PARTIZAN SUBTRACTION GAMES[1]

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Abstract

Partizan subtraction games are combinatorial games where two players, say Left and Right, alternately remove a number n of tokens from a heap of tokens, with $n \in S_{\mathcal{L}}$ (resp. $n \in S_{\mathcal{R}}$) when it is Left's (resp. Right's) turn. The first player unable to move loses. These games were introduced by Fraenkel and Kotzig in 1987, where they introduced the notion of dominance, i.e., an asymptotic behavior of the outcome sequence where Left always wins if the heap is sufficiently large. In the current paper, we investigate the other kinds of behaviors for the outcome sequence. In addition to dominance, three other disjoint behaviors are defined, namely *weak dominance*, *fairness* and *ultimate impartiality*. We consider the problem of computing this behavior with respect to $S_{\mathcal{L}}$ and $S_{\mathcal{R}}$, which is connected to the well-known Frobenius coin problem. General results are given, together with arithmetic and geometric characterizations when the sets $S_{\mathcal{L}}$ and $S_{\mathcal{R}}$ have size at most 2.

1. Introduction

Partizan subtraction games were introduced by Fraenkel and Kotzig in 1987 [2]. They are 2-player combinatorial games played on a heap of tokens. Each player is assigned a finite set of integers, respectively denoted $S_{\mathcal{L}}$ (for the Left player), and $S_{\mathcal{R}}$ (for the Right player). A move consists in removing a number m of tokens from

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the heap, provided m belongs to the set of the player. The first player unable to move loses. When $S_{\mathcal{L}} = S_{\mathcal{R}}$, the game is impartial and known as the standard SUBTRACTION GAME—see [1].

We now recall the useful notations and definitions coming from combinatorial game theory. More information can be found in the reference book [7]. There are two basic *outcome* functions: for a position g,

$$o_L(g) = \begin{cases} L, & \text{if Left moving first has a winning strategy;} \\ R, & \text{otherwise;} \end{cases}$$

and

$$o_R(g) = \begin{cases} R, & \text{if Right moving first has a winning strategy;} \\ L, & \text{otherwise.} \end{cases}$$

It is usual to talk of the *outcome* of a position g and the associated outcome function $o(g)$,

- For $o_L(g) = o_R(g) = L$—Left wins regardless of who moves first, written $o(g) = \mathcal{L}$;
- For $o_L(g) = o_R(g) = R$—Right wins regardless of who moves first, written $o(g) = \mathcal{R}$;
- For $o_L(g) = L$, $o_R(g) = R$—the player who starts has a winning strategy, $o(g) = \mathcal{N}$;
- For $o_L(g) = R$, $o_R(g) = L$—the second player has a winning strategy, $o(g) = \mathcal{P}$.

In outcome function, there should be a reference to the game/rules. In this paper, the position will be a number but the rules will be clear from the context so the rules will not be included in the function.

A partizan subtraction game G with rules $(S_{\mathcal{L}}, S_{\mathcal{R}})$ will be denoted $(S_{\mathcal{L}}, S_{\mathcal{R}})$ in the rest of the paper. A game position of G will be simply denoted by an integer n corresponding to the size of the heap. The *outcome sequence* of G is the sequence of the outcomes for $n = 0, 1, 2, 3, \ldots$, i.e., $o(0), o(1), o(2), \ldots$. A well-known result ensures that the outcome sequence of any impartial subtraction game is ultimately periodic [7]. Note that in that case, the outcomes only have the values $\mathcal{P}$ and $\mathcal{N}$ since the game is impartial. In [2], this result is extended to partizan subtraction games.

Theorem 1 (Fraenkel and Kotzig [2])**.** *The outcome sequence of any partizan subtraction game is ultimately periodic.*

Example 2. Consider the partizan subtraction game $G = (\{1,2\},\{1,3\})$. The outcome sequence of G is

$$\mathcal{P}\,\mathcal{N}\,\mathcal{L}\,\mathcal{N}\,\mathcal{L}\,\mathcal{L}\,\mathcal{L}\,\mathcal{L}\,\cdots$$

In this particular case, the periodicity of the sequence can be easily proved by showing by induction that the outcome is $\mathcal{L}$ for $n \geq 4$.

Such a behavior where the outcome sequence has period 1 is rather frequent for partizan subtraction games. In that case, the period is only $\mathcal{L}$ or $\mathcal{R}$. In their paper, Fraenkel and Kotzig called this property *dominance*. More precisely, we say that $S_\mathcal{L} \succ S_\mathcal{R}$ - or that $S_\mathcal{L}$ dominates $S_\mathcal{R}$ - if there exists an integer n_0 such that the outcome of the game $(S_\mathcal{L}, S_\mathcal{R})$ is always $\mathcal{L}$ for all $n \geq n_0$. By symmetry, a game satisfying $S_\mathcal{L} \prec S_\mathcal{R}$ is always $\mathcal{R}$ for all sufficiently large heap sizes. When a game neither satisfies $S_\mathcal{L} \succ S_\mathcal{R}$ nor $S_\mathcal{L} \prec S_\mathcal{R}$, the sets $S_\mathcal{L}$ and $S_\mathcal{R}$ are said to be *incomparable*, denoted by $S_\mathcal{L} \| S_\mathcal{R}$. In [2], several instances have been proved to satisfy the dominance property (i.e., the games $(\{1,2m\},\{1,2n+1\})$ and $(\{1,2m\},\{1,2n\})$), or to be incomparable like $(\{a\},\{b\})$. It is also shown that the dominance relation is not transitive. Note that in [5], the game values (i.e., a refinement of the outcome notion) have been computed for the games $(\{1,2\},\{1,k\})$.

In the literature, partizan taking and breaking games have not been so much considered. A more general version, where it is also allowed to split the heap into two heaps, was introduced by Fraenkel and Kotzig in [2], and is known as partizan octal games. A particular case of such games, called *partizan splittles*, was considered in [4], where, in addition, $S_\mathcal{L}$ are $S_\mathcal{R}$ are allowed to be infinite sets. Another variation with infinite sets is when $S_\mathcal{L}$ and $S_\mathcal{R}$ make a partition of $\mathbb{N}$ [3]. In such cases, the ultimate periodicity of the outcome sequence is not necessarily preserved.

In the current paper, we propose a refinement of the structure of the outcome sequence for partizan subtraction games. More precisely, when the sets $S_\mathcal{L}$ and $S_\mathcal{R}$ are incomparable, different kinds of periodicity can occur. The following definition presents a classification for them.

Definition 3. The outcome sequence of $G = \textsc{Subtraction}(S_\mathcal{L}, S_\mathcal{R})$ is:

- $\mathcal{SD}$ (Strongly Dominating) for Left (resp. Right), and we write $S_\mathcal{L} \succ S_\mathcal{R}$ (resp. $S_\mathcal{L} \prec S_\mathcal{R}$) if any position n large enough has outcome $\mathcal{L}$ (resp. $\mathcal{R}$). In other words, the period is reduced to $\mathcal{L}$ (resp. $\mathcal{R}$);
- $\mathcal{WD}$ (Weakly Dominating) for Left (resp. Right), and we write $S_\mathcal{L} >_w S_\mathcal{R}$ if the period contains at least one $\mathcal{L}$ and no $\mathcal{R}$ (or resp. one $\mathcal{R}$ and no $\mathcal{L}$);
- $\mathcal{F}$ (Fair) if the period contains both $\mathcal{L}$ and $\mathcal{R}$.
- $\mathcal{UI}$ (Ultimately Impartial) if the period contains no $\mathcal{L}$ and no $\mathcal{R}$.

Remark 4. Note that inside a period, not all the combinations of $\mathcal{P}$, $\mathcal{N}$, $\mathcal{L}$ and $\mathcal{R}$ are possible. For example, in a game that is not $\mathcal{UI}$, a period that includes $\mathcal{P}$ must include $\mathcal{N}$. Indeed, assume on the contrary it is not the case and let n be a position of outcome $\mathcal{P}$ in the period, where the period has length p. Let $a \in S_{\mathcal{L}}$. Now the position $n+a$ is in the period, and $o(n+a) = \mathcal{L}$ since Left can win going first and, by assumption, $o(n+a) \neq \mathcal{N}$. For the same reason, $o(n+2a) = \mathcal{L}$. By repeating this argument, $o(n+ka) = \mathcal{L}$ for all k. Since n is in the period, we now have $\mathcal{P} = o(n) = o(n+pa) = \mathcal{L}$, a contradiction.

If the literature detailed above give examples of $\mathcal{SD}$ and $\mathcal{UI}$ games (as impartial subtraction games are $\mathcal{UI}$), we will see later in this paper examples of $\mathcal{WD}$ games (e.g. Lemma 15). The example below shows an example of a fair game.

Example 5. Let $S_{\mathcal{L}} = \{c, c+1\}$ and $S_{\mathcal{R}} = \{1, b\}$ with $b = c(c+1)$ and $c > 1$. Then the game $(S_{\mathcal{L}}, S_{\mathcal{R}})$ is $\mathcal{F}$.

Proof. We proceed by induction on the size of the heap, in order to show there are infinitely many $\mathcal{L}$ and $\mathcal{R}$. Since $c > 1$, we have $o(1) = \mathcal{R}$, and $o(c+1) = \mathcal{L}$. Now we assume that for some n, $o(n) = \mathcal{L}$, and show that $o(n+b+c) = \mathcal{L}$. In the position $n+b+c$, Left considers these as the two heaps n and $b+c$, and if Right removes 1, Left regards this as a move in the n component else it is a move in the second heap. Left moving first applies her winning strategy on n and then, regardless of whether Left moved first or second, responds in the remnants of n heap whenever Right removes 1 token. If at some point, Right chooses to remove b tokens, then Left answers immediately by removing c tokens, eliminating the second heap. In that case, Left wins at the end by applying her winning strategy on n. On the contrary, if Right never plays b, then Left empties the n component and it is Right's turn from the $b+c$ position. Again, from $b+c$, playing b is a losing move for Right. If he plays 1, then Left plays $c+1$, leading to the position $b-2 = (c-1)(c+2)$. All the next legal moves of Right are 1, and all the answers of Left are $c+1$, which guarantees to empty the position and hence win the game.

Assume now that $o(n) = \mathcal{R}$ and we show that $o(n+b+c) = \mathcal{R}$. As previously, Right considers this position as the two heaps n and $b+c$. He applies his winning strategy on n and any move c of Left leads Right to answer by removing b tokens, leaving a winning position for Right. Hence assume that Left plays $c+1$ until Right wins on n. At this point, Left has to play from a position $k+b+c$ with $k < c$. If $k = 0$, then Left loses for the same reasons as in the above case (as the position $b+c$ is $\mathcal{P}$). Otherwise, any move c or $c+1$ of Left is followed by a move b of Right, leading to a position with at most k tokens, from which Left cannot play and loses. □

The paper is organized as follows. In Section 2, we consider the two decision problems related to the computation of the outcome of a game position and of the behavior of the outcome sequence. Links with the Frobenius coin problem and the knapsack problem are given. Then, we try to characterize the behavior of the outcome sequence ($\mathcal{SD}$, $\mathcal{WD}$, $\mathcal{F}$ or $\mathcal{UI}$) according to $S_\mathcal{L}$ and $S_\mathcal{R}$. When $S_\mathcal{L}$ is fixed, Section 3 gives general results about strong and weak dominance according to the size of $S_\mathcal{R}$. In Section 4 and 5, we characterize the behavior of the outcome sequence when $|S_\mathcal{R}| = 1$ and $|S_\mathcal{L}| \leq 2$. Section 6 is devoted to the case $|S_\mathcal{L}| = |S_\mathcal{R}| = 2$, where it is proved that the sequence is mostly strongly dominating.

2. Complexity

Computing the outcome of a game position is a natural question when studying combinatorial games. For partizan subtraction games, we know that the outcome sequence is eventually periodic. This implies that, if $S_\mathcal{L}$ and $S_\mathcal{R}$ are fixed, computing the outcome of a given position n can be done in polynomial time. However, if the subtraction sets are part of the input, then the algorithmic complexity of the problem is not so clear. This problem can be expressed as follows:

PSG OUTCOME
Input: two sets of integers $S_\mathcal{L}$ and $S_\mathcal{R}$, a game position n
Output: the outcome of n for the game $(S_\mathcal{L}, S_\mathcal{R})$

In the next result, we show that this problem is actually NP-hard.

Theorem 6. PSG OUTCOME *is NP-hard, even in the case where the set of one of the players is reduced to one element.*

Proof. We use a reduction to UNBOUNDED KNAPSACK PROBLEM defined as follows.

UNBOUNDED KNAPSACK PROBLEM
Input: a set S and an integer n
Output: can n be written as a sum of non-negative multiples of S?

UNBOUNDED KNAPSACK PROBLEM was shown to be NP-complete in [9].

Let (S, n) be an instance of UNBOUNDED KNAPSACK PROBLEM, where S is a finite set of integers, and n is a positive integer. Without loss of generality, we can assume that $1 \notin S$ since otherwise the problem is trivial. We consider the partizan subtraction game where Left can only play 1, and Right can play any number x such that $x + 1 \in S$. In other words, we have $S_L = \{1\}$ and $S_R = S - 1$. We claim that for this game, Right has a winning strategy playing second if and only if n can be written as a sum of non-negative multiples of elements of S.

Observe that during one round (i.e., one move of Left followed by one move of Right), if x is the number of tokens that were removed, then $x \in S$. Suppose that Right has a winning strategy, and consider any play where Right plays according to this strategy. Then Right makes the last move, and after this move no token remains. Indeed, if there was at least one token remaining, then Left could still remove this token and continue the game. At each round an element of S was removed, and at the end, no tokens remains. This implies that n is a sum of non-negative multiples of S.

In the other direction, if n is a sum of non-negative multiples of S, we can write $n = \sum_{x \in S} n_x x$. A winning strategy for Right is simply to play n_x times the move $x - 1$ for each $x \in S$. □

Remark 7. In the case of impartial subtraction games (i.e., $S_{\mathcal{L}} = S_{\mathcal{R}}$), there is no known result about the complexity of this problem. This is surprising as these games have been thoroughly investigated in the literature.

The second question that emerged from partizan subtraction games is the behavior of the outcome sequence, according to Definition 3. It can also be formulated as a decision problem.

PSG SEQUENCE
Input: two sets of integers $S_{\mathcal{L}}$ and $S_{\mathcal{R}}$
Output: is the game $(S_{\mathcal{L}}, S_{\mathcal{R}})$ $\mathcal{SD}$, $\mathcal{WD}$ (and not $\mathcal{SD}$), $\mathcal{F}$ or $\mathcal{UI}$?

Unlike PSG OUTCOME, the algorithmic complexity is open for PSG SEQUENCE. The next sections will consider this problem for some particular cases. In addition, one can wonder whether the knowledge of the sequence could help to compute the outcome of a game position. The answer is no, even if the game is $\mathcal{SD}$:

Proposition 8. *Let $S_{\mathcal{L}} = \{a_1, \ldots, a_n\}$ be such that $\gcd(a_1 + 1, \ldots, a_n + 1) = 1$, and let $S_{\mathcal{R}} = \{1\}$. The game $(S_{\mathcal{L}}, S_{\mathcal{R}})$ is $\mathcal{SD}$ for Left but computing the length of the preperiod is NP-hard.*

The proof will be based on the well-known COIN PROBLEM (also called Frobenius problem).

COIN PROBLEM
Input: a set of n positive integers $a_1, \ldots, a_n$ such that $\gcd(a_1, \ldots, a_n) = 1$
Output: the largest integer that cannot be expressed as a linear combination of $a_1, \ldots, a_n$.

This value is called the *Frobenius number*. For $n = 2$, the Frobenius number equals $a_1a_2 - a_1 - a_2$ [8][2]. No explicit formula is known for larger values of n. Moreover, the problem has been proved to be NP-hard in the general case [6].

Proof of Proposition 8. Under the assumptions of the proposition, we will show that the length of the preperiod is exactly the Frobenius number of $\{a_1 + 1, \dots, a_n + 1\}$. Indeed, let N be the Frobenius number of $\{a_1+1, \dots, a_n+1\}$. Then $N+1, N+2\dots$ can be written as a linear combinations of $\{a_1+1, \dots, a_n+1\}$. Note that in the game $(S_{\mathcal{L}}, S_{\mathcal{R}})$, any round (sequence of two moves) can be seen as a linear combination of $\{a_1 + 1, \dots, a_n + 1\}$, as Left plays an a_i and Right plays 1. Hence if Right starts from $N + 1$, Left follows the linear combination for $N + 1$ to choose her moves, so as to play an even number of moves until the heap is empty. For the same reasons, if Right starts from $N + 2$, Left has a winning strategy as a second player. Since Right's first move is necessarily 1, it means that Left has a winning strategy as a first player from $N + 1$. Thus the position satisfies $o(N + 1) = \mathcal{L}$. Using the same arguments, this remains true for all positions greater than $N + 1$. In other words, it proves that the game is $\mathcal{SD}$ for Left. Now, we consider the position N and show that $o(N) \neq \mathcal{L}$. Indeed, assume that Right starts and Left has a winning strategy. It means that an even number of moves will be played. According to the previous remark, the sequence of moves that is winning for Left is necessarily a linear combination of $\{a_1 + 1, \dots, a_n + 1\}$. This contradicts the Frobenius property of N. □

This correlation between partizan subtraction games and the coin problem will be reused further in this paper.

3. When $S_{\mathcal{L}}$ is Fixed

In this section, we consider the case where $S_{\mathcal{L}}$ is fixed and study the behaviour of the sequence when $S_{\mathcal{R}}$ varies. In particular, we look for sets $S_{\mathcal{R}}$ that make the game $(S_{\mathcal{L}}, S_{\mathcal{R}})$ favorable for Right. This can be seen as a prelude to the game where players would choose their sets before playing: if Left has chosen her set $S_{\mathcal{L}}$, can Right force the game to be asymptotically more favorable for him?

3.1. The Case $|S_{\mathcal{R}}| > |S_{\mathcal{L}}|$

If $S_{\mathcal{R}}$ can be larger than $S_{\mathcal{L}}$, then it is always possible to obtain a game favorable for Right, as it is proved in the following theorem.

[2]Although not germane to this paper, Sylvester's solution is central to the strategy stealing argument that proves that naming a prime 5 or greater is a winning move in SYLVER COINAGE[1], chapter 18.

Theorem 9. *Let $S_{\mathcal{L}}$ be any finite set of integers. Let p be the period of the impartial subtraction game played with $S_{\mathcal{L}}$ and let $S_{\mathcal{R}} = S_{\mathcal{L}} \cup \{p\}$. Then Right strongly dominates the game $(S_{\mathcal{L}}, S_{\mathcal{R}})$, i.e., the game $(S_{\mathcal{L}}, S_{\mathcal{R}})$ is ultimately $\mathcal{R}$.*

Proof. Let n_0 be the preperiod of the impartial subtraction game played on $S_{\mathcal{L}}$ and m be the maximal value of $S_{\mathcal{L}}$. We prove that Right wins if he starts on any heap of size $n > n_0 + p$, which implies that the outcome on $(S_{\mathcal{L}}, S_{\mathcal{R}})$ is $\mathcal{R}$ for any heap of size $n > n_0 + p + m$.

If n is an $\mathcal{N}$-position for the impartial subtraction game on $S_{\mathcal{L}}$, then Right follows the strategy for the first player, never uses the value p, and wins.

If n is a $\mathcal{P}$-position, Right takes p tokens, leaving Left with a heap of size $n-p > n_0$ which is, using periodicity, also a $\mathcal{P}$-position in the impartial game. After Left's move, we are in the case of the previous paragraph and Right wins. □

Note that in the previous theorem, $S_{\mathcal{R}}$ contains the set $S_{\mathcal{L}}$, and thus has a large common intersection. We prove in the next theorem that if $S_{\mathcal{R}}$ cannot contain any value in $S_{\mathcal{L}}$, then it is still possible to have a game that is at least fair for Right (i.e., it contains an infinite number of $\mathcal{R}$-positions). Note that we do not know if for any set $S_{\mathcal{L}}$, there is always a set $S_{\mathcal{R}}$ with $|S_{\mathcal{R}}| = |S_{\mathcal{L}}| + 1$ and $S_{\mathcal{R}} \cap S_{\mathcal{L}} = \emptyset$ that is (weakly or strongly) dominating for Right.

Theorem 10. *For any set $S_{\mathcal{L}}$, there exists a set $S_{\mathcal{R}}$ with $S_{\mathcal{L}} \cap S_{\mathcal{R}} = \emptyset$ and $|S_{\mathcal{R}}| = |S_{\mathcal{L}}| + 1$ such that the resulting game contains an infinite number of $\mathcal{R}$-positions.*

Proof. Let n be any integer such that the set $A = \{n - m, m \in S_{\mathcal{L}}\}$ is a set of positive integers that is disjoint from $S_{\mathcal{L}}$. Putting $S_{\mathcal{R}} = A \cup \{n\}$ gives a set that satisfies the condition of the theorem and the game $(S_{\mathcal{L}}, S_{\mathcal{R}})$.

We claim that $o(kn) = \mathcal{R}$ for $k = 1, 2, \ldots$. If Left starts on a position kn by removing m tokens, then Right can answer by taking $n - m$ tokens and leaves $(k-1)n$ tokens, and by induction, Right wins. If Right starts, he takes n tokens and, again, Left has a multiple of n and loses. □

Consequently, if Right has a small advantage on the size of the set, he can ensure that the sequence of outcomes contains an infinite number of $\mathcal{R}$-positions. So having a larger subtraction set seems to be an important advantage. However, having a larger set is not always enough to guarantee dominance. Indeed, we have the following result.

Theorem 11. *Let $G = (S_L, S_R)$ be a partizan subtraction game. Assume that $|S_L| \geq 2$ and that G is eventually L, with preperiod at most p. Let $x_1, x_2 \in S_L$, with $x_1 < x_2$, and let d be an integer with $d > p + \max(S_R \cup \{x_2 - x_1\})$, then $G_d = (S_L, S_R \cup \{d\})$ is eventually L with preperiod at most $(d + x_2)\lceil \frac{d+x_2}{x_2-x_1} \rceil$*

Proof. Let G, d, x_1 and x_2 be as in the statement of the theorem. We first prove the following claim.

Claim 12. *In the game G_d, if $o_L(n) = \mathcal{L}$ (resp. $o_R(n) = \mathcal{L}$) then Left has a winning strategy on $n + (d + x)$ as first (resp. second player), with $x \in S_L$.*

Proof of Claim 12. We will show the result by induction on n.

First, assume $o_R(n) = \mathcal{L}$. We will show that there is a winning strategy for Left playing second on $n + d + x$. Starting from the position $n + d + x$, there are three possible cases:

- Right plays $y \in S_R$, with $y \leq n$. By the assumption on n, Left wins as first player on $n-y$, and using the induction hypothesis, he also wins as first player on $n - y + d + x$. Therefore, Left wins as second player on $n + d + x$.
- Right plays $y \in S_R$, with $y > n$. Now Left answers by playing x. This leads to the position $(n - y) + d$, with $(n - y) + d > p$ by assumption on d, and $n-y+d < d$ by assumption on y. Since $n-y+d < d$, Right can no longer play his move d, and the outcome of G_d on $n - y + d$ is the same as the outcome of G on this position. Since $n - y + d > p$ Left wins playing second on this position.
- Right plays d, then Left answers by playing x, leading to the position n on which Left wins as second player by assumption.

Suppose now that Left wins playing first on n, and let $y \in S_L$ be a winning move for Left. Then Left wins playing second on $n - y$, and using the induction hypothesis, she wins playing second on $n - y + d + x$. Consequently, y is a winning move for Left on $n + d + x$. □

For $i \geq 0$, denote by X_i the set of integers $k < d + x_2$ such that the position $i(d+x_2)+k$ is $\mathcal{L}$ for G_d. To prove the theorem, it is enough to show that if i is large enough, then $X_i = [0, x_2 + d[$. From the claim above, we know that $X_i \subseteq X_{i+1}$.

Additionally, using the hypothesis on d, we have that $[p+1, d-1] \subseteq X_0$. Finally, we have the following property. For any $x \geq 0$, if $x \in X_i$, then $x - (x_2 - x_1) \mod (d + x_2) \in X_{i+1}$. Indeed, if $x \in X_i$, then $i(d + x_2) + x$ is an $\mathcal{L}$-position, and using the claim above, so is $i(d + x_2) + x + d + x_1 = (i + 1)(d + x_2) + x - (x_2 - x_1)$.

Let $0 \leq x < d + x_2$, and write $(d - x) \mod (d + x_2) = \alpha(x_2 - x_1) + \beta$ the euclidian division of $(d - x) \mod (d + x_2)$ by $(x_2 - x_1)$. We have $0 < \beta \leq x_2 - x_1$, and $\alpha \leq \lceil \frac{d+x_2}{x_2-x_1} \rceil$. This can be rewritten as:

$$x = (d - \beta) - \alpha(x_2 - x_1) \mod (d + x_2)$$

Since we know that $d - \beta \geq p$ by assumption on d, we have that $(d - \beta) \in X_0$, and using the observation above, this implies that $x \in X_\alpha \subseteq X_{\lceil \frac{d+x_2}{x_2-x_1} \rceil}$.

Consequently, G_d is ultimately $\mathcal{L}$, and the preperiod is at most $(d+x_2)\lceil \frac{d+x_2}{x_2-x_1} \rceil$. □

By applying iteratively Theorem 11 with a game that is $\mathcal{SD}$ for Left (like the game of Example 2), we obtain the following corollary.

Corollary 13. *There are sets $S_\mathcal{L}$ and $S_\mathcal{R}$ with $|S_\mathcal{L}| = 2$ and $|S_\mathcal{R}|$ arbitrarily large such that $(S_\mathcal{L}, S_\mathcal{R})$ is $\mathcal{SD}$ for Left .*

Remark 14. The condition on $d \geq p + \max(S_R \cup \{x_2 - x_1\})$ in Theorem 11 is optimal. Indeed, take $S_\mathcal{L} = \{c, c+1\}$ and $S_\mathcal{R} = \{1\}$. As seen in the proof of Proposition 8, the game $(S_\mathcal{L}, S_\mathcal{R})$ is $\mathcal{SD}$ for Left, with preperiod the Froebenius number of $\{c+1, c+2\}$, which is $p = c^2 + 2c - 1 = c(c+1) - 1$. Thus, by Theorem 11, the game $(\{c, c+1\}, \{1, d\})$ with $d > c(c+1)$ is also $\mathcal{SD}$ for Left. But, as proved in Example 5, this is not true for $d = c(c+1)$ since the game is then $\mathcal{F}$.

3.2. The Case $|S_\mathcal{R}| \leq |S_\mathcal{L}|$

We first consider the case $S_\mathcal{L} = \{1, \ldots, k\}$ and prove that the game is always favorable to Left and that $S_\mathcal{L}$ strongly dominates in all but a few cases.

Lemma 15. *Let $S_\mathcal{L} = \{1, \ldots, k\}$, and $|S_\mathcal{R}| = k$. Then:*

1. *If $S_\mathcal{R} = \{c+1, c+2, \ldots c+k\}$ for some integer c, then Left weakly dominates if $c > 0$ and the game is impartial if $c = 0$,*

2. *otherwise, Left strongly dominates.*

Proof. 1. In this case, the game is purely periodic, with period $\mathcal{P}\mathcal{L}^c\mathcal{N}^k$. This can be proved by induction on the size of the heap n. If $0 < n \leq c$, only Left can play and the game is trivially $\mathcal{L}$. Otherwise, let $x = n \bmod c + k + 1$. If $x = 0$, then if the first player removes i tokens, the second player answers by removing $c + k + 1 - i$ tokens, leading to the position $n - c - k - 1$ which is $\mathcal{P}$ by induction, and so is n. If $0 < x < c + 1$, when Left starts she takes one token, leading to a $\mathcal{L}$ or a $\mathcal{P}$-position, and wins. If she is second, she plays as before to $n - c - k - 1$ which is a $\mathcal{L}$-position. Finally, if $x \geq c + 1$, both players win playing first by playing $x - c$ for Left and x for Right.

2. We show that if $n > 0$ is such that Right wins playing second on n, this implies that $S_\mathcal{R}$ contains k consecutive integers. Let n_0 be the smallest positive integer such that $o_L(n_o) = R$. We know that $n_0 > k$ since otherwise Left can win playing first by playing to zero. Since Right has a winning strategy playing second then Right has a winning first move on all the position $n - i$ for $1 \leq i \leq k$. This means that for each of these positions, Right has a winning move to some position m_i where $o_L(m_i) = R$. By minimality of n_0, this implies

that $m_i = 0$, and consequently $n - i \in S_{\mathcal{R}}$ for all $1 \leq i \leq k$. Consequently, if $S_{\mathcal{R}}$ does not contain k consecutive integers, there is no position $n > 0$ such that Right wins playing second. In particular, there are neither $\mathcal{R}$ nor $\mathcal{P}$-positions in the period. By Remark 4, this implies that the period only contains $\mathcal{L}$-positions, meaning that the game is strongly dominating for Left. □

The set $S_{\mathcal{L}} = \{1, \ldots, k\}$ is somehow optimal for Left, since the exceptions of strongly domination for Left in the previous lemma appear for any set of k elements:

Lemma 16. *For any set $S_{\mathcal{L}}$, there is a set $S_{\mathcal{R}}$ with $|S_{\mathcal{R}}| = |S_{\mathcal{L}}|$ and $S_{\mathcal{R}} \cap S_{\mathcal{L}} = \emptyset$ such that Left does not strongly dominate.*

Proof. Let $S_{\mathcal{R}} = n_0 - S_{\mathcal{L}}$ for an integer n_0 larger than all the values of $S_{\mathcal{L}}$ and such that $S_{\mathcal{R}} \cap S_{\mathcal{L}} = \emptyset$. Then Right wins playing second in all the multiples of n_0. □

4. When One Set Has Size 1

We now consider the case where one of the set, say $S_{\mathcal{R}}$ has size 1. As seen in Section 2, the study of the game is closely related to Unbounded Knapsack Problem and to the coin problem. Indeed, Right does not have any choice and thus the result is only depending on the possibility or not for n to be decomposed as a combination of the values in $S_{\mathcal{L}} + S_{\mathcal{R}}$. Our aim in this section is to exhibit the precise periods.

4.1. Case $|S_{\mathcal{L}}| = |S_{\mathcal{R}}| = 1$

In this really particular case, the game is always $\mathcal{WD}$ for the player that have the smallest integer.

Lemma 17. *Let $S_L = \{a\}$ and $S_R = \{b\}$ with $a < b$. The outcome sequence of $S = (S_L, S_R)$ is purely periodic, the period length is $a + b$ and the period is $\mathcal{P}^a\mathcal{L}^{b-a}\mathcal{N}^a$. In particular, the game is weakly dominating for Left.*

Proof. We prove that for all $n \geq 0$, if one of the player has a winning move playing first (resp. second) on n, then he also has one playing first (resp. second) on $n+a+b$. Indeed, suppose for example that Left has a winning move on position n playing first (the other cases are treated in the same way). If Left plays first on position $n + a + b$, then after two moves, it's again Left 's turn to play, and the position is now n, and Left wins the game.

The result then follows from computing the outcome of the positions $n \leq a + b$. These outcomes are tabulated in Table 1.

□

Heap sizes	Left move range	Right move range	Outcome
$[0, a-1]$	no moves	no moves	$\mathcal{P}$
$[a, b-1]$	$[0, b-a-1]$	no moves	$\mathcal{L}$
$[b, b+a-1]$	$[b-a, b-1]$	$[0, a-1]$	$\mathcal{N}$
$[b+a, b+2a-1]$	$[b, b+a-1]$	$[a, 2a-1]$	$\mathcal{P}$
$[b+2a, 2b+2a-1]$	$[b+a, 2b+a-1]$	$[2a, b+2a-1]$	$\mathcal{L}$

Table 1: Outcomes with $S_L = \{a\}$ and $S_R = \{b\}$ for first values

4.2. Case $|S_{\mathcal{L}}| = 2$ and $|S_{\mathcal{R}}| = 1$

In these cases, we are able to give the complete periods.

Theorem 18. *Let a, b and c be three positive integers, and let $g = \gcd(a+c, b+c)$. The game $(\{a, b\}, \{c\})$ is:*

- *strongly dominated by Left if $g \leq c$,*
- *weakly dominated by Left with period $(\mathcal{P}^{g-c}\mathcal{L}^{2c-g}\mathcal{N}^{g-c})$ if $c < g < 2c$,*
- *ultimately impartial with period $(\mathcal{P}^{c}\mathcal{N}^{c})$ if $g = 2c$,*
- *weakly dominated by Right with period $(\mathcal{P}^{c}\mathcal{R}^{g-2c}\mathcal{N}^{c})$ if $g > 2c$.*

Proof. Throughout this proof we write $n = qg + r$, with $0 \leq r < c$.

We start by proving the following claim which holds in all four cases.

Claim 19. *If $(n \bmod g) < c$ then $o_R(n) = L$ for large enough n.*

Proof. After both players play once, the number of tokens decreases by either $a + c$ or $b + c$ depending on which move Left played. By the results on the coin problem, we know that if q is large enough, then qg can be written as $\alpha(a + c) + \beta(b + c)$, with α and β two non-negative integers. If Left is playing second, a strategy can be to play a α times, and b β times. After these moves, it is Right's turn to play, and the position is $r < c$. Consequently Right now has no move and loses the game. □

We will now use this claim to prove the result in the four different cases.

For the first case, we have $g \leq c$. For any integer n, we have $(n \mod g) < g \leq c$. Consequently, by Claim 19, there is an integer n_0 such that for any $n \geq n_0$, $o_R(n) = L$. This also implies that for any $n \geq n_0 + a$, $0_L(n) = L$ since she plays to $n - a > n_0$ and, by the claim, $o_R(n - a) = L$. Thus the outcome is $\mathcal{L}$ for any position n large enough.

For the three remaining cases, we will show that the following four properties holds when n is large enough. The result of the theorem immediately follows from these four properties.

1. if $r < c$, then Left wins playing second,
2. if $r \geq g - c$, then Left wins playing first,
3. if $r \geq c$, then Right wins playing first,
4. if $r < g - c$, then Right wins playing second.

We now prove these four points.

1. This point is exactly the claim above.
2. If $r \geq g-c$, and n is large enough, then Left can play a. The position after the move is such that $n-a \equiv r-a \equiv r+c \mod g$. Moreover, since $g-c \leq r < g$, we know that $g \leq r+c < g+c$. From item 1, we know that $o_R(n) = L$ if $n-a$ is large enough, so Left has a winning strategy as a first player if $r \geq g - c$.
3. If $r \geq c$, and Right plays first, then whatever Left plays, after an even number of moves, Right still has a move available. Indeed, let n' be the position reached after an even number of moves. The number of tokens removed, $n-n'$, is a multiple of g. Consequently, $n' = (n \mod g)$. Since $(n \mod g) \geq c$, this implies that $n' \geq c$, and Right can play c. This proves that Right will never be blocked, and Left will eventually lose the game.
4. Finally, if $r < g - c$, then Left playing first can move to a position n' equal to either $n - a$ or $n - b$. Since $a \equiv b \equiv -c \mod g$, in both cases, we have $n' \equiv r + c \mod g$. Since $c \leq r + c < g$, by the argument above, we know that Right playing first on n' wins. Consequently, Left playing first on n loses.

□

When $c > b$ and $b \geq 2a$, which is included in the first case, we know the whole outcome sequence. This will be useful in next section.

Theorem 20. *The outcome sequence of the game* $(\{a, b\}, \{c\})$*, with* $c > b$ *and* $b \geq 2a$*, is the following:*

$$\mathcal{P}^a \mathcal{L}^{c-a} \mathcal{N}^a \mathcal{L}^\infty.$$

Proof. We show the result by induction on n, the position of the game.

- If $n < a$, then neither player has a move and thus $o(n) = \mathcal{P}$.
- If $a \leq n < c$, then only Left has a valid move and thus $o(n) = \mathcal{L}$.
- If $c \leq n < a + c$, then Right has a winning move to a position $n - c \leq a$ which has outcome $\mathcal{P}$, and Left has a winning move to a position with outcome either $\mathcal{P}$ or $\mathcal{L}$, and thus $o(n) = \mathcal{L}$.

- Finally, if $n \geq a + c$, then Right has no winning move, and Left has at least one winning move. Indeed, since $k \geq a$, we cannot have at the same time $n - a$ and $n - a - k$ in the interval $[c, a + c[$. So at least one of $n - a$ and $n - a - k$ is not in this interval, and is either a $\mathcal{P}$-position or a $\mathcal{L}$-position by induction.

□

5. When Both Sets Have Size 2

The goal of this section is to investigate the sequence of outcomes for the game $G = (S_{\mathcal{L}}, S_{\mathcal{R}})$ with $S_L = \{a, b\}$ and $S_R = \{c, d\}$. In particular, if we suppose that a and b are fixed, we would like to characterize for which positions the game G is eventually $\mathcal{L}$. The picture on Figure 1 gives an insight of what is happening. On the figure on the left, we have an example with $b \geq 2a$. In this case, the game G is almost always eventually $\mathcal{L}$, except when the point (c, d) is close to the diagonal, i.e., when $|d - c|$ is close to zero. When (c, d) is close to the diagonal, the behavior seems more complicated, and we will not give a characterization here.

When $b < 2a$, the behavior is more complicated, but shares some similarities with the previous case. From the picture on the right in Figure 1 we can see that there are some lines such that if the point (c, d) is far enough from these lines, then the game is eventually $\mathcal{L}$. Again, when the point is close to these lines, the behavior is more complex, and we will not try to characterize it here. In all cases, we can see that if a and b are fixed, for almost all of the choices of c and d, Left dominates.

In the rest of this section, we will assume that we have $d > c > b$. We start by the case $b \geq 2a$ which is easier to analyze.

5.1. Case $b \geq 2a$

We start by the case where $b \geq a$, and show that in this case G is ultimately $\mathcal{L}$ if (c, d) is far enough from the diagonal.

Theorem 21. *If $b \geq 2a$, and $d > c + b$, then $S_{\mathcal{L}} \succ S_{\mathcal{R}}$. More precisely, the outcome sequence is:*

$$\mathcal{P}^a \mathcal{L}^{c-a} \mathcal{N}^a \mathcal{L}^{d-c-a} \mathcal{N}^a \mathcal{L}^{\infty}.$$

Proof. Again, we will show this result by induction on n, the starting position of the game. Let G' be the game $(\{a, a + k\}, \{c\})$. If $n < d$, then G played on n has the same outcome as G', since playing d is not a valid move for Right in this case. Consequently we can just apply Theorem 20, and get the desired result. Otherwise, there are two possible cases:

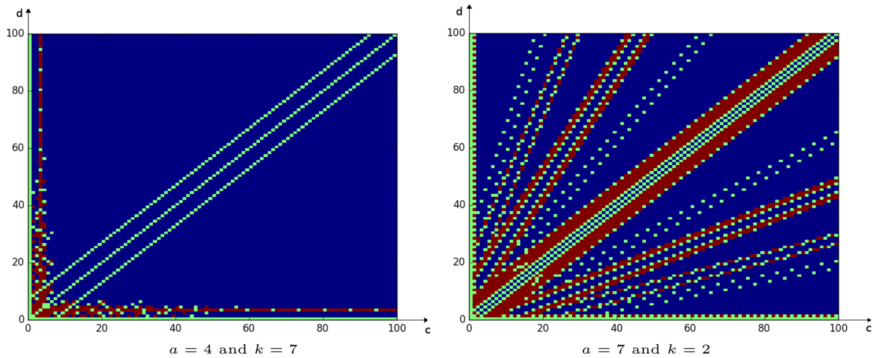

Figure 1: Properties of the outcome sequences for $G = (\{a, a+k\}, \{c, d\})$. The parameters a and k are fixed, and the pictures are obtained by varying the parameters c and d. The point at coordinate (c, d) is blue if the corresponding game is eventually $\mathcal{L}$, red if it is eventually $\mathcal{R}$, and green if there is a mixed period.

- If $d \leq n < d + a$, then Right has a winning move to the position $n - d < a$, and Left has a winning move by playing his strategy for the game G' on n. Indeed, this leads to a position $n - x < d$ for some $x \in \{a, a+k\}$ with outcome either $\mathcal{P}$ or $\mathcal{L}$ for G' and consequently also for G, since d cannot be played anymore at this point.

- If $n \geq d + a$, denote by I_1 and I_2 the two intervals containing the $\mathcal{N}$-position, i.e., $I_1 = [c, c+a[$, and $I_2 = [d, d+a[$. Since $k \geq a$, we cannot have that $n - a$ and $n - a - k$ are both in I_1, or both in I_2. Additionally, since $d > c + a + k$, we cannot have both $n - a - k \in I_1$ and $n - a \in I_2$ at the same time. Consequently, one of $n - a$ and $n - a - k$ has outcome either $\mathcal{L}$ or $\mathcal{P}$, and Left has a winning move on n.

□

5.2. General Case

In the general case, we will again prove that if we fix a and b, for most choices of c and d the outcome is ultimately $\mathcal{L}$. The exceptional cases are slightly more complicated to characterize. The characterization is related to the following definition.

Definition 22. Given an integer a, and a real number $\alpha \geq 1$, we denote by $T_{a,\alpha}$ the set of points defined by:

- $T_{0,\alpha} = \{(c, d) \ : \gcd(c, d) \geq \frac{\max(c,d)}{\alpha}\}$;
- for $a \geq 1$, $T_{a,\alpha}$ is obtained from $T_{0,\alpha}$ by a translation of vector $(-a, -a)$.

We can remark that, for any α and β with $\beta \geq \alpha$, we have $T_{0,\alpha} \subseteq T_{0,\beta}$. We now prove some properties of the sets $T_{a,\alpha}$ which will be usefull for the proofs later on.

Lemma 23. *Assume that there are some positive integers x, y, u and v such that $xu - yv = 0$ with $(u, v) \neq (0, 0)$. Then $(x, y) \in T_{0,\max(u,v)}$.*

Proof. Up to dividing u and v by $\gcd(u, v)$, we can assume that u and v are coprime. Then, the equation is $xu = yv$. Consequently, u is a divisor of yv, and since u and v are coprimes, this means that u is a divisor of v. We can write $y = gu$, and consequently we have $xu = yv = vgu$. This means that $x = vg$, and $g = \gcd(x, y)$. Consequently, $\frac{\max(x,y)}{\gcd(x,y)} = \max(u, v)$, and $(x, y) \in T_{0,\max(u,v)}$. □

Given two points $p = (x, y)$ and $p' = (x', y')$, we denote by $\mathbf{d}(p, p')$ the distance between these two points according to the 1-norm: $\mathbf{d}(p, p') = |x - x'| + |y - y'|$. If $\mathcal{D}$ is a subset of $\mathbb{N}^2$, we denote by $\mathbf{d}(p, \mathcal{D}) = \min\{\mathbf{d}(p, p''), p'' \in \mathcal{D}\}$ the distance of the point p to the set $\mathcal{D}$.

Lemma 24. *Assume that there are some positive integers x, y, u, v and a such that $|xu - yv| \leq a$, then $\mathbf{d}((x, y), T_{0,\max(u,v)}) \leq a(u + v)$.*

Proof. Let $r = xu - yv$, with $|r| \leq a$, and $g = \gcd(u, v)$. By definition, r is a multiple of g, and we can write $r = qg$ for some integer q. Additionally, by Bézout's identity, we know that there exists two integers u' and v' such that $uu' + vv' = g$, and $|u'| \leq u$ and $|v'| \leq v$. Consider the point (x', y'), with $x' = x - qu'$, and $y' = y + qv'$. We have the following:

$$x'u - y'v = xu + yv - q(uu' + vv') = r - qg = 0.$$

By Lemma 23, we know that $(x', y') \in T_{0,\max(u,v)}$. Additionally, $\mathbf{d}((x, y), (x', y')) = |qu'| + |qv'| \leq |r|(u + v) \leq a(u + v)$. This proves the Lemma. □

For any a and α, the set $T_{a,\alpha}$ satisfies the following properties.

Lemma 25. *For any a and α, the set $T_{a,\alpha}$ is the union of a finite set of lines.*

Proof. Since $T_{a,\alpha}$ can be obtained from $T_{0,\alpha}$ by a translation, we only need to prove the result in the case $a = 0$. Let $\mathcal{D}$ be the union of the lines with equation $xu - yv = 0$, for all $u, v \leq \alpha$. The set $\mathcal{D}$ is the union of a finite number of lines. By Lemma 23, we know that $\mathcal{D} \subseteq T_{0,\alpha}$. Reciprocally, let (x, y) be a point in $T_{0,\alpha}$, and let $g = \gcd(x, y)$. We can write $x = x'g$, and $y = y'g$ for some integers x' and y'. We have the following:

$$xy' - yx' = x'y'g - y'gx = 0$$

Additionally, we have $x' = \frac{x}{g} \leq x\frac{\max(x,y)}{\alpha} \leq \alpha$, and similarly for y'. Consequently, $(x, y) \in \mathcal{D}$, and $T_{0,\alpha} = \mathcal{D}$. □

The goal in the remaining of this section is to prove the following theorem.

Theorem 26. *Let a, b, c and d be positive integers, let $A = \lceil \frac{a}{b-a} \rceil + 1$. Assume that $\mathbf{d}((c,d), T_{a,A}) \geq 2A(a+2b)$, then the partizan subtraction game with $S_L = \{a,b\}$, and $S_R = \{c,d\}$ is ultimately $\mathcal{L}$.*

Given two integers i and j, we define the following intervals:

- $I^{\mathcal{P}}_{i,j} = [\alpha_{i,j}, \alpha_{i,j} + a - (i+j)(b-a)[$
- $I^{\mathcal{N}}_{i,j} = [\beta_{i,j}, \beta_{i,j} + a - (i+j-1)(b-a)[$

where

- $\alpha_{i,j} = i(d+b) + j(c+b)$,
- and $\beta_{i,j} = \alpha_{i,j} - b$.

Denote by $I^{\mathcal{P}}$ the set $\cup_{i,j} I^{\mathcal{P}}_{i,j}$, and similarly, $I^{\mathcal{N}} = \cup_{i,j} I^{\mathcal{N}}_{i,j}$. Note that $I^{\mathcal{P}}_{i,j}$ is empty if $i + j \geq \lceil \frac{a}{k} \rceil$, and $I^{\mathcal{N}}_{i,j}$ is empty if $i + j \geq \lceil \frac{a}{k} \rceil + 1$. Our goal is to show that, under the conditions in the statement of the theorem, the set $I^{\mathcal{N}}$ is the set of $\mathcal{N}$-positions, $I^{\mathcal{P}}$ the set of $\mathcal{P}$-positions, and all the other positions have outcome $\mathcal{L}$. In particular, since both $I^{\mathcal{P}}$ and $I^{\mathcal{N}}$ are finite, this will imply that the outcome sequence is eventually $\mathcal{L}$. Before showing this, we prove that under the conditions of the theorem the intervals $I^{\mathcal{P}}_{i,j}$ and $I^{\mathcal{N}}_{i,j}$ satisfy the following properties.

Lemma 27. *Fix the parameters a and b, and let $A = \lceil \frac{a}{b-a} \rceil + 1$. Assume that c and d are such that $\mathbf{d}((c,d), T_{b,A}) \geq 2A(a+2b)$, then the intervals $I^{\mathcal{N}}_{i,j}$ and $I^{\mathcal{P}}_{i,j}$ satisfy the following properties:*

(i) they are pairwise disjoint,

(ii) there is no interval $I^{\mathcal{P}}_{i',j'}$ or $I^{\mathcal{N}}_{i',j'}$ intersecting any of the b positions preceding $I^{\mathcal{N}}_{i,j}$,

(iii) $I^{\mathcal{P}}_{i,j} + c = I^{\mathcal{N}}_{i,j+1}$,

(iv) $I^{\mathcal{P}}_{i,j} + d = I^{\mathcal{N}}_{i+1,j}$,

(v) $(I^{\mathcal{N}}_{i,j} + a) \cap (I^{\mathcal{N}}_{i,j} + b) = I^{\mathcal{P}}_{i,j}$.

Proof. The points (iii), (iv) and (v) are just consequences of the definitions of $I^{\mathcal{P}}_{i,j}$ and $I^{\mathcal{N}}_{i,j}$. Consequently, we only need to prove the two other points.

We know that $I^{\mathcal{N}}_{i,j}$ and $I^{\mathcal{P}}_{i,j}$ are empty when $i+j \geq \lceil \frac{a}{b-a} \rceil + 1 = A$, consequently, we will assume in all the following that the indices i, j, i' and j' are all upper bounded by A. We first show the following claim. The rest of the proof will simply consists in applying this claim several times.

Claim 28. *Assume that there is an integers B, and indices $i, j, i', j' \leq A$, such that one of the following holds:*

- $|\alpha_{i,j} - \alpha_{i',j'}| \leq B$
- $|\beta_{i,j} - \beta_{i',j'}| \leq B$
- $|\alpha_{i,j} - \beta_{i',j'}| \leq B$

Then in all three cases we have $\mathbf{d}((c, d), T_{b,A}) \leq 2A(B + b)$.

Proof. The first two cases are equivalent to the inequality $|(i-i')(d+b)+(j-j')(c+b)| \leq B$, and the result follows by applying Lemma 24. The third case is equivalent to $|(i-i')(d+b)+(j-j')(c+b)+b| \leq B$. Using the triangle inequality, this implies $|(i-i')(d+b)+(j-j')(c+b)| \leq B+b$, and the result follows from Lemma 24. □

We will prove the points (i) and (ii) by proving their contrapositives. In other words, assuming that one of these two conditions does not hold, we want to show that $\mathbf{d}((c, d), T_{b,\alpha}) \leq 2\alpha(a + b)$.

We first consider the point (i). First, assume that there are two intervals $I^{\mathcal{P}}_{i,j}$ and $I^{\mathcal{P}}_{i',j'}$ such that the two intervals intersect. Then, the Left endpoint of one of these two intervals is contained in the other interval. Without loss of generality, we can assume that $\alpha_{i,j} \in I^{\mathcal{P}}_{i',j'}$. This implies:

$$\alpha_{i',j'} \leq \alpha_{i,j} \leq \alpha_{i',j'} + a - (b - a)(i' + j')$$
$$0 \leq \alpha_{i,j} - \alpha_{i',j'} \leq a - (b - a)(i' + j') \leq a$$

By claim 28, this implies $\mathbf{d}((c, d), T_{b,A}) \leq 2A(a + b)$.

Similarly, if we assume that $I^{\mathcal{N}}_{i,j}$ and $I^{\mathcal{N}}_{i',j'}$ intersect, then this implies without loss of generality that $\beta_{i,j} \in I^{\mathcal{N}}_{i',j'}$, and consequently, $0 \leq \beta_{i,j}-\beta_{i',j'} \leq a-(i+j-1)k \leq a$. Again, using Claim 28, this implies $\mathbf{d}((c, d), T_{b,A}) \leq 2(a + b)A$.

Finally, if $I^{\mathcal{N}}_{i',j'}$ and $I^{\mathcal{P}}_{i,j}$ intersect, then either $0 \leq \alpha_{i,j} - \beta_{i',j'} \leq a$ if $\alpha_{i,j} \in I^{\mathcal{N}}_{i',j'}$ or $0 \leq \beta_{i',j'} - \alpha_{i,j} \leq a$ if $\beta_{i',j'} \in I^{\mathcal{P}}_{i,j}$. In both cases, the claim 28 gives the desired result.

The proof for the point (ii) is essentially the same as above. If $I^{\mathcal{N}}_{i',j'}$ intersects one of the b positions preceding $I^{\mathcal{N}}_{i,j}$, then we have the two inequalities:

$$\beta_{i',j'} + a - (i' + j' - 1)k \geq \beta_{i,j} - b \qquad\qquad \beta_{i',j'} \leq \beta_{i,j}$$

From these inequalities we can immediately deduce $-a - b \leq \beta_{i',j'} - \beta_{i,j} \leq 0$. The inequality $\mathbf{d}((c, d), T_{a+k,A}) \leq 2A(3a + 2k)$ follows immediately from Claim 28. Similarly, if the interval $I^{\mathcal{P}}_{i',j'}$ intersects one of the b positions preceding $I^{\mathcal{N}}_{i,j}$, then we have the two inequalities:

$$\alpha_{i',j'} + a - (i'+j')(b-a) \geq \beta_{i,j} - b \qquad\qquad \alpha_{i',j'} \leq \beta_{i,j}$$

This implies $-(a+b) \leq \alpha_{i',j'} - \beta_{i,j} \leq 0$, and again the result holds by claim 28. □

We now have all the tools needed to prove the theorem.

Proof of Theorem 26. Let a, b, c, d be integers, and let $A = \lceil \frac{a}{b-a} \rceil$, and assume that $\mathbf{d}((c,d), T_{b,A}) \geq 2A(a+2b)$. We know that the four properties of Lemma 27 hold. We will show by induction on n that for any position $n \geq 0$, if $n \in I^{\mathcal{P}}$, then n is a $\mathcal{P}$-position, if $n \in I^{\mathcal{N}}$, then it is an $\mathcal{N}$-position, and otherwise it is an $\mathcal{L}$-position. The inductive case is treated in the same way as the base case.

First, assume that $n \in I^{\mathcal{N}}_{i,j}$ for some indices i and j such that $i+j \geq 1$. Left has a winning move by playing a. Indeed, the interval $I^{\mathcal{N}}_{i,j}$ has length at most a, and using the condition (ii) from Lemma 27 and the induction hypothesis, $n-a$ is a $\mathcal{L}$-position. If $i > 0$, then Right playing c leads to the position $n - c \in I^{\mathcal{P}}_{i-1,j}$ by condition (iii). This position is a $\mathcal{P}$-position using the induction hypothesis. If $j > 0$, then similarly, Right can play d, and put the game in the position $n - d \in I^{\mathcal{P}}_{i,j-1}$ by condition (iv). This position is a $\mathcal{P}$-position using the induction hypothesis.

Suppose now that $n \in I^{\mathcal{P}}_{i,j}$. If i and j are both zero, then neither player has any move, and n is a $\mathcal{P}$-position. Otherwise, if Left plays either a or b, this leads to a position $n' \in I^{\mathcal{N}}_{i,j}$ by condition (v). Using the induction hypothesis, n' is an $\mathcal{N}$-position, and Left has no winning move. Right's only possible winning move would be to a $\mathcal{P}$-position n'. Using the induction hypothesis this means $n' \in I^{\mathcal{P}}$. However, this would mean by conditions (iii) and (iv) that $n \in I^{\mathcal{N}}$, which is a contradiction of the property (i) that $I^{\mathcal{N}}$ and $I^{\mathcal{P}}$ are disjoint. Consequently, Right has no winning move.

Finally, suppose that $n \notin I^{\mathcal{P}} \cup I^{\mathcal{N}}$. We will show that Left has a winning move on n, and Right does not. Since $I^{\mathcal{P}}_{0,0} = [0, a[$, we can assume $n \geq a$, and Left can play a. Suppose that Left's move to $n - a$ is not a winning move, and let us show that Left has a winning move to $n - a - k$. Since Left's move to $n - a$ is not a winning move, this means that $n - a \in I^{\mathcal{N}}_{i,j}$ for some integer i, j with $i + j \geq 1$. Consequently we have $n \geq b$, and playing b is a valid move for Left. By condition (v), we cannot have $n - b \in I^{\mathcal{N}}_{i,j}$ since otherwise we would have $n \in I^{\mathcal{P}}_{i,j}$. Moreover, we cannot have either $n - b \in I^{\mathcal{N}}_{i',j'}$ for some $(i', j') \neq (i, j)$ since it would contradict condition (ii). Consequently, $n - b \in I^{\mathcal{L}}$, and using the induction hypothesis, this is a winning move for Left. The only possible winning move for Right would be to play to a position n' which is a $\mathcal{P}$-position. Using the induction hypothesis, this means that $n' \in I^{\mathcal{P}}$. However using the conditions (iii) and (iv) this would also imply $n \in I^{\mathcal{N}}$, a contradiction. □

Corollary 29. *Under the conditions of the theorem, the game G is ultimately $\mathcal{L}$.*

Proof. Since $I^{\mathcal{N}}_{i,j}$ and $I^{\mathcal{P}}_{i,j}$ are both empty if $i + j > a$, the two sets $I^{\mathcal{L}}$ and $I^{\mathcal{N}}$ are finite, and the result follows from the theorem. □

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