TRANSVERSE WAVE: AN IMPARTIAL COLOR-PROPAGATION GAME INSPRIED BY SOCIAL INFLUENCE AND QUANTUM NIM

Kyle Burke[1]
Department of Computer Science, Plymouth State University, Plymouth, New Hampshire

Matthew Ferland
Department of Computer Science, University of Southern California, Los Angeles, California

Shang-Hua Teng
Department of Computer Science, University of Southern California, Los Angeles, California
`https://viterbi-web.usc.edu/~shanghua/`

Received: 1/15/21, Revised: 9/14/21, Accepted: 12/3/21, Published: 12/20/21

Abstract

In this paper, we study Transverse Wave, a colorful, impartial combinatorial game played on a two-dimensional grid. We are drawn to this game because of its apparent simplicity, contrasting intractability, and intrinsic connection to two other combinatorial games, one about social influences and another inspired by quantum superpositions. More precisely, we show that Transverse Wave is at the intersection of two other games, the social-influence-derived Friend Circle and superposition-based Demi-Quantum Nim. Transverse Wave is also connected with Schaefer's logic game Avoid True from the 1970's. In addition to analyzing the mathematical structures and computational complexity of Transverse Wave, we provide a web-based version of the game. Furthermore, we formulate a basic network-influence game, called Demographic Influence, which simultaneously generalizes Node-Kayles and Demi-Quantum Nim (which in turn contains Nim, Avoid True, and Transverse Wave as special cases). These connections illuminate a *lattice order* of games, induced by special-case/generalization relationships, fundamental to both the design and comparative analyses of combinatorial games.

1. The Game Dynamics of Transverse Wave

Elwyn Berlekamp, John Conway, and Richard Guy were known not only for their deep mathematical discoveries but also for their elegant minds. Their love of math-

[1] Webpage: `https://turing.plymouth.edu/~kgb1013/`

ematics and their life-long efforts of making mathematics fun and approachable led to their masterpiece, "Winning Ways for your Mathematical Plays" [3], a book that has inspired many. The field that they pioneered—*combinatorial game theory*—reflects their personalities. Popular combinatorial games are usually:

- ***Approachable***: they have easy to remember and understand rules, and
- ***Elegant***: they have attractive game boards, yet
- ***Deep***: they are challenging to play optimally and have intriguing strategies.

1.1. Combinatorial Games and Computational Complexity

The last property has a characterization based on computational complexity. In order to make the competition *interesting*, we want the winnability to be *computationally intractable*, meaning there is no known efficient algorithm to always calculate a position's outcome class. One way to argue this is to show that the problem of finding the outcome class is hard for a common complexity class. Many such combinatorial games are found to be PSPACE-hard, meaning that finding a polynomial-time algorithm automatically leads to a polynomial-time solution to *all* problems in PSPACE.[2]

On the other hand, for games where the winner can be determined algorithmically in polynomial-time, optimal players can be programmed that will run efficiently. In a match with one of these players, there is no need to play the game out to determine whether a winning move exists; just run the program and use that output.

In popular games from Go [21] to Hex [25, 23], determining the winnability becomes computationally intractable as the dimensions of the board grow. These sorts of elegant combinatorial games with easy to follow rulesets and intractable complexity are the gold standard for combinatorial game design [11, 5].

This argument does not entirely settle the debate about whether a ruleset is "interesting". Indeed, it could be the case that from common starting positions, there is a strategy for the winning player to avoid the computationally-hard positions. Finding computational-hardness for general positions in a ruleset is only a minimum requirement. Improvements can be made by finding hard positions that are "naturally" reachable from the start. The best proofs of hardness yield positions that are starting positions themselves[3].

[2]Note that NP is contained in PSPACE.

[3]This requires some variance on these starting positions. "Empty" or well-structured initial boards do not have a large enough descriptive size to be computationally hard in the expected measures.

1.2. A Colorful Propagation Game over 2D Grids

In this paper, we consider a simple, colorful, impartial combinatorial game over two-dimensional grids.[4] We call this game TRANSVERSE WAVE. We became interested in this game during our study of quantum combinatorial games [5], particularly, in the complexity-theoretical analysis of a family of games formulated by NIM superpositions. In addition to Schaefer's logic game, AVOID TRUE, TRANSVERSE WAVE is also fundamentally connected with several social-influence-inspired combinatorial games. As we shall show, TRANSVERSE WAVE is PSPACE-hard on some possible starting positions.

Ruleset 1 (TRANSVERSE WAVE)**.** For a pair of integer parameters $m, n > 0$, a game position of TRANSVERSE WAVE is an m by n grid G, in which the cells are colored either `green` or `purple`.[5]

For the game instance starting at this position, two players take turns selecting a column of this colorful grid. A column $j \in [n]$ is feasible for G if it contains at least one `green` cell. The selection of j transforms G into another colorful m by n grid $G \otimes [j]$ by recoloring column j and every row with a `purple` cell at column j with `purple`.[6] In the normal-play convention, the player without a feasible move loses the game.

See Figure 1 for an example of a TRANSVERSE WAVE move.

Note that purple cells cannot change to green, and each move changes all green cells in one column to purple (and possibly some in other columns as well). Thus, any position with dimension m by n must end in at most n turns, and the height of a TRANSVERSE WAVE game tree is at most n. Consequently, TRANSVERSE WAVE is solvable in polynomial space.

Note also that the selection of column j could make some other feasible columns infeasible[7]. In Section 4, we will show that the interaction among columns introduces sufficiently rich mathematical structures for TRANSVERSE WAVE to efficiently encode any PSPACE-complete game such as HEX [17, 22, 12, 23], AVOID TRUE [25], NODE KAYLES [25], GO [21], or GEOGRAPHY [21]. In other words, TRANSVERSE WAVE is a PSPACE-complete impartial game.

We have implemented TRANSVERSE WAVE in HTML/JavaScript.[8]

[4]We tested the approachability of this game by explaining its ruleset to a bilingual eight-year-old second-grade student—in Chinese—and she turned to her historian mother and flawlessly explained the ruleset in English.

[5]or any pair of easily-distinguishable colors.

[6]Think of purple paint cascading down column j and inducing a purple "transverse wave" whenever the propagation goes through an already-purple cell.

[7]This perpendicular propagation gives rise to the name "Transverse Wave".

[8]Web version: `https://turing.plymouth.edu/~kgb1013/DB/combGames/transverseWave.html`

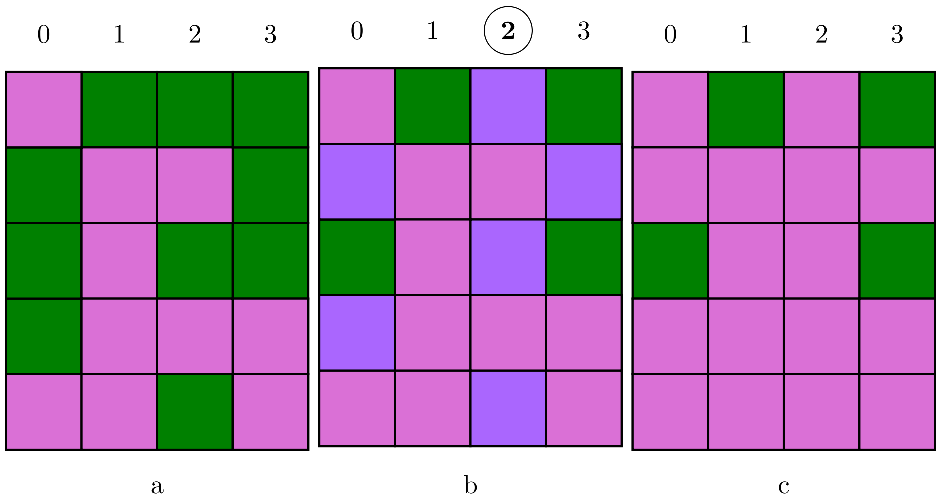

Figure 1: An example move for Transverse Wave. **a**: A position, from which the first player chooses column 2. (This is a legal choice, because column 2 has a green cell.) **b**: Indigo cells denote those that will become purple. These include the previously-green cells in column 2 as well as the green cells in rows where column 2 had purple cells. **c**: The new position after all cells are changed to be purple.

1.3. Dual Logical Interpretations of Transverse Wave

Transverse Wave uses only two colors; a position can be expressed naturally with a Boolean matrix. Furthermore, making a move can be neatly captured by basic Boolean functions. Let us consider the following two combinatorial games over Boolean matrices that are isomorphic to Transverse Wave. While these logic associations are straightforward, they set up stimulating connections to combinatorial games inspired by social influence and quantum superpositions.

We use the following standard notation for matrices. For an $m \times n$ matrix $\mathbf{A}$, $i \in [m]$ and $j \in [n]$, let $\mathbf{A}[i,:]$, $\mathbf{A}[:,j]$, $\mathbf{A}[i,j]$ denote, respectively, the i^{th} row, j^{th} column, and the $(i,j)^{th}$ entry in $\mathbf{A}$.

Ruleset 2 (Crosswise AND)**.** For integers $m, n > 0$, Crosswise AND plays on an $m \times n$ Boolean matrix $\mathbf{B}$.

During the game, two players alternatively select $j \in [n]$, where j is feasible for $\mathbf{B}$ if $\mathbf{B}[:,j] \neq \vec{0}$. The move with selection j then changes the Boolean matrix to the following: for all $i \neq j \in [m]$, the i^{th} row takes a component-wise AND with its j^{th} bit, $\mathbf{B}[i,j]$, then the $(i,j)^{th}$ entry is set to 0. Under normal play, the player with no feasible column to choose loses the game.

By mapping purple cells to Boolean 0 (i.e., `false`) and green cells to Boolean 1 (i.e., `true`), we see the following proposition.

Proposition 1. Transverse Wave *and* Crosswise AND *are isomorphic games.*

Ruleset 3 (Crosswise OR)**.** For integer parameters $m, n > 0$, Crosswise OR plays on an $m \times n$ Boolean matrix $\mathbf{B}$.

During the game, two players alternatively select $j \in [n]$, where j is feasible for $\mathbf{B}$ if $\mathbf{B}[:, j] \neq \vec{1}$. The move with selection j then changes the Boolean matrix to the following: for all $i \neq j \in [m]$, the i^{th} row takes a component-wise OR with its j^{th} bit, $\mathbf{B}[i, j]$, then the $(i, j)^{th}$ entry is set to 1. Under normal play, the player with no feasible column to choose loses the game.

By mapping purple cells to Boolean 1, and green cells to Boolean 0, we see another isomorphism.

Proposition 2. Transverse Wave *and* Crosswise OR *are isomorphic games.*

1.4. Transverse Wave Game Values

As we will discuss later, Transverse Wave is PSPACE-complete. So, we have no hope for an efficient complete characterization for the game values. In this subsection, we show that we can fully characterize the *Grundy values* (i.e., *nimbers*) for the specific case of Transverse Wave where each column has either 0 or 1 purple tiles. Because we are able to classify the moves into two types of options, and by extension can define the game by two parameters, a fun and interesting Pascal-like triangle of nimbers arises.

Before stating the theorem, we will define some terms. These are named based on starting positions, but also relate to any position resulting during the play from one of these positions. We call rows/columns that contain only purple titles *all-purple.* We will call the rows with an odd number of purple tiles the *odd parity* rows (not counting tiles in all-purple columns) and the ones with an even number of purple tiles the *even parity* rows (again, not counting tiles in all-purple columns). We will also refer to columns that contain only green tiles (except for all-purple rows) as *all-green.* These terms make more sense in the case where there are no all-purple columns or all-purple rows, but we note that we can remove those rows and columns, resulting in an isomorphic game position.

Theorem 1 (Pascal-Like Nimber Triangle)**.** *Let p be the number of rows that are not all-purple, k be the number of rows with odd parity, and $q = 0$ if there are an even number of columns with only green tiles, and 1 otherwise. We define G' to be*

$$G' = \begin{cases} 0, & (k \text{ is even and } p > 2k) \text{ or } (k \text{ is odd and } p < 2k) \\ *, & (k \text{ is even and } p < 2k) \text{ or } (k \text{ is odd and } p > 2k) \\ *2, & p = 2k \end{cases}$$

If G is a Transverse Wave *position where for every column we can select, there is no more than one purple tile (discounting rows with only purple tiles), then $G = G' + *q$.*

We include example applications of Theorem 1 in Figure 2.

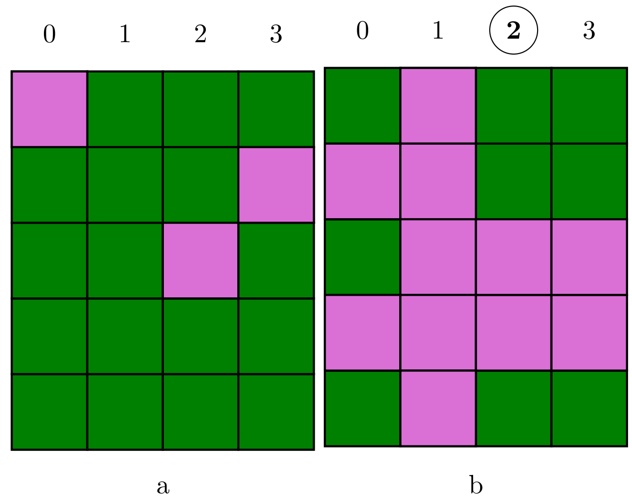

Figure 2: Two Transverse Wave positions where we can apply Theorem 1. In (a), $p = 3$, $k = 3$, and $q = 1$, so $G = G' + *q = * + * = 0$. In (b), there is one all-purple row and one all-purple column, which we ignore/remove. Then, $p = 2$, $k = 1$, and $q = 0$, so $G = G' + *q = *2 + 0 = *2$.

Proof. Note that all-green columns cannot be fully colored purple by a move from another column, as that other column would either have to be all purple (so it can not be chosen) or it would have to have green cells in a row where the all-green column is purple (then the all-green column is not all-green). This means that each all-green column contributes an additive $*$ to the game; it can be replaced with a single independent move.

We claim that G' is the game value without including the all-green columns, and G is the game value with them.

In the beginning, each column has at most one purple cell. Thus, the difference between the number of purple cells in each column is at most one. After each play, either only the cells in the chosen column become purple, or it will additionally turn one extra entire row purple.

Assuming G' is as we claim, it is not difficult to see that G is correct. As previously noted, each all-green column is just a $*$, so $G = G'$ if there are an even number of all-green columns and $G = G' + *$ if there are an odd number of them.

We have an illustrative triangle of cases of G' of up to 8 rows in Figure 3.

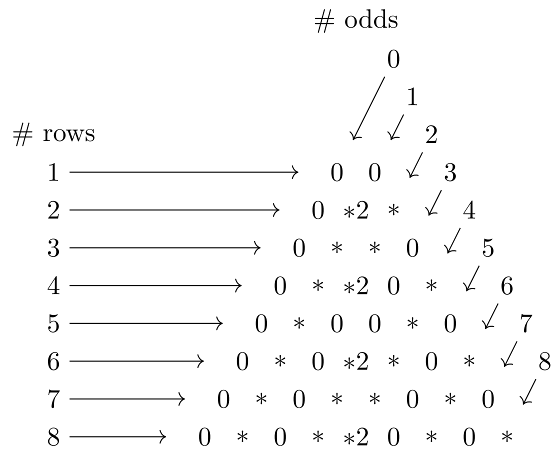

Figure 3: A Pascal-like Nimber Triangle of the values of Transverse Wave positions where each column has no more than 1 purple tile. The levels of the triangle are the number of rows in the game board (with at least one purple tile) and the diagonal columns marked indicate the number of odd parity rows. Each entry can be determined from the two above, by taking the mex of the above-left entry and $*+$ the above-right entry.

If the player chooses a column where the purple tile is in an odd parity row, then an even number $(2t)$ of other (playable) columns also have a single purple cell in that row; all of those columns become all-green. As mentioned above, each of these columns additively contributes $*$ to the value, for a total of $2t \times * = 0$. Thus, the resulting option's value is just the same as one with $p-1$ rows and $k-1$ rows with odd parity. This is just the value above and left in the triangle previously referenced.

If the player instead chooses a column where the purple tile is in a row with even parity, then an odd number $(2t+1)$ of other rows also have that single purple cell. Thus, the result is $(2t+1) \times * = *$ added to the option. Thus, the resulting game is the value above and right in the triangle (the same number of odd rows and one less row overall) plus $*$.

By inspection, note that Table 1 represents the correct value for each possible pair of parents in the game tree.

Now, we have 5 cases which invoke those table cases.

1. k even and $p > 2k$: case a, b, h, or j (thus the value is 0)

2. k odd and $p < 2k$: case a, g, h, or k (the value is 0)

Case	Above left	Above right	Value
a			0
b		0	0
c	0		$*$
d	0	0	$*2$
e	0	$*$	$*$
f	0	$*2$	$*$
g	$*$		0
h	$*$	0	0
i	$*$	$*$	$*2$
j	$*$	$*2$	0
k	$*2$	0	0
l	$*2$	$*$	$*$

Table 1: Values for a game with a given position based on what is above and left in the triangle and what is above and right. The position has value based on options to the above right plus $*$ and to the above left.

3. k even and $p < 2k$: case c, e, or l (the value is $*$)

4. k odd and $p > 2k$: case e or f (the value is $*$)

5. $p = 2k$: case d or i (the value is $*2$)

We prove the correctness of these cases by induction on the number of rows. Our base case, a single row, holds by inspection.

For our inductive case, we assume that it holds for row $p - 1$.

We first show that it holds when k is even and $p > 2k$ (case 1 in our list). This is either the base case (case a), or there are 0 odd rows (case b), or k is some other even number less than $\frac{1}{2}p$. In this last case, the left parent will have $k - 1$ with $p - 1 > 2(k - 1)$, which, by induction, must be $*$. Then the right parent has $p - 1 \geq 2k$, and thus is either 0 or $*2$. Thus, it must be either case h or j.

Now, we examine case 2 in our list, where k is odd and $p < 2k$. This is either the base case (case a), or $k = p$ (which must be case g, since it only has a single left parent with even k), or it is some other odd k such that $p < 2k$. Then it has a right parent with odd k and $p < 2k$, which is inductively 0. The left parents have even k and $p - 1 \leq 2(k - 1)$, thus a left parent is either $*$ or $*2$, which are cases h and k, respectively.

For case 3 in our list, if k is even and $p < 2k$, then either $k = p$, in which case it has a single left parent in case 2, which is case c, or it is some other even k with $p < 2k$. In that case, the right parent will have the same k, be in case 3, and have value $*$. The left parent will have $k - 1$ and $n - 1 \leq 2(k - 1)$, and thus be 0 (case e) or $*2$ (case l).

For case 4, if k is odd and $p-1 > 2(k-1)$, then we know the left parent has $p > 2(k-1)$ which is 0 (since it is case 1). The right parent has $p-1 \geq 2k$, and is thus 0 (case e) or $*2$ (case f).

Finally, in case 5, if $p = 2k$, then k can either be odd or even. If it is odd, then the left parent has even k and $p-1 > 2(k-1)$ and is thus 0, and the right parent has even k and $p < 2k$ and is thus 0, putting this in case d. If k is even, then the left parent is odd and thus $*$, and the right parent is even and thus $*$, putting us in case 1. □

For more general Transverse Wave positions, we provide examples with values up to $*7$, as shown in in Table 2.

Nimber	Rows (shorthand)	Other Columns
0	(0)	
$*$	(0) (01)	
$*2$	(01) (2)	
$*3$	(01) (2)	3
$*4$	(01) (234) (035)	
$*5$	(01) (234) (035)	6
$*6$	(012) (034) (0156) (2578)	
$*7$	(012) (034) (0156) (2578)	9

Table 2: Instances of values up to $*7$. The shorthand uses parentheses to indicate rows and numbers to indicate purple columns. So, for example, in the $*3$ case, the first row has columns 0 and 1 colored purple, row two has column 2 colored purple, and column 3 has no purple cells.

These results can be extended to the related rulesets that are exact embeddings of this, e.g., Avoid True and Demi-Quantum Boolean Nim. We discuss these transformations in more detail later.

Although we can only characterize Transverse Wave in very special cases, the Pascal-like formation of these game values provides us a glimpse of a potential elegant structure. Something interesting to explore in the future is whether one can cleanly characterize the game values when the game is restricted to have only two purple tiles in each column. If so, then one would also like to see how large of the parameter one can characterize in order to encounter intractability. Answering these questions can also tell us about the values for the other related games.

2. Connection to a Social-Influence-Inspired Combinatorial Game

Because mathematical principles are ubiquitous, combinatorial game theory is a field intersecting many disciplines. Combinatorial games have drawn wide inspiration from logic [25] to topology [22], from military combat [13, 21] to social sciences [8], and from graph theory [25, 2] to game theory [7]. Because the field cherishes challenging games with simple rulesets and elegant game boards, combinatorial game design is also a distillation process, aiming to derive elementary moves and transitions in order to capture the essence of complex phenomena that inspire the designers.

In this and the next sections, we discuss two other rulesets whose intersection contains Transverse Wave. While they have found a common ground, these games were inspired by separate research fields. The first game, called Friend Circle, is motivated by viral marketing [24, 20, 9], while the second, called Demi-Quantum Nim, was inspired by quantum superpositions [18, 10, 5]. In this section, we first focus on Friend Circle.

2.1. Viral-Marketing Broadcasting: Friend Circle

In many ways, viral marketing itself is a game of financial optimization. It aims to convert more people through network-propagation-based social influence by strategically investing in a seed group of people [24, 20]. In the following combinatorial game inspired by viral marketing, Friend Circle, we use a high-level perspective of social influence and social networks. Consider a social-network universe like Facebook, where people have their "circles" of friends. They can *broadcast* to all people in their friend circles (with a single post), or they can individually interact with some of their friends (via various personalized means). We will use individual interaction to set up the game position. In Friend Circle, only the broadcast-type of interaction is exploited. We will use the following traditional graph-theory notation: In an undirected graph $G = (V, E)$, for each $v \in V$, the neighborhood of v in G is $N_G(v) = \{u \mid (u, v) \in E\}$.

Ruleset 4 (Friend Circle)**.** For a ground set $V = [n]$ (of n people), a Friend Circle position is defined by a triple (G, S, w) where the properties are as follows.

- $G = (V, E)$ is an undirected graph. An edge between two vertices represents a friendship between those people.
- $S \subset V$ denotes the seed set.
- $w : E \to \{\mathbf{f}, \mathbf{t}\}$ (false and true) represents whether those friends have already spoken about the target product (with at least one recommending it to the other).

To choose their move, a player (a viral marketing agent) picks a person from the seed set, $v \in S$, such that there exists $e = (v, x) \in E$ where $w(e) = \mathbf{f}$. This represents choosing someone who has not spoken about the product to at least one of their friends.

The result of the move choosing v is a new position (G, S, w'), where w' is the same as w except that for all $x \in N_G(v)$:

- $w'((v, x)) = \mathbf{t}$, and
- if $w((v, x)) = \mathbf{t}$ then for all $y \in N_G(x) : w'((x, y)) = \mathbf{t}$.

An example of a Friend Circle move is shown in Figure 2.1.

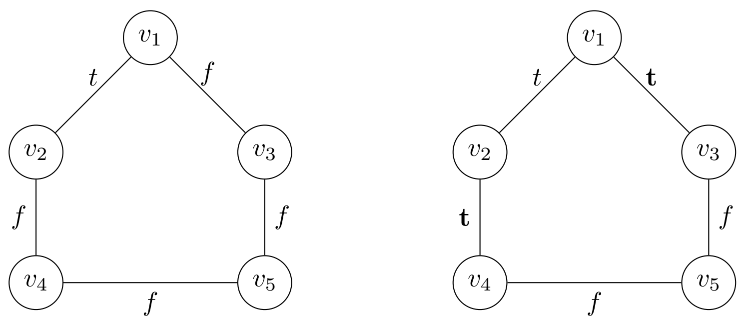

Figure 4: Example of a Friend Circle move. In the position on the left, let the seed set $S = \{v_1, v_2, v_3, v_4\}$, all of which are acceptable to choose because they all have an incident false edge. If a player chooses v_2, then the result is the right-hand position. In the second position, v_2 has had all of its incident edges become true. In addition, since (v_1, v_2) was true, all of v_1's incident edges have also changed to true. The altered edges in the figure are represented in bold. Note that in the resulting position, the next player can only choose to play at either v_3 or v_4, as v_1 and v_2 have only true edges and $v_5 \notin S$.

By inducing people in the seed's friend circle who had an existing intersection with the chosen seed to broadcast, Friend Circle emulates an elementary two-step cascading network of social influence.

2.2. Intractability of Friend Circle

We first connect Friend Circle to the classic graph-theory game Node-Kayles.

Ruleset 5 (Node-Kayles)**.** The starting position of Node-Kayles is an undirected graph $G = (V, E)$.

During the game, two players alternate turns selecting vertices, where a vertex $v \in V$ is feasible if it has neither already been selected in the previous turns nor any of its neighbors has already been selected. The player who has no more feasible moves loses the game.

When the selected vertices form a *maximal independent set* of G, the next player cannot make a move, and hence loses the game. It is well-known that Node-Kayles is PSPACE complete [25].

Theorem 2 (Friend Circle is PSPACE-complete). *The problem of determining whether a* Friend Circle *position is winnable is* PSPACE*-complete.*

Proof. First, we show that Friend Circle is PSPACE-solvable. During a game of Friend Circle starting at (G, S, w), once a node $s \in S$ is selected by one of the players, all edges incident to s become $\mathbf{t}$. Since true edges can never later become $\mathbf{f}$, s can never again be chosen for a move and the height of the game tree is at most $|S|$. Then, by the standard depth-first-search (DFS) procedure for evaluating the game tree for (G, S, w) in Friend Circle, we can determine the outcome class in polynomial space.

To establish that Friend Circle is PSPACE-hard, we reduce Node-Kayles to Friend Circle.

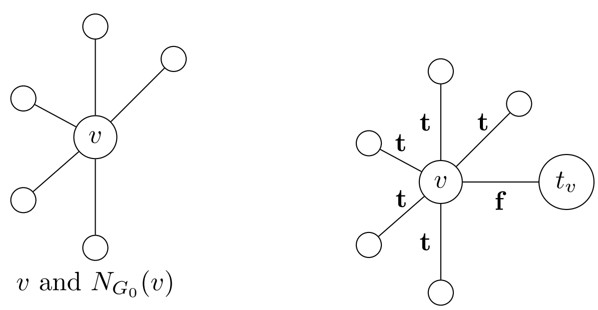

Figure 5: Example of the reduction from Node Kayles to Friend Circle. On the left is a Node Kayles vertex and its neighborhood. On the right are those same vertices, along with t_v with $\mathbf{t}$-weights on all the old edges and $\mathbf{f}$ on the new edge with (v, t_v).

Suppose we have a Node-Kayles instance at graph $G_0 = (V_0, E_0)$. For the reduced Friend Circle position, we create a new graph $G = (V, E)$ as follows. First, for each $v \in V_0$, we introduce a new vertex t_v. Let $T_0 = \{t_v | v \in V_0\}$, so $V = V_0 \cup T_0$. In addition, let $E_1 = \{(v, t_v) \mid v \in V_0\}$, so $E = E_0 \cup E_1$. Next, we set the weights:

- for all $e \in E_0 : w(e) = \mathbf{t}$, and
- for all $e \in E_1 : w(e) = \mathbf{f}$

as shown in Friend Circle. Last, we set $S = V_0$.

We now prove that Friend Circle is winnable on (G, S, w) if and only if Node Kayles is winnable at G_0. Note that because for all $v \in V_0$, $w(v, t_v) = \mathbf{f}$, all vertices in V_0 are feasible choices for the current player in Friend Circle. As the game progresses, vertices in V_0 are no longer able to be chosen when their edge to the t vertex becomes true. From here the argument is very simple: each play on vertex $v \in V_0$ in Node Kayles corresponds exactly to the play on v in Friend Circle. In Node Kayles, when v is chosen, itself and all its neighbors, $N_{G_0}(v)$ are removed from future consideration. In Friend Circle, v is also removed because the edge (v, t_v) becomes $\mathbf{t}$. In addition, since all neighboring vertices $x \in N_{G_0}(v)$ share a $\mathbf{t}$-edge with v, their edge with t_x will also become $\mathbf{t}$, but no other vertices will be removed as future choices. $\square$

2.3. Transverse Wave in Friend Circle

We now show that Friend Circle *contains* Transverse Wave as a *special case.* In the proposition below and the rest of the paper, we say that two game instances are *isomorphic* to each other if there exists a *bijection* between their moves such that their game trees are isomorphic under this bijection.

Proposition 3 (Social-Influence Connection of Transverse Wave)**.** *For any complete bipartite graph $G = (V_1, V_2, E)$ over two disjoint ground sets V_1 and V_2 (i.e., with $E = V_1 \times V_2$), any weighting $w : E \to \{\boldsymbol{f}, \boldsymbol{t}\}$, and seeds $S = V_1$,* Friend Circle *position (G, S, w) is isomorphic to* Crosswise OR *over a pseudo-adjacency matrix $\mathbf{A}_G$ for G with V_1 as columns and V_2 as rows. In this matrix, we will have the entry at column $x \in V_1$ and row $y \in V_2$ be 0 if $w((x, y)) = \boldsymbol{f}$ and 1 if the weight is $\boldsymbol{t}$.*

Note that by varying $w : E \to \{\mathbf{f}, \mathbf{t}\}$, one can realize any $|V_1| \times |V_2|$ Boolean matrix with $\mathbf{A}_G$. Thus, Friend Circle *generalizes* Transverse Wave.

Proof. Imagine these two games are played in tandem. We map the selection of a vertex $v \in S = V_1$ to the selection of the column associated with v in the matrix $\mathbf{A}_G$ of G. Because G is a complete bipartite graph, v is feasible for Friend Circle if there exists $u \in V_2$ such that $w(u, v) = \mathbf{f}$. Thus, the column associated with v in $\mathbf{A}_G$ is not all 1s. This is precisely the condition for v to be feasible in Crosswise OR over $\mathbf{A}_G$. The Direct Influence at v in Friend Circle over G changes all v's edges to $\mathbf{t}$ and the subsequent Cascading Influence on v's initially $\mathbf{t}$ neighbors in V_2 is isomorphic to crosswise ORs. Thus, Friend Circle on G is isomorphic to Crosswise OR over $\mathbf{A}_G$. $\square$

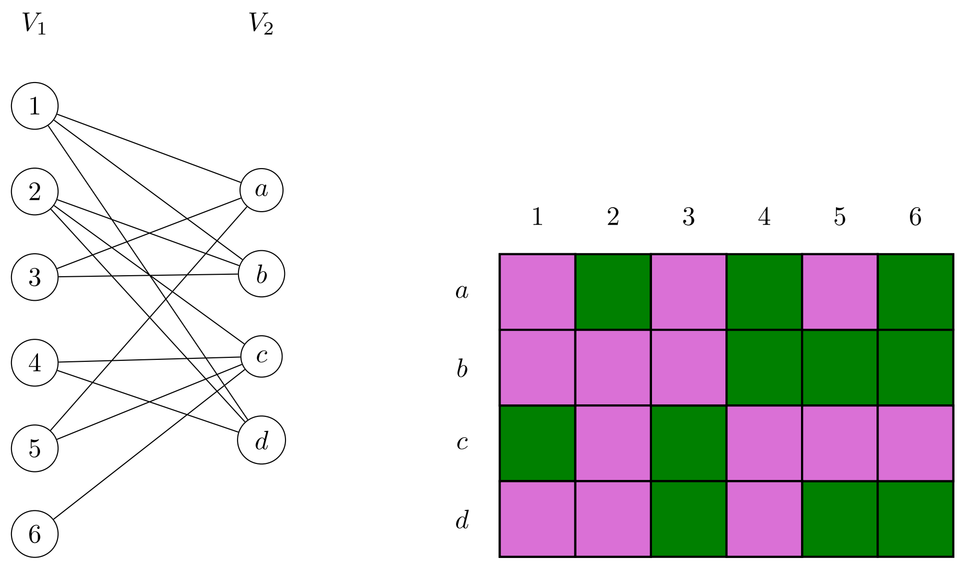

Figure 6: On the left is a FRIEND CIRCLE position on the complete bipartite graph between V_1 and V_2, where the seed set $S = V_1$. Instead of labelling edges, we have removed all false edges and are including only true edges. On the right is the equivalent TRANSVERSE WAVE position. The purple cells correspond to the (true) edges in the bipartite graph.

3. Connection to Quantum Combinatorial Game Theory

In this section, we discuss the connection of TRANSVERSE WAVE to a basic quantum-inspired combinatorial game.

Quantum computing is inspirational not only because the advances of quantum technologies have the potential to drastically change the landscape of computing and digital security, but also because the quantum framework—powered by superpositions, entanglements, and collapses—has fascinating mathematical structures and properties. Not surprisingly, quantumness has already found a way to enrich combinatorial game theory. In early 2000s, Allan Goff introduced basic quantum elements into TIC-TAC-TOE, as a conceptual illustration of quantum physics [18]. The quantum-generalization of TIC-TAC-TOE expands the strategy space by allowing superpositions of classical moves, creating game boards with entangled components. Consistency-based conditions for collapsing can then reduce the degree of possible realizations in the potential parallel game scenarios. In 2017, Dorbec and Mhalla [10] presented a general framework, motivated by Goff's concrete adventures, for a quantum-inspired extension applicable to all classical combinatoral games. Their framework enabled our recent work [5] on the structures and complexity of quantum-inspired games, which also led to the creation of Transverse Wave and Friend Circle in this paper.

3.1. Superposition of Moves and Game Realizations

In this subsection, we briefly discuss quantum combinatorial game theory (QCGT) to introduce needed concepts and notations for this paper. More detailed discussions of QCGT can be found in [18, 10, 5].

- ***Quantum Moves***: A quantum move is a superposition of two or more distinct classical moves. The superposition of w classical moves $\sigma_1, ..., \sigma_w$—called a w-wide quantum move—is denoted by $\langle \sigma_1 \mid ... \mid \sigma_w \rangle$.
- ***Quantum Game Position***: A quantum position is a superposition of two or more distinct classical game positions. The superposition of s classical positions $b_1, ..., b_s$—called an s-wide quantum superposition—is denoted by $\mathbb{B} = \langle b_1 \mid ... \mid b_s \rangle$. We call $b_1, ..., b_s$ the *realizations* of $\mathbb{B}$. We sometimes refer to classical moves and positions as 1-wide superpositions.

Classical/quantum moves can be applied to classical/quantum positions. Variants of the Dorbec-Mhalla framework differ in the condition when classical moves are allowed to engage with quantum positions. In this paper, we will focus on the least restrictive flavor—referred to by variant D in [10, 5]—in which moves, classical or quantum, are allowed to interact with game positions, classical or quantum, provided that they are feasible. There are some subtle differences between these variants, and we direct interested readers to [10, 5]. In this least restrictive flavor, a superposition of moves (including 1-wide superpositions) is *feasible* for a quantum position (including 1-wide superpositions) if each move is feasible for some realization in the quantum position. A superposition of moves and a quantum position of realizations creates a "tensor" of classical interactions in which infeasible classical interactions introduce collapses in realizations.

Quantum moves can have an impact on the outcome class of games, even on classical positions. In Figure 7, we borrow an illustration from [5] showing that quantum moves can change the outcome class of a basic Nim position. $(2, 2)$ becomes a fuzzy ($\mathcal{N}$, a first-player win) position instead of a zero ($\mathcal{P}$, a second-player win) position. Quantumness matters for many combinatorial games, as investigated in [5].

Ruleset 6 (Nim)**.** A Nim position is described by a non-negative integer vector, e.g., $(3, 5, 7) = G$, representing heaps of objects (here pebbles). A turn consists of removing pebbles from exactly one of those heaps. We describe these moves as a non-positive vector with exactly one non-zero element, e.g., $(0, -2, 0)$. Each move

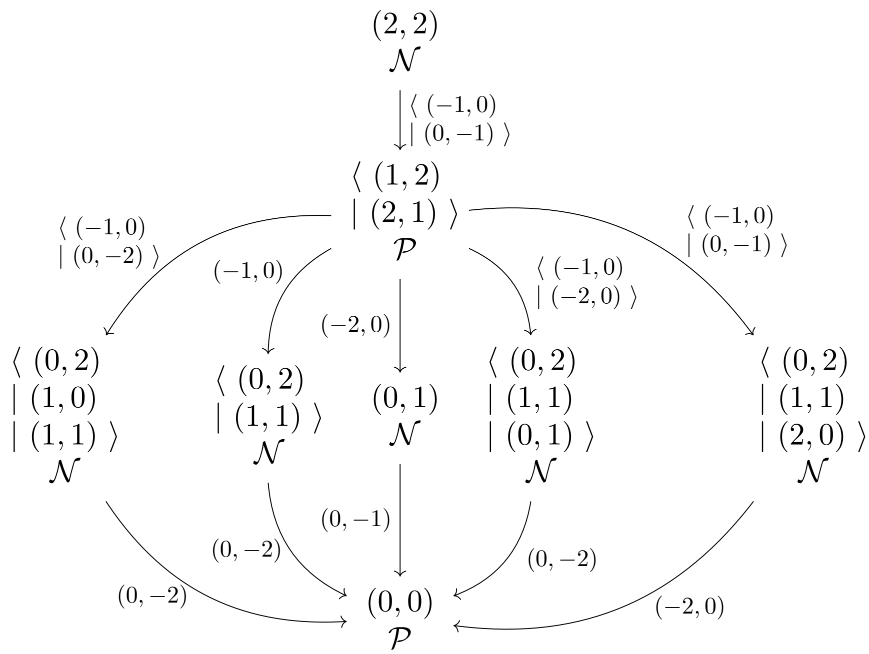

Figure 7: Illustration from [5]: Winning strategy for Next player in Quantum Nim $(2,2)$, showing that quantum moves impact the game's outcome. Only one option from $(2,2)$ is shown because it is a winning move. (There are four additional move options from $\langle(1,2) \mid (2,1)\rangle$ that are not shown because they are symmetric to moves given.)

cannot remove more pebbles from a heap than already exist there. Thus, the move $(-7,0,0)$ is not a legal move from G, above. When all heaps are zero, there are no legal moves.

To readers familiar with combinatorial game theory, it may seem odd that we explicitly define the description of moves in the game. However, it is integral for playing quantum combinatorial games, as move description affects quantum collapse. For more information, see [5].

Quantum interactions between moves and positions, as demonstrated in [18, 10], can have a significant impact on Nimber Arithmetic. In additon, as shown in [5], quantum moves can also fundamentally impact the complexity of combinatorial games.

3.2. Demi-Quantum Nim: Superposition of Nim Positions

The combinatorial game that contains Transverse Wave as a special case is derived from Nim [4, 16], in a framework motivated by a practical implementation of quantum combinatorial games [5].

For integer $s > 1$, an s-wide quantum Nim position of n heaps can be specified by an $s \times n$ integer matrix, where each row defines a single Nim realization. For example, the 4-wide quantum Nim position with 6 piles,

$$\langle (5,3,0,4,2,2) \mid (1,3,3,2,1,0) \mid (0,0,4,6,5,7) \mid (4,2,5,0,1,2) \rangle,$$

can be expressed in the following matrix form.

$$\begin{pmatrix} 5 & 3 & 0 & 4 & 2 & 2 \\ 1 & 3 & 3 & 2 & 1 & 0 \\ 0 & 0 & 4 & 6 & 5 & 7 \\ 4 & 2 & 5 & 0 & 1 & 2 \end{pmatrix} \tag{1}$$

Like the quantum generalization of combinatorial games, this *demi-quantum generalization* systematically extends any combinatorial game by expanding its game positions [5]. The intuitive difference here is that players may not introduce new quantum moves, they may only make classical moves, which apply to all (and may collapse some) of the realizations in the current superposition.

Definition 1 (Demi-Quantum Generalization of Combinatorial Games)**.** For any game ruleset R, the demi-quantum generalization of R, denoted by, Demi-Quantum-R, is a combinatorial game defined by the interaction of classical moves of R with quantum positions in R.

Central to the demi-quantum transition is the rule for *collapses*. Given a quantum superposition $\mathbb{B}$ and a classical move σ of R, σ is feasible if it is feasible for at least one realization in $\mathbb{B}$, and σ collapses all realizations in $\mathbb{B}$ for which σ is infeasible, meanwhile transforming each of the other realizations according to ruleset R.

For example, the move $(0,0,0,0,-2,0)$ applied to the quantum Nim position in Equation (1) collapses realizations 2 and 4, and transforms realizations 1 and 3, according to Nim as in the following matrices.

$$\begin{pmatrix} 5 & 3 & 0 & 4 & \{2-2\} & 2 \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ 0 & 0 & 4 & 6 & \{5-2\} & 7 \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \end{pmatrix} = \begin{pmatrix} 5 & 3 & 0 & 4 & 0 & 2 \\ 0 & 0 & 4 & 6 & 3 & 7 \end{pmatrix}$$

Note that for any impartial ruleset R, Demi-Quantum-R remains impartial. We now show that Demi-Quantum-Nim contains Crosswise AND, and thus, Transverse Wave, as a special case.

Proposition 4 (QCGT Connection of TRANSVERSE WAVE)**.** *Let* BOOLEAN NIM *denote* NIM *in which each heap has either one or zero pebbles.* DEMI-QUANTUM BOOLEAN NIM *is isomorphic to* CROSSWISE AND*, and hence isomorphic to* TRANSVERSE WAVE*.*

Proof. The proof uses the following equivalent "numerical interpretation" of collapses in the (demi-)quantum generalization of NIM. When a realization collapses, we can either remove it from the superposition or replace it with a NIM position in which all piles have zero pebbles. For example, the following two NIM superpositions are equivalent for subsequent game dynamics.

$$\begin{pmatrix} 5 & 3 & 0 & 4 & 0 & 2 \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \\ 0 & 0 & 4 & 6 & 3 & 7 \\ \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes & \boxtimes \end{pmatrix} \equiv \begin{pmatrix} 5 & 3 & 0 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 6 & 3 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In BOOLEAN NIM, each move can only remove one pebble from a pile. So, we can simplify the specification of the move by the index i alone. Note also that each quantum BOOLEAN NIM position can be specified by a Boolean matrix. Let $\mathbf{B}$ denote the Boolean matrix of DEMI-QUANTUM-BOOLEAN-NIM under consideration. With the above numerical interpretation of collapses in demi-quantum generalization, the collapse of a realization of $\mathbf{B}$ when applying a move i corresponding to the case that the corresponding row in $\mathbf{B}$ has 0 at the i^{th} entry, and the row is replaced by the crosswise AND with that column selection. Thus, DEMI-QUANTUM BOOLEAN NIM with position $\mathbf{B}$ is isomorphic to CROSSWISE AND with position $\mathbf{B}$. □

As an aside, notice that positions with all green tiles are trivial, so they are not appropriate starting positions. Interesting games need to be primed with some arbitrary purple tiles. Thus the hard positions given by the reduction could be natural starting positions, and the hardness statement is particularly meaningful for TRANSVERSE WAVE.

4. The Graph Structures Underlying Demi-Quantum Nim

As the basis of Nimbers and Sprague-Grundy theory [3, 26, 19], NIM holds a unique place in combinatorial game theory. It is also among the few non-trivial-looking games with a polynomial-time solution. Over the past few decades, multiple efforts have been made to introduce graph-theoretical elements into the game of NIM [15, 27, 6]. In 2001, Fukuyama [15] introduced an edge-version of GRAPH NIM, with NIM piles placed on edges of undirected graphs. Stockman [27] analyzed several versions with piles on the nodes. Both use the graph structure to capture the locality of the piles players can take from. Burke and George [6] then formulated a version called Neighboring Nim, for which classical Nim corresponds to Neighboring Nim over the complete graph, where each vertex hosts a pile.

The graph structures have a profound impact on the game of Nim both mathematically [15, 27] and computationally [6]. By a reduction from Geography, Burke and George proved that Neighboring Nim on some graphs is PSPACE-hard while on others (such as complete graphs) it is polynomial-time solvable [6]. However, Neighboring Boolean Nim, where each pile has at most one pebble, is equivalent to Undirected Geography, and thus can be solved in polynomial time [14].

In contrast, Demi-Quantum Boolean Nim is intractable.

Theorem 3 (Intractability of Demi-Quantum Boolean Nim)**.** Demi-Quantum Boolean Nim*, and hence* Transverse Wave (Crosswise AND*;* Crosswise OR)*, is a PSPACE-complete game.*

4.1. The Logic and Graph Structures of Demi-Quantum Boolean Nim

The intractability follows from the next theorem, which connects Demi-Quantum Boolean Nim to Schaefer's elegant PSPACE-complete game, Avoid True [25]. The reduction also reveals the bipartite and hyper-graph structures of Demi-Quantum Boolean Nim.

Ruleset 7 (Avoid True)**.** A game position of Avoid True is defined by a positive CNF F (`and` of a set of `or`-clauses of only positive variables) over a ground set V and a subset $T \subset V$, the "true" variables, (which is usually the empty set at the beginning of the game) .

A turn consists of selecting one variable from $V \setminus T$, where a variable $x \in V \setminus T$ is *feasible* for position (F, V, T) if assigning all variables in $T \cup \{x\}$ to `true` does not make F `true`. If x is feasible, then the position resulting from that move is $(F, V, T \cup \{x\})$. Under normal play, the next player loses if the position has no feasible move.

Theorem 4 ([5])**.** Demi-Quantum Boolean Nim *and* Avoid True *are isomorphic games.*

Proof. The part of the proof in [5] showing that Quantum Nim is Σ_2-hard also establishes the above theorem. Because establishing this theorem is not the main focus of [5], we reformulate the proof here to make this theorem more explicit as well as to provide a complete background of our discussion in this section.

We first establish the direction from Demi-Quantum Boolean Nim to Avoid True. Given a position $\mathbb{B}$ in Demi-Quantum Boolean Nim, we can create an `or`-clause from each realization in $\mathbb{B}$. Suppose $\mathbb{B}$ has m realizations and n piles.

We introduce n Boolean variables, $V = \{x_1, ..., x_n\}$. For each realization in $\mathbb{B}$, the `or`-clause consists of all variables corresponding to piles with zero pebbles. The reduced CNF $F_{\mathbb{B}}$ is the `and` of all these `or`-clauses. Taking a pebble from a pile collapses a realization for which the pile has no pebble. Such a move is mapped to selecting the corresponding Boolean variable making the `or`-clause associated with the realization `true`. Thus, playing Demi-Quantum Boolean Nim at position $\mathbb{B}$ is isomorphic to playing Avoid True starting at position $(F_{\mathbb{B}}, V, \emptyset)$. Note that the reduction can be set up in polynomial time.

For example, consider the following Demi-Quantum Boolean Nim position (with heaps (columns) labelled by their indices and realizations (rows) labelled A, B, and C).

$$\begin{array}{cc} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} & \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} & \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$$

This reduces to the Avoid True position with formula:

$$\underbrace{(x_2 \vee x_3 \vee x_6)}_{A} \wedge \underbrace{(x_1 \vee x_3 \vee x_7)}_{B} \wedge \underbrace{(x_2 \vee x_3 \vee x_7)}_{C}$$

and $T = \emptyset$. The three clauses are labelled by their respective realization. Those variables that appear in each clause are those with a zero in the matrix. Notice the following properties.

- The third heap is empty in all Nim realizations, so no player can legally play there. That is the same in the resulting Avoid True position; no player can pick x_3 as it is in all clauses and would make the formula true.
- The fourth and fifth heaps have a pebble in all three realizations in Nim, so a player can play in either of them without any collapses. Because of this, those Boolean variables do not occur in any of the Avoid True clauses.

For the reverse direction, consider an Avoid True position (F, V, T). Assume $V = \{x_1, ..., x_n\}$, and F has m clauses, $C_1, ..., C_m$. We reduce it to a Boolean Nim superposition $\mathbb{B}_{(F,V,T)}$ with m realizations and n piles. In the realization for C_i, we set piles corresponding to variables in C_i to zero to set up the mapping between collapsing the realization with making the clause `true`. We also set all piles associated with variables in T to zero, to set up the mapping between collapsing the realization with selecting a selected variable. Again, we can use these two mappings to inductively establish that the game tree for Demi-Quantum Boolean Nim at $\mathbb{B}_{(F,V,T)}$ is isomorphic to the game tree for Avoid True at (F, V, T). Note that the reduction also runs in polynomial time.

We demonstrate this reduction on the following AVOID TRUE position, with formula:

$$\underbrace{(x_1 \vee x_2 \vee x_3 \vee x_4)}_{A} \wedge \underbrace{(x_1 \vee x_5 \vee x_6 \vee x_7)}_{B} \wedge \underbrace{(x_1 \vee x_3 \vee x_6)}_{C} \wedge \underbrace{(x_2 \vee x_5 \vee x_8)}_{D}$$

and already-chosen variables, $T = \{x_8\}$. Following the reduction, we produce the following DEMI-QUANTUM BOOLEAN NIM position.

$$\begin{array}{c c}
\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} & \\
\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix} & \begin{matrix} A \\ B \\ C \end{matrix}
\end{array}$$

Since x_8 has already been made true ($x_8 \in T$), the following properties hold.

- The 8th column is all zeroes.
- Since x_8 appears in clause D, that clause does not have a corresponding realization in the quantum superposition (i.e., a row in the matrix).

□

Theorem 4 presents the following bipartite-graph interpretation of DEMI-QUANTUM BOOLEAN NIM.

Ruleset 8 (POWER STATION)**.** The game is defined by a bipartite graph $G = (U, V, E)$ (with edges E between vertex sets U and V) and a token on a vertex $s \in U$, where there is a battery at each vertex in $U \setminus \{s\}$ and s has 0 batteries. We say a vertex ("station") $u \in U$ is *reachable* from s if s and u can be connected by a length-two path in G, i.e., there exists a $v \in V$ (a "bridge") such that (s, v) and (v, u) are both edges of G. During the game, the players takes turns selecting a vertex $u \in U$ with a battery reachable from the current vertex s. The token is then moved to u, and u's battery is expended (removed) to "power" all stations in V connected to u. All vertices in V not connected to u are removed from the graph, because they are not powered.

Theorem 4 also gives the following simple hypergraph interpretation of DEMI-QUANTUM BOOLEAN NIM. Recall that a hypergraph H over a groundset $V = [n]$ is a collection of subsets of V. We write $H = (V, E)$, where for each hyperedge $e \in E$, $e \subseteq V$.

Ruleset 9 (HYDROPIPE)**.** The game is defined by a hypergraph $H = (V, E)$, in which each vertex $v \in V$ has one dose of drain cleaner and there is a token representing water flow on vertex $c \in V$. Players alternate turns moving the (water flow) token to another vertex v connected to c by a hyperedge ("pipe"), where v still has its drain cleaner. After moving the token, the cleaner at v is expended, used to clean all incident pipes. All other pipes that were not cleaned are clogged, and can no longer be cleaned or traversed through for the rest of the game. A player that can not reach a drain cleaner loses the game.

Because both Power Station and Hydropipe are simply graph-theoretical interpretations of the proof for Theorem 4, the proof also provides the following corollary.

Corollary 1. Power Station *and* Hydropipe *are isomorphic to* Transverse Wave *and are therefore PSPACE-complete combinatorial games.*

4.2. A Social Influence Game Motivated by Demi-Quantum Nim

The connection between Demi-Quantum Boolean Nim and Friend Circle motivates the following social-influence-inspired game, Demographic Influence, which mathematically generalizes Demi-Quantum Nim. In a nutshell, the setting has a demographic structure over a population, in which individuals have their own friend circles. Members in the population are initially *un-influenced* but *receptive* to ads, and (viral marketing) influencers try to target their ads at demographic groups to influence the population. People can be influenced either by influencers' ads or by "enthusiastic endorsement" cascading through friend circles.

The following combinatorial game is distilled from the above scenario.

Ruleset 10 (Demographic Influence)**.** A Demographic Influence position is defined by a tuple $Z = (G, D, \Theta)$, where

- $G = (V, E)$ is an undirected graph representing a symmetric social network on a population V,
- $D = \{D_1, \ldots, D_m\}$ is the set of demographics, each a subset of V,
- $\Theta : V \to \mathbb{Z}$ represents how resistant each individual is to the product (i.e., their threshold to being influenced).
  - If $\Theta(v) > 0$, then v is *uninfluenced*,
  - if $\Theta(v) = 0$, then v is *weakly influenced*, and
  - if $\Theta(v) < 0$, then v is *strongly influenced*.

A player's turn consists of choosing a demographic, D_k and the amount they want to influence, $c > 0$ where there exists $v \in D_k$ where $\Theta(v) \geq c$. (Since $c > 0$, there must be an uninfluenced member of D_k.)

- $\Theta(v)$ decreases by c (all individuals are influenced by c).

- If $\Theta(v)$ became negative by this subtraction (if it went from $\mathbb{Z}^+ \cup \{0\}$ to $\mathbb{Z}^-$), then for all $x \in N_G(v) : \Theta(x) = -1$.[9]

Importantly, we perform all the subtractions and determine which individuals are newly-strongly influenced before they go and strongly influence their friends. We include an example move in Figure 8.

Note that when influencing a demographic, D_k, since c cannot be greater than the highest threshold, that highest-threshold individual will not be strongly influenced by the subtraction step. (If one of their neighbors does get strongly influenced, then they will be strongly influenced in that manner.)

Since a player needs to make a move on a demographic group with at least one uninfluenced individual, the game ends when there are no remaining groups to influence.

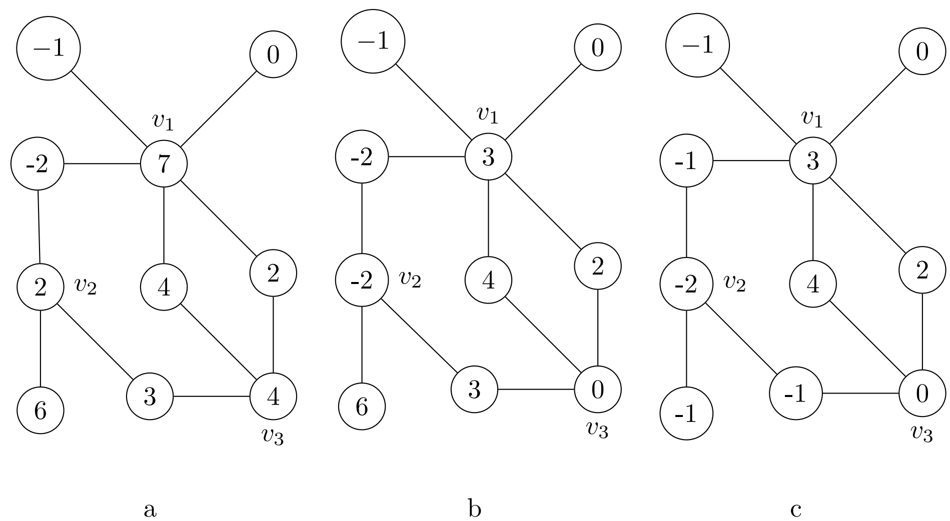

Figure 8: A Demographic Influence move, influencing $D_3 = \{v_1, v_2, v_3\}$ by 4. Panel (a) shows G, Θ, and D_3 prior to making the move. Panel (b) shows the first part of the move: subtracting from the thresholds of v_1, v_2, and v_3. Panel (c) shows the final results of the move: since v_2 went negative, its neighbors are set to -1 to show that they have been strongly influenced as well. (The magnitude of negativity does not matter, so it is okay that the vertex at -2 "goes back" to -1.)

The following theorem shows that Demographic Influence generalizes Demi-Quantum Nim.

[9]Thematically, if an individual becomes strongly influenced directly by the marketing campaign, then they enthusiastically recommend it to their friends and strongly influence them as well.

Theorem 5 (Demi-Quantum Nim Generalization: Social-Influence Connection)**.** Demographic Influence *contains* Demi-Quantum Nim *as a Special Case. Therefore,* Demographic Influence *is a* PSPACE-*complete game.*

Proof. For every Demi-Quantum Nim instance Z with m realizations of n piles, we construct the following Demographic Influence instance Z', in which, (1) $V = \{(r, c) \mid r \in [m], c \in [n]\}$, (2) $E = \{((r_1, c_1), (r_2, c_2)) \mid r_1 = r_2\}$ (i.e., vertices from all piles from the same realization are a clique), (3) for all $(r, c) \in V$, $\Theta((r, c))$ is set to be the number of pebbles that the c^{th} pile has in the r^{th} realization of Nim, (4) $D = (D_1, ..., D_n)$, where $D_c = \{(r, c) \mid r \in [m]\}$, i.e., nodes associated with the c^{th} Nim pile.

We claim that Z and Z' are isomorphic games. Imagine the two games are played in tandem. Suppose the player in Demi-Quantum Nim Z makes move (k, q), removing q pebbles from pile k. In its Demographic Influence counterpart, Z', the corresponding player also plays (k, q), investing q units in demographic group k. Note that in Z, (k, q) is feasible if and only if in at least one of the realizations, the k^{th} Nim pile has at least q pebbles. This is same as $q \leq \max_{i \in [m]} \Theta((i, q))$. Therefore, (k, q) is feasible in Z if and only if (k, q) is feasible in Z'.

When (k, q) is feasible, then for any realization $i \in [m]$, there are three cases: (1) if the k^{th} Nim pile has more pebbles than q, then in that realization, a classical transition is made, f, the q pebbles are removed from the pile. This corresponds to the reduction of the threshold at node (i, k) by q. (2) if the k^{th} Nim pile has exactly q pebbles, then all pebbles are removed from the pile. This corresponds to the case where node (i, k) becomes weakly influenced. (3) if q is more than the number of pebbles in the k^{th} Nim pile, then the move collapses realization i. This corresponds to the case in Demographic Influence where (i, k) become strongly influenced, and then strongly influences all other vertices in the same row (i). Therefore, Z and Z' are isomorphic games, with a connection between the collapse of a realization in the quantum version and the cascading of influence by endorsement in friend circle. □

The proof of Theorem 5 illustrates that Demographic Influence can be viewed as a graph-theoretical extension of Nim. Recall Burke-George's Neighboring Nim, which extends both classical Nim (when the underlying graph is a clique) and Undirected Geography (when all Nim heaps have at most one item in them, i.e., Boolean Nim). The next theorem complements Theorem 5 by showing that Demographic Influence also generalizes Node-Kayles.

Theorem 6 (Social-Influence Connection with Node-Kayles)**.** Demographic Influence *contains* Node-Kayles *as a special case.*

Proof. Consider a Node-Kayles instance defined by an n-node undirected graph $G_0 = (V_0, E_0)$ with $V_0 = [n]$. We define a Demographic Influence instance

$Z = (G, D, \Theta)$ as the following. (1) For each $v \in V_0$, we introduce a new vertex t_v. Let $V = V_0 \cup T_0$, where $T_0 := \{t_v \mid v \in V_0\}$. (2) For all $v \in V$, $\Theta(v) = 0$ and $\Theta(t_v) = 1$. (3) For all $v \in V$, $C(v) = N_G(v) \cup \{t_w \mid w \in N_G(v)\} \cup \{t_v\}$ and $C(t_v) = \{v\}$. (4)$D = \{D_1, ..., D_n\}$, where $D_v = \{v, t_v\}$ for all $v \in V$.

We show an example of this transformation in Figure 9.

Note that because $\Theta(v) \in \{0, 1\}$, for all $v \in V$, the space of moves in this Demographic Influence is $\{(v, 1) \mid v \in [n]\}$, whereas in Node-Kayles, a move consists of selecting one of the vertices from $[n]$.

We now show that Demographic Influence over Z is isomorphic to Node-Kayles over G under the mapping of moves $(v, 1) \Leftrightarrow v$.

For a Node Kayles move at v, it removes v and all $x \in N_{G_0}(v)$ from future move choices. In Demographic Influence, choosing the $(v, 1)$ move means that $\Theta(t_v)$ becomes 0 and $\Theta(v)$ becomes -1. Then v is strongly influenced and it strongly influences its neighbors at $N_G(v) = N_{G_0}(v) \cup \{t_x \mid x \in N_{G_0}(v)\}$, so all those vertices also get a threshold of -1. Those include both the x and t_x vertices for each $x \in N_{G_0}(v)$, so it removes all those neighboring demographics from future moves, $(x, 1)$, the set of which is isomorphic to those removed from the corresponding Node Kayles move.

Therefore, with induction, we establish that Demographic Influence over Z is isomorphic to Node-Kayles over G under the mapping of moves from $v \Leftrightarrow (v, 1)$. □

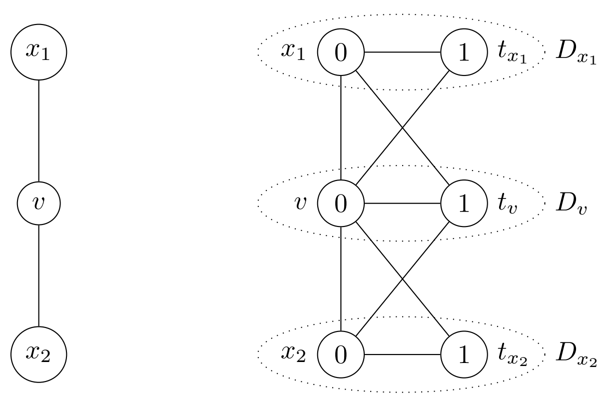

Figure 9: An example of the reduction from Node Kayles to Demographic Influence.

Therefore, Demographic Influence simultaneously generalizes Node-Kayles and Demi-Quantum Nim (which in turn generalizes classical Nim, Avoid True, and Transverse Wave).

We can also establish that DEMOGRAPHIC INFLUENCE Generalizes FRIEND CIRCLE defined in the earlier section.

Theorem 7 (DEMOGRAPHIC INFLUENCE Generalizes FRIEND CIRCLE)**.** *DEMOGRAPHIC INFLUENCE contains* FRIEND CIRCLE *as a special case.*

Proof. For a FRIEND CIRCLE position, $Z = (G, S, w)$, where $G = (V, E)$, we construct DEMOGRAPHIC INFLUENCE instance $Z' = (G', D, \Theta)$ using the following properties.

- For each edge $e \in E$, we create a new vertex v_e. Then $V' = \{v_e \mid e \in E\}$
- $E' = \{(v_{e_1}, v_{e_2}) \mid \text{there exists } v \in V : e_1, e_2 \text{ both incident to } v\}$
- $G' = (V', E')$
- $D = \{D_s\}_{s \in S}$, where $D_s = \{v_e \mid e \text{ is incident to } s\}$
- $\Theta : V' \to \{0, 1\}$, where if $w(e) = \mathbf{f}$, then $\Theta(e) = 1$; otherwise, if $w(e) = \mathbf{t}$, then $\Theta(e) = 0$.

In other words, the connections are built on the line graph of the underlying graph in FRIEND CIRCLE. Each seed vertex defines the demographic group and associates with all edges incident to it. Targeting this demographic group influences all these edges and edges adjacent to $\mathbf{t}$-edges in this set. See Figure 10 for an example of the reduction.

We complete the proof by showing that a play on FRIEND CIRCLE position s is isomorphic to playing $(D_s, 1)$ on DEMOGRAPHIC INFLUENCE, meaning choosing demographic D_s and investing $c = 1$. In FRIEND CIRCLE, playing at s means that for all e incident to s : (1) $w(e)$ becomes $\mathbf{t}$, and (2) if $w(e)$ was already $\mathbf{t}$, then for all f adjacent to $e : w(f)$ becomes $\mathbf{t}$.

In DEMOGRAPHIC INFLUENCE, the corresponding play, $(D_s, 1)$ means that for all $v_e \in D_s$:

- $\Theta(v_e)$ is reduced by 1, which corresponds to setting $w(e)$ to $\mathbf{t}$;
- if $\Theta(v_e)$ becomes -1, then for all $v_f \in N_{G'}(v_e)$, we have that $\Theta(v_f)$ also becomes -1; by our definition of E', these v_f are exactly those where both $w(e)$ were previously $\mathbf{t}$ and e and f are adjacent in G.

Thus, following analogous moves, $w(e) = \mathbf{t}$ if and only if $\Theta(v_e) \leq 0$. A seed vertex, $s' \in S$ is surrounded by $\mathbf{t}$-edges (and ineligible as a move) exactly when all vertices $v_e \in D_s$ are influenced, also making D_s ineligible as a move. This mapping of moves shows that the games are isomorphic.

□

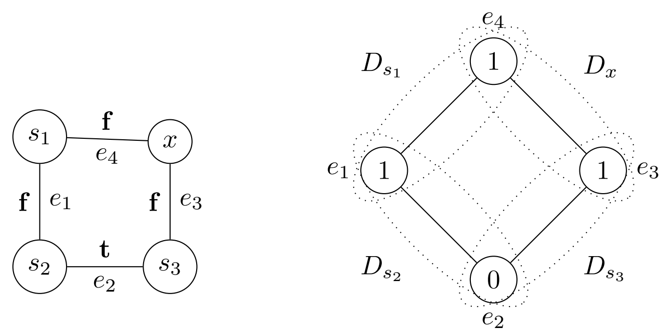

Figure 10: Example of the reduction. On the left is a Friend Circle instance. On the right is the resulting Demographic Influence position.

5. Conclusions and Future Work

One of the beautiful aspects of Winning Ways [3] is the relationships between games, especially when positions in one ruleset can be transformed into equivalent instances of another ruleset. As examples, Dawson's Chess positions are equivalent to Node Kayles positions on paths, Wythoff's Nim is the one-queen case of Wyt Queens, and Subtraction-$\{1, 2, 3, 4\}$ positions exist as instances of many rulesets, including Adders and Ladders with one token after the top of the last ladder and last snake.

Transforming instances of one ruleset to another (reductions) is a basic part of Combinatorial Game Theory[10], just as it is vital to computational complexity. Transverse Wave arose not only by reducing to other things, but more concretely as a special case of other games we explored.

"Ruleset A is a special case of ruleset B" (i.e., "B is a generalization of A") not only proves that computational hardness of A results in the computational hardness of B, but also:

- if A is deep, then it is a fundamental part of B's strategies; and
- if B's rules are straightforward, then A could be a basic building block in creating other fun rulesets.

Relevant special-case/generalization relationships between games presented here include the following.

- Demi-Quantum Nim is a generalization of Nim. (See Section 3.) (This is true of any ruleset R and Demi-Quantum R.)

[10] "Change the Game!" is the title of a section in Lessons in Play, Chapter 1, "Basic Techniques" [1].

- Demi-Quantum Nim is a generalization of Transverse Wave. (Via Demi-Quantum Boolean Nim, see Section 3.2.)
- Friend Circle is a generalization of both Transverse Wave (Section 2.3) and Node Kayles (Section 2.2).
- Demographic Influence is a generalization of both Demi-Quantum-Nim and Friend Circle (Section 4.2).

We show these relationships in a lattice-manner in Figure 11. Understanding Transverse Wave is a key piece of the two other rulesets, which also include the impartial classics Nim and Node Kayles.

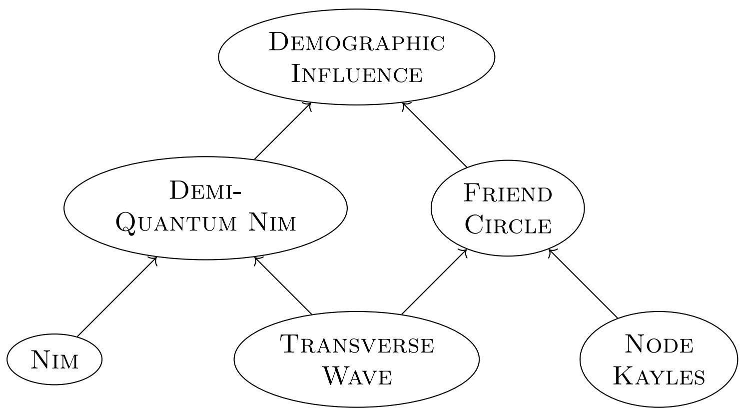

Figure 11: Generalization relationships of the rulesets in this paper. $A \to B$ means that A is a special case of B and B is a generalization of A.

Furthermore, several of the relationships outside of Figure 11 that were discussed in this paper were completely isomorphic, preserving the game values, not just winnability (as in Section 1.4). More explicitly, any new findings on the Grundy values for Transverse Wave also give those exact same results for Crosswise OR, Crosswise AND, Demi-Quantum Boolean Nim, Avoid True, Power Station, and Hydropipe.

We have been drawn to Transverse Wave not only because it is colorful, approachable, and intriguing, but also because the relationships with other games have inspired us to discover more connections among games. Our work offers us a glimpse of the *lattice order* induced by *special-case/generalization relationships* over mathematical games, which we believe is an instrumental framework for both the design and comparative analysis of combinatorial games. In one direction of this lattice, when given two combinatorial games A and B, it is a stimulating and creative process to design a game with the simplest ruleset that generalizes both A and B.[11] For example, in generalizing both Nim and Undirected Geography, Neighboring Nim highlights the role of "self-loops" in Graph-Nim. In our work, the aim of capturing both Node Kayles and Demi-Quantum Nim has contributed to our design of Demographic Influence. In the other direction, identifying a well-formulated basic game at the intersection of two seemingly unconnected games may greatly expand our understanding of game structures. It is also a refinement process for identifying intrinsic building blocks and fundamental games. By exploring the lattice order of game relationships, we will continue to improve our understanding of combinatorial game theory and identify new fundamental games inspired by the rapidly evolving world of data, networks, and computing.

Acknowledgements. We thank the anonymous reviewers for helpful suggestions. This research was supported in part by an NSF grant (CCF-1815254), and by a Simons Investigator Award.

References

[1] M. H. Albert, R. J. Nowakowski, and D. Wolfe, *Lessons in Play: An Introduction to Combinatorial Game Theory*, A. K. Peters, Wellesley, Massachusetts, 2007.

[2] G. Beaulieu, K. G. Burke, and É. Duchêne, Impartial coloring games, *Theoret. Comput. Sci.* **485** (2013), 49-60.

[3] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning Ways for your Mathematical Plays*, volume 1, A. K. Peters, Wellesley, Massachsetts, 2001.

[4] C. L. Bouton, Nim, a game with a complete mathematical theory, *Annals of Mathematics* **3(1/4)** (1901), 35-39.

[5] K. Burke, M. Ferland, and S.-H. Teng, Quantum combinatorial games: structures and computational complexity, *CoRR* abs/2011.03704, 2020.

[6] K. W. Burke and O. George, A PSPACE-complete graph nim, *Games of No Chance 5* (2019), 259-270.

[7] K. W. Burke and S.-H. Teng, Atropos: a PSPACE-complete sperner triangle game, *Internet Mathematics* **5(4)** (2008), 477-492.

[8] K. W. Burke, *Science for Fun: New Impartial Board Games*, PhD thesis, USA, 2009.

[9] W. Chen, S.-H. Teng, and H. Zhang, A graph-theoretical basis of stochastic-cascading network influence: characterizations of influence-based centrality, *Theor. Comput. Sci.* **824-825** (2020), 92-111.

[10] P. Dorbec and M. Mhalla, Toward quantum combinatorial games, *arXiv preprint arXiv:1701.02193*, 2017.

[11] It is also a relevant pedagogical question to ask when introducing students to combinatorial game theory.

[11] D. Eppstein, Computational complexity of games and puzzles, 2006, http://www.ics.uci.edu/∼eppstein/cgt/hard.html.

[12] S. Even and R. E. Tarjan, A combinatorial problem which is complete in polynomial space, *J. ACM* **23(4)** (1976), 710–719.

[13] A. S. Fraenkel and D. Lichtenstein, Computing a perfect strategy for n x n chess requires time exponential in n, *J. Comb. Theory, Ser. A* **31(2)** (1981), 199-214.

[14] A. S. Fraenkel, E. R. Scheinerman, and D. Ullman, Undirected edge geography, *Theor. Comput. Sci.* **112(2)** (1993), 371-381.

[15] M. Fukuyama, A nim game played on graphs, *Theor. Comput. Sci.* **1-3(304)** (2003), 387-399.

[16] D. Gale, A curious nim-type game, *American Mathematical Monthly* **81** (1974), 876-879.

[17] D. Gale, The game of hex and the Brouwer fixed-point theorem, *American Mathematical Monthly* **10** (1979), 818-827.

[18] A. Goff, Quantum tic-tac-toe: a teaching metaphor for superposition in quantum mechanics, *American Journal of Physics* **74(11)** (2006), 962-973.

[19] P. M. Grundy, Mathematics and games, *Eureka* **2** (1939), 198-211.

[20] D. Kempe, J. Kleinberg, and E. Tardos, Maximizing the spread of influence through a social network, In *Proceedings of the 9th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, KDD '03 (2003), 137-146.

[21] D. Lichtenstein and M. Sipser, Go is polynomial-space hard, *J. ACM* **27(2)** (1980), 393-401.

[22] J. F. Nash, *Some Games and Machines for Playing Them*, RAND Corporation, Santa Monica, CA 1952.

[23] S. Reisch, Hex ist PSPACE-vollständig, *Acta Inf.* **15** (1981), 167-191.

[24] M. Richardson and P. Domingos, Mining knowledge-sharing sites for viral marketing, In *Proceedings of the 8th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, KDD '02 (2002), 61-70.

[25] T. J. Schaefer, On the complexity of some two-person perfect-information games, *Journal of Computer and System Sciences* **16(2)** (1978), 185-225.

[26] R. P. Sprague, Über mathematische kampfspiele, *Tôhoku Mathematical Journal* **41** (1935-36), 438-444.

[27] G. Stockman, Presentation: The game of nim on graphs: nimg, 2004, Available at `http://www.aladdin.cs.cmu.edu/reu/mini_probes/papers/final_stockman.ppt`.