should play G and H with a "normal-play strategy", trying to get the last move and force the opponent to play first on the zugzwang part. Thus, if we don't have $G \geqslant^+ H$, we cannot have $G \geqslant^-_{\mathcal{U}} H$.

Corollary 1. *For any absolute universe* $\mathcal{U}$*, if* $G \geqslant^-_{\mathcal{U}} H$ *then* $G \geqslant^+ H$*.*

Proof. Suppose you have $G \geqslant^-_{\mathcal{U}} H$, and so you also have conditions (1) and (2) from Theorem 1. We need to show Left wins second on $G + \overline{H}$ under normal play. Right's options are of the form $G^R + H$ or $G + \overline{H^L}$; in each case, the Maintenance property guarantees the existence of a response for Left. Moreover, that response always brings the game to another position of the form $X + \overline{Y}$ where $X \geqslant^-_{\mathcal{U}} Y$. Thus, in all followers of $G + \overline{H}$, Left will always be able to reply to any Right move, and so Left will win $G + \overline{H}$ under normal play. □

3.2. The Proviso for Dicots

In the universe of Dicots, $\mathcal{D}$, the only end is the zero game. Thus, the Proviso in $\mathcal{D}$ reduces to "Left wins G first if Left wins H first, and Right wins H first if Right wins G first." This is precisely equivalent to $\mathrm{o}(G) \geqslant \mathrm{o}(H)$, where these functions indicate misère outcome. Thus, we have the recursive comparison test for $\mathcal{D}$ stated below. As we omitted the proof of Theorem 1, we include here a partial proof of the result specifically for $\mathcal{D}$.

Theorem 2 (Comparison in $\mathcal{D}$)**.** *Let* $G, H \in \mathcal{D}$*. Then* $G \geqslant_{\mathcal{D}} H$ *if and only if*

1. $o(G) \geqslant o(H)$*;*

2. $\forall H^L \in H^{\mathcal{L}}, \exists G^L \in G^{\mathcal{L}} \colon G^L \geqslant_{\mathcal{D}} H^L$ *or* $\exists H^{LR} \in H^{L\mathcal{R}} \colon G \geqslant_{\mathcal{D}} H^{LR}$*;*

3. $\forall G^R \in G^{\mathcal{R}}, \exists H^R \in H^{\mathcal{R}} \colon G^R \geqslant_{\mathcal{D}} H^R$ *or* $\exists G^{RL} \in G^{R\mathcal{L}} \colon G^{RL} \geqslant_{\mathcal{D}} H$*.*

Proof. ($\Rightarrow$) It is trivial that $G \geqslant_{\mathcal{D}} H$ implies condition 1. Since $\mathcal{D}$ is absolute, $G \geqslant_{\mathcal{D}} H$ implies conditions 2 and 3 by Theorem 1 (Maintenance).

($\Leftarrow$) Assume $G, H \in \mathcal{D}$ and conditions 1, 2, and 3 are satisfied. We need to show $\mathrm{o}(G + X) \geqslant \mathrm{o}(H + X)$ for all $X \in \mathcal{D}$. We proceed by induction. For the base case, when $X = 0$, we need only $\mathrm{o}(G) \geqslant \mathrm{o}(H)$; this is given by condition 1.

Let X be any game of rank greater than 0, and assume $\mathrm{o}(G + X') \geqslant \mathrm{o}(H + X')$ for all X' of smaller rank than X. To show $\mathrm{o}(G + X) \geqslant \mathrm{o}(H + X)$, we show that when Left wins $H+X$ going first (second), Left also wins $G+X$ going first (second). So suppose Left wins $H + X$ going first. Since $X \neq 0$ and X is a dicot, we know Left has a move in $H + X$.

If Left wins $H + X$ with a move to $H + X^L$, then Left wins $G + X$ with a move to $G + X^L$, by the induction hypothesis. Otherwise, Left wins $H + X$ with a move to $H^L + X$. By condition 2, there is either a $G^L \geqslant_{\mathcal{D}} H^L$, in which case Left wins $G + X$ with $G^L + X$, or there is an $H^{LR} \leqslant_{\mathcal{D}} G$. But $H^{LR} + X$ is left-win or next-win, because $H^L + X$ is a good first left move, so this means $G + X$ is left-win or next-win, as required.

The argument for Left playing second is similar, using condition 3 instead of condition 2. □

3.3. The Proviso for Dead-ending Games

Recall, the universe of dead-ending games, $\mathcal{E}$, is a superset of the dicots. In $\mathcal{D}$, the only end is 0; in $\mathcal{E}$, there are nonzero ends, but they must be *dead* ends: e.g., a left end in $\mathcal{E}$ has no options for left now and no options for left later. The comparison test for dicots is not quite strong enough to give inequality modulo $\mathcal{E}$, even if both games are dicots.

To illustrate the complication, consider $G = \{* \mid *\} = * + *$ and $H = 0$. It is easy to check that all three conditions of Theorem 2 are satisfied, so that $G \geqslant_{\mathcal{D}} H$ (and, by symmetry, in fact $G \equiv_{\mathcal{D}} H$). However, it is not true that $G \geqslant_{\mathcal{E}} H$. Consider the logic for dicots. If Left can win $0 + X$, then Left can follow the same strategy to win $* + * + X$: if Right plays in $* + *$, Left can respond to bring that component to zero and resume winning on X; otherwise Left ignores $* + *$ and eventually runs out of moves in X, at which point X must be 0, and then Left wins playing next on $* + *$.

The problem for dead-ending games is the conclusion “at which point X must be 0”. In $\mathcal{E}$, when Left runs out of moves in X, the position is a left end, but not necessarily 0. In our example above, suppose Right has a single move remaining in

X when Left runs out of moves in X; now Left moves in one of the stars to leave $* + \overline{1}$, and Right wins from here moving to $*$. So $* + * \not\geqslant_{\mathcal{E}} 0$ (because Left prefers $0 + \overline{1}$ over $* + * + \overline{1}$).

For dead-ending games, the base case will be that X is any left end, not necessarily 0; this brings us back to the Proviso:

1. $\mathrm{o}_L(G + X) \geqslant \mathrm{o}_L(H + X)$ for all Left ends $X \in \mathcal{U}$;

2. $\mathrm{o}_R(G + X) \geqslant \mathrm{o}_R(H + X)$ for all Right ends $X \in \mathcal{U}$;

Our goal is to find a sufficient, constructive condition for the Proviso in $\mathcal{E}$.

We begin by introducing new notation for the worst possible outcome (for left) of "$G + X$ for any left end".

Definition 2. The *strong Left-outcome* and *strong Right-outcome* of $G \in \mathcal{E}$ is

$$\hat{o}_L(G) = \underbrace{\min}_{\textit{Left-end } X} \{o_L(G + X)\},$$

$$\hat{o}_R(G) = \underbrace{\max}_{\textit{Right-end } Y} \{o_R(G + Y)\},$$

respectively.

The strong Left- or Right-outcome can be determined by a general argument. For example, if $G = \{0, * \mid 0\}$, the strong left-outcome is L because Left wins first on ($G+$ any nonzero left end X) by playing to $0 + X$, and left wins first on $G + 0$ by playing to $*$.

However, we claim there is a direct, constructive way to compute strong outcome and thereby verify the Proviso in $\mathcal{E}$. For this, we introduce a new family of dead ends, and we will prove that these are the 'worst' left-ends for Left and the worst right-ends for Right. Consider a left-end over which Right has total control: Right can terminate the end at any point (i.e., Right always has a move to zero). We call this type of game a *perfect murder*. In a sum of G and a perfect murder, it is as if Right uses the murder game to 'pass' in G, until it is advantageous for Right to terminate that end and, if possible, force Left into the last move in G.

Definition 3. The *perfect murder* of rank n, $M_n \in \mathcal{E}$, is recursively defined by:

$$M_n = \begin{cases} \mathbf{0}, & \text{if } n = 0; \\ \{\cdot \mid \mathbf{0}, M_{n-1}\}, & \text{if } n > 0. \end{cases}$$

Thus, $M_0 = \mathbf{0}$, $M_1 = \{\,\mid \mathbf{0}\}$, $M_2 = \{\,\mid \mathbf{0}, \{\,\mid \mathbf{0}\}\}$, and so on. Figure 1 shows perfect murders of rank up to 4.

We are aiming to prove that perfect murder games can be used to easily determine the strong outcome of a given game. This is done across three results (stated here

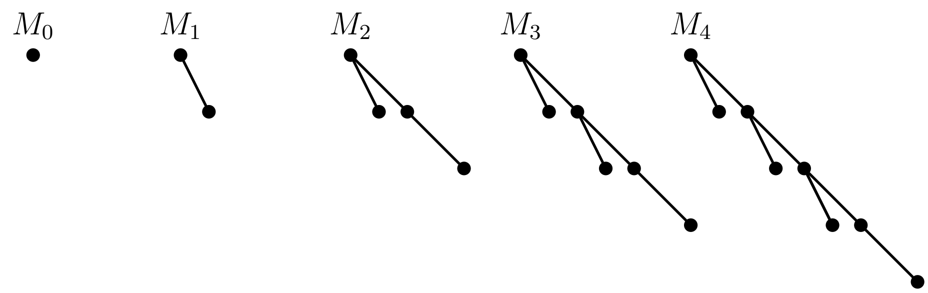


Figure 1: Perfect murder games of rank 0 to 4.

from Left's perspective): Lemma 1 shows Left prefers M_n to M_{n+1}; Lemma 2 shows that Left prefers any left end to M_n, provided the rank is at most n; and finally, Theorem 3 shows that the strong left-outcome of G with rank k is precisely the smaller of $\mathrm{o}_L(G)$ and $\mathrm{o}_L(G + M_{k-1})$.

Lemma 1. *For all $n > 0$, $M_n \geqslant_{\mathcal{E}} M_{n+1}$.*

Proof. We need to show that $o(M_n + X) \geqslant o(M_{n+1} + X)$, for all $n > 0$ and for all $X \in \mathcal{E}$. Let $n > 0$ and suppose Right wins $M_n + X$ (first or second or both). We need to show that Right can win $M_{n+1} + X$.

Right's winning move in $M_n + X$ must be to $\mathbf{0} + X'$, for a follower X' of X (in which Left moves to a Right end). Say this move to $\mathbf{0}$ occurs at level k of M_n. Then Right can win $M_{n+1} + X$ by following exactly the same strategy, moving to $\mathbf{0} + X'$ at level k of M_{n+1}. □

Lemma 2. *If G is a Left-end of rank $k > 0$, then $G \geqslant_{\mathcal{E}} M_n$, for all $n \geqslant k$.*

Proof. Let $G \in \mathcal{E}$ be a fixed Left-end of rank $k > 0$. By Lemma 1, it suffices to show $G \geqslant_{\mathcal{E}} M_k$.

Let X be an arbitrary game in $\mathcal{E}$. We must prove that $\mathrm{o}(G + X) \geqslant o(M_k + X)$. If $X = \mathbf{0}$, then $\mathrm{o}(G + X) = \mathscr{L} = \mathrm{o}(M_k + X)$, because both G and M_k are nonzero left-ends. Now assume $\mathrm{rank}(X) > 0$.

Suppose Left wins $M_k + X$ going first. Then Left's good first move is to $M_k + X^L$. By induction, $G + X^L$ is at least as good as this move, so Left can also win $G + X$ going first.

Suppose Left wins $M_k + X$ going second; so all Right moves in $M_k + X$ are left- or next-win. Consider Left playing second in $G + X$. There are three possibilities:

(1) If there is a Right option $G^R = \mathbf{0}$, then since this is also an option of M_k, we know Left wins $\mathbf{0} + X$.

(2) If Right moves to $G^R + X$, with $G^R \neq \mathbf{0}$, then by induction $G^R \geqslant_{\mathcal{E}} M_{k-1}$. Since Left wins from $M_{k-1} + X$, Left also wins from $G^R + X$.

(3) If Right moves to $G + X^R$, then by induction this is at least as good for Left as $M_k + X^R$, which Left wins.

□

Recall, the strong left-outcome $\hat{o}_L$ of a game G is the minimum left-outcome (L or R, with L>R) of $G+X$, where X ranges over all possible left ends in $\mathcal{E}$. Lemma 2 establishes that perfect murders are the worst (nonzero) left ends for Left. It makes sense, then, that when calculating the strong left-outcome, we need only consider the outcome of $G +$ a perfect murder left-end, and also $G + 0$. Theorem 3 shows that M_{k-1} will yield the minimum outcome of G with a nonzero left end.

Theorem 3. *Let* $G \in \mathcal{E}$. *If* $\operatorname{rank}(G) = k$ *then*

$$\hat{o}_L(G) = \min \{o_L(G), o_L(G + M_{k-1})\},$$
$$\hat{o}_R(G) = \max \left\{o_R(G), o_R(G + \overline{M_{k-1}})\right\}.$$

Proof. We prove the result for $\hat{o}_L$, and $\hat{o}_R$ follows analogously. If $o_L(G + 0) = R$, then the result is clear. Otherwise, let X be a nonzero left end such that $o_L(G+X)$ is a minimum.

We need to show $o_L(G+M_{k-1})$ is not greater than $o_L(G+X)$; i.e., that $o_L(G+X) \geqslant o_L(G+M_{k-1})$. If $\operatorname{rank}(X) \leqslant k-1$, then this follows by Lemma 2. So assume $\operatorname{rank}(X) \geqslant k$. Let X' be the game X 'trimmed' to rank $k-1$ (that is, X' is obtained from the game tree of X by deleting all nodes of levels larger or equal than k). Then by Lemma 2, $X' \geqslant_{\mathcal{E}} M_{k-1}$, and so $o_L(G+X') \geqslant o_L(G+M_{k-1})$. We will now show that $o_L(G+X) \geqslant o_L(G+X')$, by showing $o_L(G+X') = L \Rightarrow o_L(G+X) = L$.

Suppose $o_L(G+X') = \mathrm{L}$. To win $G+X$ playing first, Left can follow the same strategy as in $G+X'$. The only way Right can foil this is if Right makes a move in $G+X$ that was not possible in $G+X'$; but that move would be in level k or higher in X, and at that point, Left has made k moves in G, and so there are no moves remaining in G (which has rank k) and no left moves in X. So Left wins.

Thus, $o_L(G+X) \geqslant o_L(G+X') \geqslant o_L(G+M_{k-1})$, for any nonzero left end X, and so the minimum left-outcome of G with any left-end is the minimum of $o_L(G+M_{k-1})$ and $o_L(G+0)$. □

We now have a constructive way to compute the strong left-outcome and strong right-outcome. We can pair the two outcomes to give the *strong outcome* of the game.

Definition 4. The *strong outcome* of $G \in \mathcal{E}$ is

$$\hat{o}(G) = \begin{cases} \mathscr{L}, & \text{if } (\hat{o}_L(G), \hat{o}_R(G)) = (\mathrm{L}, \mathrm{L}); \\ \mathscr{N}, & \text{if } (\hat{o}_L(G), \hat{o}_R(G)) = (\mathrm{L}, \mathrm{R}); \\ \mathscr{P}, & \text{if } (\hat{o}_L(G), \hat{o}_R(G)) = (\mathrm{R}, \mathrm{L}); \\ \mathscr{R}, & \text{if } (\hat{o}_L(G), \hat{o}_R(G)) = (\mathrm{R}, \mathrm{R}). \end{cases}$$

Observation 4. Let E be a non-zero dead Left-end. Then $\hat{o}(E) = \mathscr{L}$, because if Left goes first then she has no move and wins, and if Right goes first then the position is still a Left-end (because the original position was a dead end) and so Left also wins going second.

With the concept of strong outcome, we now have a recursive comparison test for dead-ending games.

Theorem 5 (Comparison in $\mathcal{E}$)**.** *Let $G, H \in \mathcal{E}$. Then $G \geqslant_{\mathcal{E}} H$ if and only if*

1. $\hat{o}(G) \geqslant \hat{o}(H)$*;*
2. *For all $H^L \in H^{\mathcal{L}}, \exists G^L \in G^{\mathcal{L}} \colon G^L \geqslant_{\mathcal{E}} H^L$ or $\exists H^{LR} \in H^{L\mathcal{R}} \colon G \geqslant_{\mathcal{E}} H^{LR}$;*
3. *For all $G^R \in G^{\mathcal{R}}, \exists H^R \in H^{\mathcal{R}} \colon G^R \geqslant_{\mathcal{E}} H^R$ or $\exists G^{RL} \in G^{R\mathcal{L}} \colon G^{RL} \geqslant_{\mathcal{E}} H$.*

Proof. ($\Rightarrow$) Since $\mathcal{E}$ is absolute, $G \geqslant_{\mathcal{E}} H$ implies conditions 2 and 3 by Theorem 1 (Maintenance). Also, $G \geqslant_{\mathcal{E}} H$ implies condition 1: if not, then (without loss of generality) $\hat{o}_L(G) = \mathrm{R}$ and $\hat{o}_L(H) = \mathrm{L}$, so there is a Left-end X such that $o_L(G + X) = \mathrm{R}$, but for X (and all Left-ends), $o_L(H + X) = \mathrm{L}$; this contradicts $G \geqslant_{\mathcal{E}} H$.

($\Leftarrow$) Assume $G, H \in \mathcal{E}$ and conditions 1, 2, and 3 are satisfied. We need to show $\mathrm{o}(G+X) \geqslant \mathrm{o}(H+X)$ for all $X \in \mathcal{E}$. We use induction on the 'left-rank' (the depth of the game tree beginning with a Left move) of X. In fact, the proof is identical to the proof of Theorem 2, except for the base case. Here, the base case is X is a left end. We will prove that conditions 1,2,3 together imply $\mathrm{o}(G+X) \geqslant \mathrm{o}(H+X)$ for all *left ends* $X \in \mathcal{E}$.

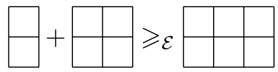
Assume $\hat{\mathrm{o}}(G) \geqslant \hat{\mathrm{o}}(H)$ and let X be any left end in $\mathcal{E}$. We need to show $\mathrm{o}(G + X) \geqslant \mathrm{o}(H + X)$. Suppose Left wins $H + X$ moving first (moving second follows analogously). If this is because H is also a left end, then $\hat{\mathrm{o}}(H) = \mathscr{L} \Rightarrow \hat{\mathrm{o}}(G) \in \mathscr{L} \Rightarrow \mathrm{o}_L(G + Y) = \mathrm{L}$ for all left ends Y, so Left wins $G + X$ going first.

If H is not a left end, then Left wins $H+X$ with a move to H^L+X. By condition 2, either (1) there is a $G^L \geqslant_{\mathcal{E}} H^L$ or (2) there is an $H^{LR} \leqslant_{\mathcal{E}} G$. If (1), then since Left wins $H + X$ with $H^L + X$, we know Left will win $G + X$ with $G^L + X$; if (2), since Left wins moving next on $H^{LR} + X$ (as it is a right option of $H^L + X$), we know Left wins moving next on $G + X$. □

We end with a few examples of Theorem 5, including the Domineering inequality from the introduction.

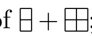
Example 6. Consider $G = \{-\mathbf{1} \mid \mathbf{1}\}$. Then, $\hat{o}_L(G) = \mathrm{L}$ and $\hat{o}_R(G) = \mathrm{R}$. Therefore, $\hat{o}(G) = \mathscr{N} = \hat{o}(\mathbf{0})$. Note that $G^R = \mathbf{1}$ and $G^{RL} = \mathbf{0} \geqslant_{\mathcal{E}} \mathbf{0}$, so Theorem 5 gives $G \geqslant_{\mathcal{E}} \mathbf{0}$. Symmetrically, $G \leqslant_{\mathcal{E}} \mathbf{0}$. Hence $G \equiv_{\mathcal{E}} \mathbf{0}$.

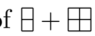
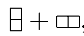
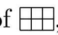
Example 7. Recall the conjectured Domineering inequality from Section 1:



This follows easily by Theorem 5, since

(1) the strong outcome of both sides is $\mathscr{P}$;

(2) all Left options of  are also Left options of ; and

(3) the single Right option of  is , which is trivially greater than or equal to the Right option  of , by hand-tying.

4. Summary and Future Directions

We have adapted general results from absolute CGT [8] to develop recursive comparison tests for restricted misère play in the dicot and dead-ending universes, $\mathcal{D}$ and $\mathcal{E}$. The main construction for $\mathcal{E}$ is the *perfect murder* family of games, which function as the 'worst' nonzero ends for a given player. Perfect murder positions are used to directly calculate the *strong outcome* of a game.

The recursive comparison tests can be used to help analyze specific rule sets within the universes of $\mathcal{D}$ and $\mathcal{E}$. Many commonly studied rulesets, including all *placement games* [7], are dead-ending. Example 14 illustrates how our comparison test for $\mathcal{E}$ can be used to find results for the game of Domineering (other similar inequalities for Domineering are presented in [4]).

Since conditions (2) and (3) of Theorem 5 are recursive, and since strong outcome can be calculated directly using perfect murder games, our comparison tests can be implemented computationally. In [5], a computer program has been written to apply Theorem 5 to determine all nontrivial comparisons of rank-2 dead-ending games.

A direction for future work is to find and study other parental universes, so that we can use absolute game theory to develop analogous comparison tests for those games. We can consider universes between $\mathcal{D}$ and $\mathcal{E}$ by considering the closure (under sums, conjugates, and parentality) of $\mathcal{D}$ union some other non-dicot dead-ending position(s) [10]. We can also consider extensions of $\mathcal{E}$ — are there parental universes between $\mathcal{E}$ and the full universe? Would something stronger than strong outcome be sufficient for comparison in such a universe? The answers to these questions will allows us to continue to develop the theory of restricted misère play.

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