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GRUNDY NUMBERS OF IMPARTIAL THREE-DIMENSIONAL CHOCOLATE-BAR GAMES

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Abstract

Chocolate-bar games are variants of the Chomp game. Let $Z_{\geq 0}$ be a set of nonnegative numbers and $x, y, z \in Z_{\geq 0}$. A three-dimensional chocolate bar is comprised of a set of $1 \times 1 \times 1$ cubes, with a "bitter" or "poison" cube at the bottom of the column at position $(0, 0)$. For $u, w \in Z_{\geq 0}$ such that $u \leq x$ and $w \leq z$, and the height of the column at position (u, w) is $\min(F(u, w), y) + 1$, where F is a monotonically increasing function. We denote such a chocolate bar as $CB(F, x, y, z)$. Two players take turns to cut the bar along a plane horizontally or vertically along the grooves, and eat the broken pieces. The player who manages to leave the opponent with a single bitter cube is the winner. In a prior work, we characterized function f for a two-dimensional chocolate-bar game such that the Sprague–Grundy value of $CB(f, y, z)$ is $y \oplus z$. In this study, we characterize function F such that the Sprague–Grundy value of $CB(F, x, y, z)$ is $x \oplus y \oplus z$.

1. Introduction

Chocolate-bar games are variants of the Chomp game. A two-dimensional chocolate bar is a rectangular array of squares in which some squares are removed throughout the course of the game. A "poisoned" or "bitter" square, typically printed in black, is included in some part of the bar. Figure 1 shows an example of a two-dimensional chocolate bar. Each player takes turns breaking the bar in a straight line along the grooves, and then "eats" a broken piece. The player who manages to leave the opponent with a single bitter block (the black block) wins the game.

A three-dimensional chocolate bar is a three-dimensional array of cubes in which a poisoned cube printed in black is included in some part of the bar. Figure 2 shows an example of a three-dimensional chocolate bar.

Each player takes turns dividing the bar along a plane that is horizontal or vertical along the grooves, and then eats a broken piece. The player who manages to leave the opponent with a single bitter cube wins the game. Examples of cut chocolate bars are shown in Figures 3, 4, and 5.

Example 1.1. Here, we provide examples of chocolate bars.
(i) Example of a two-dimensional chocolate bar.

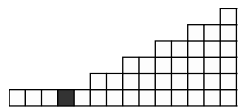

Figure 1.

(ii) Example of a three-dimensional chocolate bar.

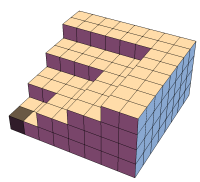

Figure 2.

Example 1.2. There are three ways to cut a three-dimensional chocolate bar.
(i) Vertical cut.

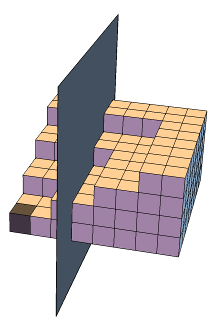

Figure 3.

(ii) Vertical cut.

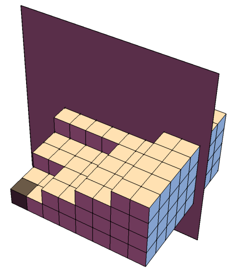

Figure 4.

(*iii*) Horizontal cut.

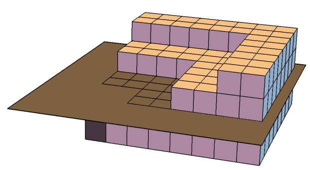

Figure 5.

The original two-dimensional rectangular chocolate bar introduced by Robin [1] is comprised of a “bitter” or “poison” corner, as shown in Figure 6. Because the horizontal and vertical grooves are independent, an $m \times n$ rectangular chocolate bar game is structured in a manner similar to that of the game Nim, which includes heaps of $m-1$ and $n-1$ stones. Therefore, the chocolate-bar game (Figure 6) is mathematically equivalent to Nim, which includes heaps of 5 and 3 stones (Figure 7). The Grundy number of the Nim game with heaps of $m-1$ and $n-1$ stones is $(m-1) \oplus (n-1)$; therefore, the Grundy number of this $m \times n$ rectangular bar is $(m-1) \oplus (n-1)$.

Extending the game to three dimensions, Robin [1] also presented a cubic chocolate bar. For example, see Figure 2. It can be easily determined that the three-dimensional chocolate bar in Figure 2 is mathematically equivalent to Nim with heaps of 5, 3, and 5 stones. Hence, the Grundy number of this $6 \times 4 \times 6$ cuboid bar is $5 \oplus 3 \oplus 5$.

Example 1.3. Here, we provide an example of the traditional Nim game and two examples of chocolate bars.

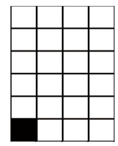

Figure 6.

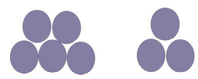

Figure 7.

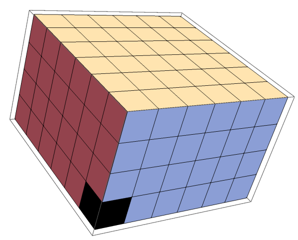

Figure 8.

In this context, it is natural to search for a necessary and sufficient condition wherein a chocolate bar may have a Grundy number calculated using the Nim-sum as the length, height, and width of the bar.

We have previously presented the necessary and sufficient condition for a two-dimensional chocolate bar in [2].

This article aims to answer the following question.

Question. What is the necessary and sufficient condition under which a three-dimensional chocolate bar may have a Grundy number $(x-1) \oplus (y-1) \oplus (z-1)$, where x, y, and z are the length, height, and width of the bar, respectively?

The remainder of this article is organized as follows. In Section 2, we briefly review some necessary concepts of combinatorial game theory.

In Section 3, we present a summary of the research results on the two-dimensional chocolate-bar game provided in [2] and utilize this result in Section 4.

In Section 4, we study a three-dimensional chocolate bar such as that shown in Figure 2 and answer the abovementioned research question. The proof of the sufficient condition for a three-dimensional chocolate bar is straightforward from the result of the two-dimensional chocolate bar presented in [2]; however, the proof of the necessary condition for a three-dimensional chocolate bar is more difficult to obtain.

2. Combinatorial Game Theory Definitions and Theorem

Let $Z_{\geq 0}$ be a set of nonnegative integers.

For completeness, we briefly review some necessary concepts from combinatorial game theory; further details may be found in [5] or [6].

Definition 2.1. Let x and y be nonnegative integers. Expressing both in base 2, $x = \sum_{i=0}^{n} x_i 2^i$ and $y = \sum_{i=0}^{n} y_i 2^i$ with $x_i, y_i \in \{0, 1\}$. We define the *nim-sum*, $x \oplus y$, as

$$x \oplus y = \sum_{i=0}^{n} w_i 2^i, \tag{1}$$

where $w_i = x_i + y_i \pmod 2$.

Lemma 1. *Let x, y, $z \in Z_{\geq 0}$. If $y \neq z$, then $x \oplus y \neq x \oplus z$.*

Proof. If $x \oplus y = x \oplus z$, then $y = x \oplus x \oplus y = x \oplus x \oplus z = z$. □

As chocolate-bar games are impartial and without draws, only two outcome classes are possible.

Definition 2.2. (a) A position is referred to as a $\mathcal{P}$*-position* if it is a winning position for the previous player (the player who just moved), as long as the player plays correctly at every stage.
(b) A position is referred to as an $\mathcal{N}$*-position* if it is a winning position for the next player, as long as the player plays correctly at every stage.

Definition 2.3. The *disjunctive sum* of the two games, denoted by $\mathbf{G} + \mathbf{H}$, is a super-game in which a player may move either in $\mathbf{G}$ or $\mathbf{H}$, but not in both.

Definition 2.4. For any position $\mathbf{p}$ of game $\mathbf{G}$, there exists a set of positions that can be reached in precisely one move in $\mathbf{G}$, which we denote as $move(\mathbf{p})$.

Remark 2.1. Note that 3.1 and 3.2 are examples of a *move*.

Definition 2.5. (i) The *minimum excluded value* (mex) of a set S of nonnegative integers is the least nonnegative integer that is not in S.
(ii) Let $\mathbf{p}$ be a position in an impartial game. The associated *Grundy number* is denoted as $G(\mathbf{p})$ and is recursively defined as $G(\mathbf{p}) = mex\{G(\mathbf{h}) : \mathbf{h} \in move(\mathbf{p})\}$.

Lemma 2. *Let S be a set of nonnegative integers and $mex(S) = m$ for some $m \in Z_{\geq 0}$. Then, $\{k : k < m$ and $k \in Z_{\geq 0}\} \subset S$.*

Proof. This also follows directly from Definition 2.5. □

Lemma 3. *If $G(\mathbf{p}) > x$ for some $x \in Z_{\geq 0}$, then $\mathbf{h} \in move(\mathbf{p})$ exists such that $G(\mathbf{h}) = x$.*

Proof. This follows directly from Lemma 2 and Definition 2.5. □

The next result demonstrates the usefulness of the Sprague–Grundy theory in impartial games.

Theorem 1. *Let $\mathbf{G}$ and $\mathbf{H}$ be impartial rulesets, and $G_{\mathbf{G}}$ and $G_{\mathbf{H}}$ respectively be the Grundy numbers of game $\mathbf{g}$ played under the rules of $\mathbf{G}$ and game $\mathbf{h}$ played under the rules of $\mathbf{H}$. Then, the following conditions hold.*
(i) For any position $\mathbf{g}$ of $\mathbf{G}$, $G_{\mathbf{G}}(\mathbf{g}) = 0$ if and only if $\mathbf{g}$ is a $\mathcal{P}$-position.
(ii) The Grundy number of position $\{\mathbf{g}, \mathbf{h}\}$ in game $\mathbf{G} + \mathbf{H}$ is $G_{\mathbf{G}}(\mathbf{g}) \oplus G_{\mathbf{H}}(\mathbf{h})$.

For the proof of this theorem, see [5].

With Theorem 1, we can find a $\mathcal{P}$-position by calculating the Grundy numbers and a $\mathcal{P}$-position of the sum of two games by calculating the Grundy numbers of two games. Therefore, Grundy numbers are an important research topic in combinatorial game theory.

3. Two-Dimensional Chocolate Bar

Here, we define two-dimensional chocolate bars and present some related results. Because the operations of cutting and defining Grundy numbers are difficult to understand in the case of three-dimensional bars, we present examples 3.1 and 3.2 of two-dimensional chocolate bars. We present the previously reported Theorem 2 and a new lemma with a proof as Lemma 4. We use Theorem 2 to prove Lemma 4 and Theorem 4 in Section 4. Further, we use Lemma 4 to prove Theorem 3 in Section 4. The employed method involves cutting three-dimensional chocolate bars into sections and then applying Theorem 2 and Lemma 4 to these sections. Note that a section of a three-dimensional chocolate bar is a two-dimensional chocolate bar.

We have previously determined that the necessary and sufficient condition for the Grundy number is $(m-1) \oplus (n-1)$ when the width of the chocolate bar monotonically increases with respect to the distance from the bitter square, where m is the maximum width of the chocolate bar, and n is the maximum horizontal distance from the bitter part. This result was previously published in [2] and is presented in Theorem 2 in this section.

Definition 3.1. A function f of $Z_{\geq 0}$ into itself is said to be *monotonically increasing* if $f(u) \leq f(v)$ for $u, v \in Z_{\geq 0}$ with $u \leq v$.

Definition 3.2. Let f be a monotonically increasing function defined by Definition 3.1. For $y, z \in Z_{\geq 0}$, the chocolate bar has $z + 1$ columns, where the 0-th column is the bitter square, and the height of the i-th column is $t(i) = \min(f(i), y) + 1$ for i = 0,1,...,z, which is denoted as $CB(f, y, z)$.

Thus, the height of the i-th column is determined by the value of $\min(f(i), y)+1$, which is determined by f, i, and y.

Definition 3.3. Each player takes turns breaking the bar in a straight line along the grooves into two pieces, and eats the piece without the bitter part. The player who breaks the chocolate bar and eats it, leaving the opponent with a single bitter block (black block), is the winner.

We define a function f for a chocolate bar $CB(f, y, z)$, and denote y, z as the coordinates of $CB(f, y, z)$.

Example 3.1. Let $f(t) = \lfloor \frac{t}{2} \rfloor$, where $= \lfloor\ \rfloor$ is the floor function. Here, we present examples of $CB(f, y, z)$-type chocolate bars. Note that the function f defines the shape of the bar, and the two coordinates y and z represent the number of grooves above and to the right of the bitter square, respectively.

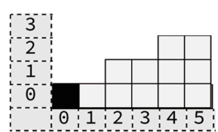

Figure 9: $\{2, 5\}$

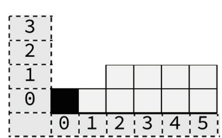

Figure 10: $\{1, 5\}$

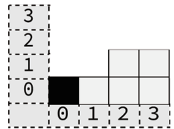

Figure 11: $\{1, 3\}$

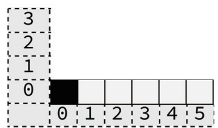

Figure 12: $\{0.5\}$

For a fixed function f, we define $move_f$ for each position $\{y, z\}$ of the chocolate bar $CB(f, y, z)$. The set $move_f(\{y, z\})$ is comprised of the positions of the chocolate bar obtained by cutting the chocolate bar $CB(f, y, z)$ once, and $move_f$ represents a special case of *move* defined by Definition 2.4.

Definition 3.4. For $y, z \in Z_{\geq 0}$, we define
$move_f(\{y, z\}) = \{\{v, z\} : v < y\} \cup \{\{\min(y, f(w)), w\} : w < z\}$, where $v, w \in Z_{\geq 0}$.

Remark 3.1. For a fixed function f, we use $move(\{y, z\})$ instead of $move_f(\{y, z\})$ for convenience.

Example 3.2. Here, we elucidate $move_f$ for $f(t) = \lfloor \frac{t}{2} \rfloor$. If we begin with position $\{y, z\} = \{2, 5\}$ in Figure 9 and reduce $z = 5$ to $z = 3$, the y-coordinate (first coordinate) becomes $\min(2, \lfloor 3/2 \rfloor) = \min(2, 1) = 1$.

Therefore, we have $\{1, 3\} \in move_f(\{2, 5\})$; i.e., we obtain $\{1, 3\}$ in Figure 11 by cutting $\{2, 5\}$. It can be easily determined that $\{1, 5\}, \{0, 5\} \in move_f(\{2, 5\})$, $\{1, 3\} \in move_f(\{1, 5\})$, and $\{0, 5\} \notin move_f(\{1, 3\})$. See Figures 9, 10, 11, and 12.

According to Definitions 2.5 and 3.4, we define the Grundy number of a two-dimensional chocolate bar.

Definition 3.5. For $y, z \in Z_{\geq 0}$, we define
$\mathcal{G}(\{y, z\}) = mex(\{\mathcal{G}(\{v, z\}) : v < y, v \in Z_{\geq 0}\} \cup \{\mathcal{G}(\{\min(y, f(w)), w\}) : w < z, w \in Z_{\geq 0}\})$.

Definition 3.6. Let h be a monotonically increasing function defined by Definition 3.1. Function h is said to have the NS property, if h satisfies condition (a).

(a) Suppose that

$$\lfloor \frac{z}{2^i} \rfloor = \lfloor \frac{z'}{2^i} \rfloor$$

for some z, $z' \in Z_{\geq 0}$, and some natural number i. Then,

$$\lfloor \frac{h(z)}{2^{i-1}} \rfloor = \lfloor \frac{h(z')}{2^{i-1}} \rfloor.$$

Theorem 2. *Let h be a monotonically increasing function defined by Definition 3.6. Let $\mathcal{G}_h$ be the Grundy number of $CB(h, y, z)$. Then, $\mathcal{G}_h(\{y, z\}) = y \oplus z$ if and only if h has the NS property as per Definition 3.6.*

Proof of this theorem is provided in Theorems 4 and 5 in [2].

A new lemma for two-dimensional chocolate bars is given below and is used for three-dimensional chocolate bars in Section 4.

Lemma 4. *Suppose that h has the NS property as per Definition 3.6, and $y \leq h(z)$ for $y, z \in Z_{\geq 0}$. Let*

$$A = \{y \oplus (z - k) : k = 1, 2, \cdots, z\}$$

and

$$B = \{\min(y, h(z - k)) \oplus (z - k) : k = 1, 2, \cdots, z\}.$$

Then, $A = B$.

Proof. For any $u, v \in Z_{\geq 0}$ with $u \leq h(v)$, let $\mathcal{G}_h(\{u, v\})$ be the Grundy number of $CB(h, u, v)$. Then, by the NS property of function h and Theorem 2,

$$\mathcal{G}_h(\{u, v\}) = u \oplus v. \tag{2}$$

Let $y, z \in Z_{\geq 0}$ such that

$$y \leq h(z). \tag{3}$$

Let n be a natural number such that

$$2^n > z, y. \tag{4}$$

An arbitrary element of A can be represented as

$$y \oplus i \in A \tag{5}$$

for some i such that $0 \leq i < z$. According to the inequality in (4),

$$\mathcal{G}_h(\{y, z + 2^n\}) = y \oplus (z + 2^n) > y \oplus i.$$

Hence, according to Lemma 3 and Equation (2),

$$\{u, v\} \in move((y, z + 2^n)) \tag{6}$$

such that

$$\mathcal{G}_h(\{u, v\}) = u \oplus v = y \oplus i. \tag{7}$$

Because $h(z)$ is monotonically increasing, according to the inequality in (3), for $w = 0,\ 1, 2, ..., 2^n - 1$,

$$y \leq h(z + w). \tag{8}$$

According to Definition 3.4,

$$\begin{aligned} move(\{y, z + 2^n\}) = &\{\{y - k, z + 2^n\} : k = 1, 2, \cdots, y\} \\ &\cup\{\{\min(y, h(z + 2^n - k)), z + 2^n - k\} : k = 1, 2, \cdots, z + 2^n\}. \end{aligned}$$

Therefore, $move(\{y, z + 2^n\})$ is the union of the sets given below as (9), (10), and (11).

$$\{\{y - j, z + 2^n\} : j = 1, 2, \cdots, y\}. \tag{9}$$

$$\begin{aligned} &\{\{\min(y, h(z + 2^n - k)), z + 2^n - k\} : k = 1, 2, \cdots, 2^n\} \\ =&\{\{y, z + 2^n - k\} : k = 1, 2, \cdots, 2^n\}, \end{aligned} \tag{10}$$

which follows from inequality (8).

$$\{\{\min(y, h(z - k)), z - k\} : k = 1, 2, \cdots, z\}. \tag{11}$$

From (4), $(y - j) \oplus (z + 2^n) \geq 2^n > y \oplus i$ for $j = 1, 2, ...y$. Hence, by (7), (u, v) does not belong to the set in (9).

Because $0 \leq i < z$, according to Lemma 1, $y \oplus i \neq y \oplus (z + j)$ for $j = 0, 1, 2, ..., 2^{n+1} - 1$; hence, (u, v) does not belong to the set expressed by (10). Therefore, according to (6), $\{u, v\}$ belongs to the set given as (11); hence,

$$u \oplus v = \min(y, h(z - t)) \oplus (z - t) \tag{12}$$

for some $t \in Z_{\geq 0}$ such that $1 \leq t \leq z$.

Therefore, according to (7) and (12)

$$y \oplus i = u \oplus v \in B. \tag{13}$$

As expressed by relation (5), $y \oplus i$ is an arbitrary element of A; hence, according to (13), $A \subset B$. According to Lemma 1, the number of elements in A is z, and the number of elements in B is less than or equal to z. Therefore, as $A \subset B$, $A = B$. $\square$

Thus far, we have considered only two-dimensional chocolate bars for monotonically increasing functions. However, we can similarly consider a three-dimensional chocolate bar $CB(f, y, z)$ for a function f, which does not monotonically increase, by forming a monotonically increasing function f' such that chocolate bars $CB(f, y, z)$ and $CB(f', y, z)$ have the same mathematical structure in the context of the game.

For example, the chocolate bar in Figure 13 is constructed by a function that does not monotonically increase, whereas the chocolate bar in Figure 14 is formed by a monotonically increasing function; however, these two chocolate bars have the same mathematical structure in terms of the game.

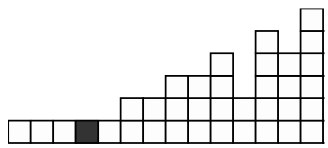

Figure 13.

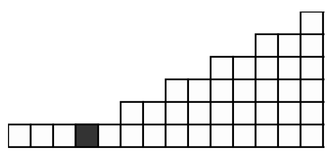

Figure 14.

Therefore, it is sufficient to study the case of a monotonically increasing function for two-dimensional chocolate bars.

4. Three-Dimensional Chocolate Bar

In this section, we answer the research question that was presented in Section 1 . Theorems 3 and 4 offer proofs of the sufficient and necessary conditions, respectively.

Definition 4.1. Suppose that $F(u, v) \in Z_{\geq 0}$ for $u, v \in Z_{\geq 0}$. F is said to be *monotonically increasing* if $F(u, v) \leq F(x, z)$ for $x, z, u, v \in Z_{\geq 0}$ for $u \leq x$ and $v \leq z$.

By generalizing Definition 3.2, we define a three-dimensional chocolate bar below.

Definition 4.2. Let F be the monotonically increasing function in Definition 4.1. Let $x, y, z \in Z_{\geq 0}$. The three-dimensional chocolate bar is comprised of a set of

$1 \times 1 \times 1$ sized boxes. For $u, w \in Z_{\geq 0}$ such that $u \leq x$ and $w \leq z$, the height of the column of position (u, w) is $\min(F(u, w), y) + 1$, where F is a monotonically increasing function. A bitter box is located in position $(0, 0)$. We denote this chocolate bar as $CB(F, x, y, z)$.

Definition 4.3. We define a three-dimensional chocolate-bar game. Each player takes turns cutting the bar along a plane oriented horizontally or vertically along the grooves, and eats the broken piece. The player who successfully leaves the opponent with a single bitter cube wins the game.

Example 4.1. Here, we provide an example of a three-dimensional coordinate system and two examples of three-dimensional chocolate bars.

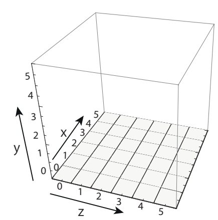

Figure 15.

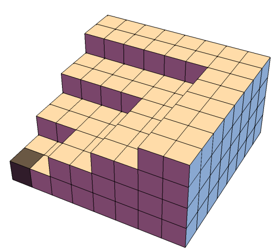

Figure 16: $CB(F, 7, 3, 7)$
$F(x, z) = \max(\lfloor \frac{x}{2} \rfloor, \lfloor \frac{z}{2} \rfloor)$.

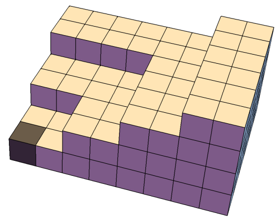

Figure 17: $CB(F, 5, 3, 7)$
$F(x, z) = \max(\lfloor \frac{x}{2} \rfloor, \lfloor \frac{z}{2} \rfloor)$.

Next, we define $move_F(\{x, y, z\})$ in Definition 4.4 as a set containing all the positions that can be directly reached from position $\{x, y, z\}$ in one step.

Definition 4.4. For $x, y, z \in Z_{\geq 0}$, we define

$$\begin{aligned} move_F(\{x, y, z\}) =& \{\{u, \min(F(u, z), y), z\} : u < x\} \cup \{\{x, v, z\} : v < y\} \\ & \cup \{\{x, \min(y, F(x, w)), w\} : w < z\}, \text{where } u, v, w \in Z_{\geq 0}. \end{aligned}$$

For example, when $F(x, z) = \max(\lfloor \frac{x}{2} \rfloor, \lfloor \frac{z}{2} \rfloor)$, then $\{5, 3, 7\} \in move_F(\{7, 3, 7\})$ because we obtain the chocolate bar shown in Figure 17 by reducing the third coordinate of the chocolate bar in Figure 16 from 7 to 5.

Remark 4.1. For a fixed function f, we use $move(\{x, y, z\})$ instead of $move_F(\{x, y, z\})$ for convenience.

Lemma 5. *For any $k, h, i \in Z_{\geq 0}$, we have*

$$\begin{aligned} k \oplus h \oplus i =& mex(\{(k - t) \oplus h \oplus i : t = 1, 2, ..., k\}, \\ & \cup \{k \oplus (h - t) \oplus i : t = 1, 2, ..., h\} \cup \{k \oplus h \oplus (i - t) : t = 1, 2, ..., i\}). \end{aligned} \tag{14}$$

Proof. The proof is omitted because this is a well-known fact regarding Nim-sum $\oplus$. See proposition 1.4. (p.181) in [6]. □

Theorem 3. *Let $F(x, z)$ be a monotonically increasing function. Let $g_n(z) = F(n, z)$ and $h_m(x) = F(x, m)$ for $n, m \in Z_{\geq 0}$. If g_n and h_m satisfy the NS property in Definition 3.6 for any fixed $n, m \in Z_{\geq 0}$, then the Grundy number of chocolate bar $CB(F, x, y, z)$ is*

$$\mathcal{G}(\{x, y, z\}) = x \oplus y \oplus z. \tag{15}$$

Proof. Let $x, y, z \in Z_{\geq 0}$ such that $y \leq F(x, z)$. We prove (15) by mathematical induction and suppose that $\mathcal{G}(\{u, v, w\}) = u \oplus v \oplus w$ for $u, v, w \in Z_{\geq 0}$, $u \leq x, v \leq$

y,$w \leq z$, $v \leq f(u, w)$, with $u + v + w < x + y + z$.
Let

$$A = \{x \oplus y \oplus (z - k) : k = 1, 2, \cdots, z\} \tag{16}$$

and

$$A' = \{x \oplus \min(y, F(x, z - k)) \oplus (z - k) : k = 1, 2, \cdots, z\}. \tag{17}$$

As $g_x(z) = F(x, z)$ satisfies the NS property, according to Lemma 4,

$$A = A'. \tag{18}$$

Let

$$B = \{(x - k) \oplus y \oplus z : k = 1, 2, \cdots, x\} \tag{19}$$

and

$$B' = \{(x - k) \oplus \min(y, F(x - k, z)) \oplus z : k = 1, 2, \cdots, x\}. \tag{20}$$

Because $h_z(x) = F(x, z)$ satisfies the NS property, according to Lemma 4

$$B = B'. \tag{21}$$

Let

$$C = \{x \oplus (y - k) \oplus z : k = 1, 2, \cdots, y\}. \tag{22}$$

By the mathematical induction hypothesis, the definition of $move_F$ in Definition 4.4 along with Equations (17), (20), and (22),

$$\begin{aligned} \mathcal{G}(\{x, y, z\}) =& mex(\{\mathcal{G}(\{x, \min(y, F(x, z - k)), z - k\}) : k = 1, 2, \cdots, z\}) \\ & \cup \{\mathcal{G}(\{x - k, \min(y, F(x - k, z)), z\}) : k = 1, 2, \cdots, x\} \\ & \cup \{\mathcal{G}(\{x, y - k, z\}) : k = 1, 2, \cdots, y\}) \\ =& mex(A' \cup B' \cup C). \end{aligned} \tag{23}$$

According to Equations (21) and(18),

$$mex(A' \cup B' \cup C) = mex(A \cup B \cup C). \tag{24}$$

From Equations (16), (19), and (22) and Lemma 5,

$$mex(A \cup B \cup C) = x \oplus y \oplus z. \tag{25}$$

From Equations (23), (24), and (25), we have Equation (15). □

Lemma 6. *Let $i \in Z_{\geq 0}$ and $z < z'$. Then, (a) and (b) hold.*
(a)

$$\lfloor \frac{z}{2^i} \rfloor = \lfloor \frac{z'}{2^i} \rfloor,$$

if and only if $d \in Z_{\geq 0}$ exists such that

$$d \times 2^i \leq z < z' < (d+1) \times 2^i.$$

(b) *Let*

$$\lfloor \frac{z}{2^i} \rfloor < \lfloor \frac{z'}{2^i} \rfloor. \tag{26}$$

Then, there exist $c, s, t \in Z_{\geq 0}$ such that $s \geq i$, $0 \leq t < 2^s$, and

$$z = c \times 2^{s+1} + t < c \times 2^{s+1} + 2^s \leq z'. \tag{27}$$

Proof. Let $z = \sum_{i=0}^{n} z_i 2^i$ and $z' = \sum_{i=0}^{n} z'_i 2^i$. (a) follows directly from the definition of the floor function; (b) falls into two cases according to the inequality in (26).

`Case (i)` Suppose that $z < 2^n \leq z'$. Let $c = 0$ and $t = z$. Then, we have the inequality in (27).

`Case (ii)` Suppose $s \in Z_{\geq 0}$ exists such that $s \geq i$ and $z_k = z'_k$ for $k = n, n-1, ..., s+1$ and $z_s = 0 < 1 = z'_s$. Then, there exist $c, t \in Z_{\geq 0}$ satisfying the inequality in (27). □

Theorem 4. *Let $F(x, z)$ be a monotonically increasing function, and let $g_n(z) = F(n, z)$ and $h_m(x) = F(x, m)$ for $n, m \in Z_{\geq 0}$. Suppose that the Grundy number of chocolate bar $CB(F, x, y, z)$ is*

$$\mathcal{G}(\{x, y, z\}) = x \oplus y \oplus z. \tag{28}$$

Then, g_n and h_m satisfy the NS property in Definition 3.6 for any fixed $n, m \in Z_{\geq 0}$.

Proof. Let $n \in Z_{\geq 0}$; to prove that g_n has the NS property, it suffices to show that

$$\lfloor \frac{g_n(a)}{2^{j-1}} \rfloor = \lfloor \frac{g_n(a+1)}{2^{j-1}} \rfloor$$

for $a \in Z_{\geq 0}$ such that

$$\lfloor \frac{a}{2^j} \rfloor = \lfloor \frac{a+1}{2^j} \rfloor. \tag{29}$$

To prove this by contradiction, we assume

$$\lfloor \frac{g_n(a)}{2^{j-1}} \rfloor < \lfloor \frac{g_n(a+1)}{2^{j-1}} \rfloor \tag{30}$$

for $a \in Z_{\geq 0}$ satisfying Equation (29). Here, we assume that $a \in Z_{\geq 0}$ is the smallest integer that satisfies Equation (29) and the inequality in (30). According to the inequality in (30) and (2) of Lemma 6, there exist $i, c \in Z_{\geq 0}$ and $t \in R$ such that $i \geq j-1$, $0 \leq t < 2^i$, and

$$g_n(a) = c \times 2^{i+1} + t < c \times 2^{i+1} + 2^i \leq g_n(a+1). \tag{31}$$

As $i+1 \geq j$, according to Equation (29),

$$\lfloor \frac{a}{2^{i+1}} \rfloor = \lfloor \frac{a+1}{2^{i+1}} \rfloor.$$

Hence, according to (1) of Lemma 6, for $d \in Z_{\geq 0}$,

$$d \times 2^{i+1} \leq a < a+1 < (d+1)2^{i+1}.$$

Therefore, we have the following inequalities, given as (32) and (33).

$$d \times 2^{i+1} \leq a < a+1 = d \times 2^{i+1} + 2^i + e < (d+1)2^{i+1} \tag{32}$$

for $d, e \in Z_{\geq 0}$ such that $0 \leq e < 2^i$.

$$d \times 2^{i+1} \leq a < a+1 = d \times 2^{i+1} + e < (d+1)2^{i+1} \tag{33}$$

for $d, e \in Z_{\geq 0}$ such that $0 < e < 2^i$.

`Case (i)` If we have the inequality in (32), then

$$\begin{aligned}
(c \times 2^{i+1} + 2^i) \oplus (a+1) =& (c \times 2^{i+1} + 2^i) \oplus (d \times 2^{i+1} + 2^i + e) \\
=& (c \oplus d)2^{i+1} + e \\
<& (c \oplus d)2^{i+1} + 2^i + (t \oplus e) \\
=& (c \times 2^{i+1} + t) \oplus (d \times 2^{i+1} + 2^i + e) \\
=& (c \times 2^{i+1} + t) \oplus (a+1).
\end{aligned} \tag{34}$$

Let $g'(z) = \min(g_n(z), c \times 2^{i+1} + t)$; we consider the two-dimensional chocolate bar $CB(g', y, z)$ for $z \leq a+1$ as defined in Section 3. Let $\mathcal{G}_{g'}(\{y, z\})$ be the Grundy number of this chocolate bar $CB(g', y, z)$. Because $g_n(z)$ is monotonically increasing, according to (31) for $z \leq a$,

$$g_n(z) \leq g_n(a) = c \times 2^{i+1} + t.$$

Therefore,

$$g'(z) = \min(g_n(z), c \times 2^{i+1} + t) = g_n(z). \tag{35}$$

Because $a \in Z_{\geq 0}$ is the smallest integer that satisfies Equation (29) and the inequality in (30), $g'(z)$ satisfies the NS-property for $z \leq a$. According to the inequality in (31) and the definition of g',

$$g'(a+1) = \min(g_n(a+1), c \times 2^{i+1} + t) = c \times 2^{i+1} + t = g'(a).$$

Therefore, $g'(z)$ satisfies the NS-property for $z \leq a+1$. Then, according to Theorem 2,

$$\mathcal{G}_{g'}(\{y, z\}) = y \oplus z \tag{36}$$

for $y, z \in Z_{\geq 0}$ such that $z \leq a+1$ and $y \leq g'(z)$. By the inequality in (34) and Equations (35) and (36),

$$\begin{aligned}(c \times 2^{i+1} + 2^i) \oplus (a+1) <&(c \times 2^{i+1} + t) \oplus (a+1) \\ =&\mathcal{G}_{g'}(\{c \times 2^{i+1} + t, a+1\}).\end{aligned} \tag{37}$$

According to the inequality in (37) and Lemma 3,

$$\begin{aligned}&(c \times 2^{i+1} + 2^i) \oplus (a+1) \\ \in&\{\mathcal{G}_{g'}(\{p, q\}) : \{p, q\} \in move_{g'}(\{c \times 2^{i+1} + t, a+1\})\}.\end{aligned} \tag{38}$$

Based on Definition 3.4, $move_{g'}(\{c \times 2^{i+1} + t, a+1\})$ is a union of two sets. The first set is created by reducing the first coordinate of point $\{c \times 2^{i+1} + t, a+1\}$ and the second is created by reducing the second coordinate of point $\{c \times 2^{i+1} + t, a+1\}$. Therefore, based on $g'(w) \leq c \times 2^{i+1} + t$, we obtain

$$\begin{aligned}&\{\mathcal{G}_{g'}(\{p, q\}) : \{p, q\} \in move_{g'}(\{c \times 2^{i+1} + t, a+1\})\} \\ =&\{G_{g'}(\{v, a+1\}) : 0 \leq v \leq c \times 2^{i+1} + t - 1\}, \\ \cup&\{G_{g'}(\{\min(c \times 2^{i+1} + t, g'(w)), w\}) : 0 \leq w \leq a\}. \\ =&\{G_{g'}(\{v, a+1\}) : 0 \leq v \leq c \times 2^{i+1} + t - 1\}, \\ \cup&\{G_{g'}(\{g'(w), w\}) : 0 \leq w \leq a\}.\end{aligned}$$

Therefore, according to (36) and (38), we have

$$\begin{aligned}&(c \times 2^{i+1} + 2^i) \oplus (a+1) \\ \in&\{G_{g'}(\{v, a+1\}) = v \oplus (a+1) : 0 \leq v \leq c \times 2^{i+1} + t - 1\}, &(39)\\ \cup&\{G_{g'}(\{g'(w), w\}) = g'(w) \oplus w : 0 \leq w \leq a\}. &(40)\end{aligned}$$

As $t < 2^i$, according to Lemma 1,

$$(c \times 2^{i+1} + 2^i) \oplus (a+1) \notin \{\mathcal{G}_{g'}(\{v, a+1\}) = v \oplus (a+1) : 0 \leq v \leq c \times 2^{i+1} + t - 1\}.$$

Therefore, by (39) and (40),

$$(c \times 2^{i+1} + 2^i) \oplus (a+1) \in \{g'(w) \oplus w : 0 \leq w \leq a\}. \tag{41}$$

According to the inequality in (31),

$$c \times 2^{i+1} + 2^i \leq g_n(a+1) = F(n, a+1);$$

hence, $\{n, c \times 2^{i+1} + 2^i, a+1\}$ is the position of chocolate bar $CB(F, x, y, z)$.

Therefore, based on Equation (28),

$$\mathcal{G}(\{n, c \times 2^{i+1} + 2^i, a+1\}) = n \oplus (c \times 2^{i+1} + 2^i) \oplus (a+1). \tag{42}$$

Then, by Relation (41),

$$n \oplus (c \times 2^{i+1} + 2^i) \oplus (a+1) \in \{n \oplus g'(w) \oplus w : 0 \leq w \leq a\}. \tag{43}$$

Then, according to Equations (28) and (35) and the definition of g_n,

$$\begin{aligned} &\{n \oplus g'(w) \oplus w : 0 \leq w \leq a\} \\ =&\{n \oplus g_n(w) \oplus w : 0 \leq w \leq a\} \\ =&\{n \oplus F(n, w) \oplus w : 0 \leq w \leq a\} \\ =&\{\mathcal{G}(\{n, F(n, w), w\}) : 0 \leq w \leq a\}. \end{aligned} \tag{44}$$

Therefore, based on Equations (42) and (44) and Relation (43), w' exists, such that $0 \leq w' \leq a$ and

$$\mathcal{G}(\{n, c \times 2^{i+1} + 2^i, a+1\}) = \mathcal{G}(\{n, F(n, w'), w'\}). \tag{45}$$

According to the inequality in (31),

$$F(n, w') \leq F(n, a) = g_n(a) = c \times 2^{i+1} + t < c \times 2^{i+1} + 2^i;$$

hence,

$$\begin{aligned} &\{n, F(n, w'), w'\} \\ =&\{n, \min(c \times 2^{i+1} + 2^i, F(n, w')), w'\} \\ \in& move(\{n, c \times 2^{i+1} + 2^i, a+1\}). \end{aligned} \tag{46}$$

Equation (45) and Relation (46) contradict the definition of the Grundy number.

`Case (ii)` If we have the inequality in (33), then, as $0 < e < 2^i$ and $0 \leq t < 2^i$,

$$\begin{aligned} (c \times 2^{i+1} + t) \oplus a =& (c \times 2^{i+1} + t) \oplus (d \times 2^{i+1} + e - 1) \\ =& (c \times 2^{i+1} + 2^i) \oplus (d \times 2^{i+1} + 2^i + t \oplus (e-1)]. \end{aligned}$$

Therefore, by Equation (28),

$$\mathcal{G}(\{n, c \times 2^{i+1} + t, a\}) = \mathcal{G}(\{n, c \times 2^{i+1} + 2^i, d \times 2^{i+1} + 2^i + t \oplus (e-1)\}). \tag{47}$$

According to inequalities (31) and (33),

$$\begin{aligned} c \times 2^{i+1} + 2^i \leq& g_n(a+1) \\ =& F(n, a+1) \\ \leq& F(n, a+1 + t \oplus (e-1)) \\ \leq& F(n, d \times 2^{i+1} + 2^i + t \oplus (e-1)); \end{aligned}$$

hence, $\{n, c \times 2^{i+1} + 2^i, d \times 2^{i+1} + 2^i + t \oplus (e-1)\}$ is the position of chocolate bar $CB(F, x, y, z)$. According to (31),

$$c \times 2^{i+1} + t = g_n(a) = F(n, a). \tag{48}$$

According to (33),

$$a < d \times 2^{i+1} + 2^i + t \oplus (e-1). \tag{49}$$

According to (48) and (49),

$$\{n, c \times 2^{i+1} + t, a\} \in \quad move(\{n, c \times 2^{i+1} + 2^i, d \times 2^{i+1} + 2^i + t \oplus (e-1)\});$$

this relation and Equation (47) lead to a contradiction. □

Thus far, we have only considered three-dimensional chocolate bars for monotonically increasing functions. However, we can similarly consider a chocolate bar $CB(F, x, y, z)$ for a function F that is not monotonically increasing by constructing a monotonically increasing function F', such that position $\{x, y, z\}$ of chocolate bar $CB(F, x, y, z)$ and position $\{x, y, z\}$ of $CB(F', x, y, z)$ have the same bar length, height, and width, and have the same mathematical structure as a game.

For example, the chocolate bar in Figure 18 is constructed by a function that does not monotonically increase, whereas the chocolate bar in Figure 19 is formed by a monotonically increasing function; however, these two chocolate bars have the same mathematical structure as a game.

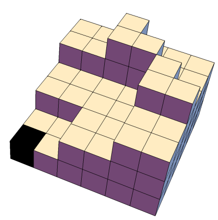

Figure 18.

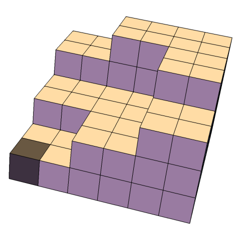

Figure 19.

Therefore, it suffices to study the case of a monotonically increasing function for three-dimensional chocolate bars.

5. Unsolved Problems

Certain chocolate bars remain that have not been considered.

Here, we present a chocolate bar with two steps; see Figure 20. This chocolate bar is represented by three coordinates x, y, z; the reduction of z may simultaneously affect the first and second coordinates x and y. The relationships between these three coordinates are expressed by the following two inequalities:

$$x \leq \lfloor \frac{z+3}{2} \rfloor$$

and

$$y \leq \lfloor \frac{z+3}{2} \rfloor.$$

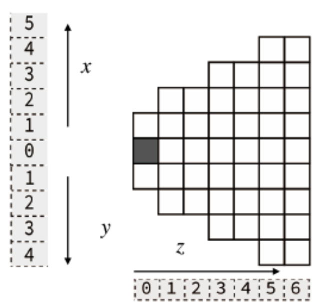

Figure 20.

The result in [2] can be applied to this type of chocolate bar; however, this proof may be complicated compared to the proof in [2].

As another type of chocolate bar, we consider a three-dimensional bar with an upper and lower structure. An example of this type of chocolate bar is shown in Figure 21.

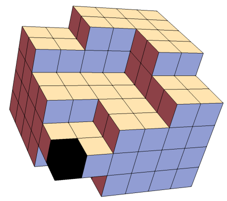

Figure 21.

The results of the present work can be used to study this type of chocolate bar. Moreover, research on the previously mentioned chocolate bar with two steps should be conducted in the future.

The chocolate bars shown in Figures 20 and 21 appear to be simple generalizations of the chocolate bars studied here and in [2]; however, they are technically complex, and their investigation may prove challenging.

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