A CODEX OF $\mathcal{N}$- AND $\mathcal{P}$-POSITIONS IN HARARY'S 'CATERPILLAR GAME'

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– This paper is dedicated in memory of Elwyn R. Berlekamp (1940-2019), John H. Conway (1937-2020) and Richard K. Guy (1916-2020), the founding fathers of combinatorial game theory.

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Abstract

Frank Harary proposed the following game: Given a caterpillar C, two players take turns removing edges of a path. The player who takes the last edge wins the game. In this paper, we completely characterize the $\mathcal{N}$- and $\mathcal{P}$-positions for all caterpillars with spine length zero, one, two and three. Furthermore, we analyze approximately 94% of the caterpillars with spine length greater than or equal to four. In those cases, they all turn out to be $\mathcal{N}$-positions.

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1. Introduction and Preliminaries

Combinatorial game theory (CGT) developed in the context of recreational mathematics. In their seminal work and with a spirit of playfulness, Berlekamp, Conway and Guy [3, 7] established the mathematical framework from which games of complete information could be studied. The power of this theory would soon become apparent and was utilized by many researchers (see Fraenkel's bibliography [11]). Along with its natural appeal, combinatorial game theory has applications to complexity theory, logic, and biology. Literature on the subject continues to increase and the interested reader can find comprehensive introductions to CGT in [2, 3, 7, 26]. Additional research articles with a theoretical flavor can be found in [1, 19, 22, 23, 24].

Combinatorial games can be played on various mathematical structures such as simplicial complexes, posets, vector spaces, groups and multisets. There are many papers on combinatorial games (played on graphs) within the research literature. They include analyses of games with rulesets based on the coloring of edges and/or vertices, "Maker-Breaker"-type constraints, the removal of specific subgraphs, restrictions on a specified graph parameter, etc. In particular, many edge-deletion games on graphs have been studied [4, 5, 10, 12, 13, 14, 18, 20, 21, 25]. Other edge-deletion games such as Arc-Kayles [8, 17] are analogs of classical combinatorial games. Further yet, graph-specific properties and/or arithmetic conditions are used to define the rulesets for other edge-deletion games [9, 16].

Before introducing Harary's *Caterpillar Game* (an edge-deletion game), we first recall some definitions and fundamental concepts from combinatorial game theory. Terms which are not explicitly defined in this paper can be found in [26]. A *combinatorial game* is one of complete information and no element of chance is involved in gameplay. Each player is aware of the game position at any point in the game. Under *normal play*, two players (P1 and P2) alternate taking turns and a player loses when he cannot make a move. An *impartial* combinatorial game is one where both players have the same options from any position. A *finite* game eventually terminates (with a winner and a loser, no draws allowed). It is understood that P1 makes the first move in any combinatorial game.

For any finite impartial combinatorial game Γ, there is an associated non-negative integer value (*Grundy-value*) $\mathcal{G}(\Gamma)$. The Grundy-value $\mathcal{G}(\Gamma)$ immediately tells us if Γ is a $\mathcal{P}$-*position* (previous player win) or an $\mathcal{N}$-*position* (next player win). In particular, $\mathcal{G}(\Gamma) = 0$ if and only if Γ is a $\mathcal{P}$-position. To compute $\mathcal{G}(\Gamma)$, we need the following definitions.

Definition 1. The *minimum excluded value* (or *mex*) of a multiset of non-negative integers is the smallest non-negative integer which does not appear in the multiset. This is denoted by $\text{mex}\{t_1, t_2, t_3, \ldots, t_k\}$.

Example 1. $\text{mex}\{0, 2, 3, 3, 5\} = 1$. ♢

Definition 2. Let Γ be a finite impartial game. Then, the *Grundy-value* of Γ (denoted by $\mathcal{G}(\Gamma)$) is defined to be

$$\mathcal{G}(\Gamma) = \text{mex}\{\mathcal{G}(\Delta) : \Delta \text{ is an option of } \Gamma\}.$$

The *sum* of finite impartial games is the game obtained by placing the individual games, side by side. On a player's turn, a move is made in a single summand. Under normal play, the last person to make a move wins. For any finite impartial game $\Gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_k$, the Grundy-value of Γ is computed in the following way. First, convert $\mathcal{G}(\gamma_i)$ into binary. Then, compute $\bigoplus \mathcal{G}(\gamma_i)$, where the sum is BitXor (NIM-addition). Finally, convert this value back into a nonnegative integer. When there is no danger of confusion, the binary representation of Grundy-values will also be used in this paper.

Example 2. Suppose that γ_1, γ_2 and γ_3 are finite impartial games with $\mathcal{G}(\gamma_1) = 1$, $\mathcal{G}(\gamma_2) = 2$ and $\mathcal{G}(\gamma_3) = 3$. Then the game $\Gamma = \gamma_1 + \gamma_2 + \gamma_3$ has Grundy-value

$$\mathcal{G}(\Gamma) = 01 \oplus 10 \oplus 11 = 00,$$

and thus has Grundy-value 0. ♢

In 2001, Frank Harary [15] introduced an edge-deletion game played on caterpillar graphs. The *Caterpillar Game* is played in the following way:

- Let C be a caterpillar, namely a path along with pendant edges connected to some (possibly all or none) of the vertices of the path. Two players take turns removing edges of a non-trivial path. The player who takes the last edge wins the game.

Example 3. Figure 1 illustrates the beginning of a sample game. After P1 makes his first move, P2 must move from a losing (i.e., Grundy-value 0) $\mathcal{P}$-position. From this point on, whatever P2 chooses to do, P1 will mimic P2's move on the other corresponding component. Hence, the starting game position is an $\mathcal{N}$-position. ♢

For whatever reason, the *Caterpillar Game* did not generate much interest within the mathematical community and was forgotten over the passage of time. We are aware of this game only because the fourth author heard Harary's lecture, so many years ago. A current literature search for the *Caterpillar Game* reveals nothing. Nevertheless, the authors find this edge-deletion game interesting in its own right.

We now give a careful analysis of the $\mathcal{N}$- and $\mathcal{P}$-positions in the *Caterpillar Game*. In Section 2, the $\mathcal{N}$- and $\mathcal{P}$-positions are characterized for all caterpillars

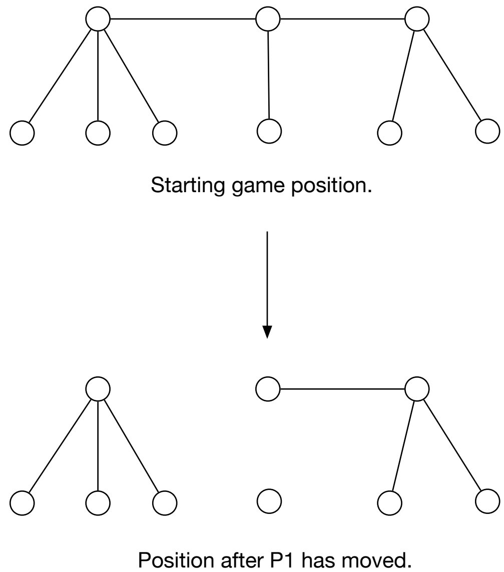

Figure 1: A first move in the *Caterpillar Game.*

with spine length zero and one. Then, general results in Section 3 are used to analyze approximately 94% of the caterpillars with spine length four or greater, all of which turn out to be $\mathcal{N}$-positions. Finally in Section 4, the $\mathcal{N}$- and $\mathcal{P}$-positions are characterized for all caterpillars with spine length two and three.

2. Stars, Paths and Caterpillars With Spine Length One

In this paper, we use standard graph-theoretic terms and concepts as found in [6]. A *star* $K_{1,n}$ $(n \geq 1)$ is a tree with one internal vertex and n leaves. The degenerate star $K_{1,0}$ is the trivial graph, consisting of a single vertex. A *path* is a trail in which all of the vertices are distinct. The path P_n $(n \geq 1)$ denotes the path containing n vertices.

Notation. When convenient, we will use the following notation: Let $n \geq 0$. Then, $\overline{n} = \{n + 3k : k \geq 0\}$.

Theorem 1. *Let* $n \geq 0$*. Then,*

$$\mathcal{G}(K_{1,n}) = \begin{cases} 1, & \textit{if } n \in \overline{1}; \\ 2, & \textit{if } n \in \overline{2}; \\ 0, & \textit{if } n \in \overline{0}. \end{cases}$$

Proof. We induct on n. For $n = 0, 1$ and 2, it is easy to check that $\mathcal{G}(K_{1,n}) = 0, 1$ and 2, respectively. Now, assume that the claim is true, for all $n \leq k$. We examine $\mathcal{G}(K_{1,k+1})$. If a player removes a path of length one, the resulting game position will have Grundy-value $\mathcal{G}(P_1) \oplus \mathcal{G}(K_{1,k})$. On the other hand, if a player removes a path of length two, the resulting game position will have Grundy-value $\mathcal{G}(P_1) \oplus \mathcal{G}(P_1) \oplus \mathcal{G}(K_{1,k-1})$. Thus, $\mathcal{G}(K_{1,k+1}) = \text{mex}\{\mathcal{G}(P_1) \oplus \mathcal{G}(K_{1,k}), \mathcal{G}(P_1) \oplus \mathcal{G}(P_1) \oplus \mathcal{G}(K_{1,k-1})\}$.

`Case 1`. $k + 1 \in \overline{4}$. Then, $\mathcal{G}(K_{1,k+1}) = \text{mex}\{0 \oplus 0, 0 \oplus 0 \oplus 2\} = 1$.
`Case 2`. $k + 1 \in \overline{5}$. Then, $\mathcal{G}(K_{1,k+1}) = \text{mex}\{0 \oplus 1, 0 \oplus 0 \oplus 0\} = 2$.
`Case 3`. $k + 1 \in \overline{3}$. Then, $\mathcal{G}(K_{1,k+1}) = \text{mex}\{0 \oplus 2, 0 \oplus 0 \oplus 1\} = 0$.

By induction, the claim is established. □

Theorem 2. $\mathcal{G}(P_n) = n - 1$*, for all* $n \geq 1$.

Proof. We induct on n. Clearly, $\mathcal{G}(P_1) = 0$ and $\mathcal{G}(P_2) = 1$. Now, assume that the claim is true, for all $n \leq k$. We examine $\mathcal{G}(P_{k+1})$. On a player's turn, a path of length l is removed, where $1 \leq l \leq k$. The resulting game position will have Grundy-value $\mathcal{G}(P_1) \oplus \mathcal{G}(P_{k+1-l})$, $\mathcal{G}(P_2) \oplus \mathcal{G}(P_{k-l})$, $\mathcal{G}(P_3) \oplus \mathcal{G}(P_{k-l-1}), \ldots,$ or $\mathcal{G}(P_{k+1-l}) \oplus \mathcal{G}(P_1)$.

Thus,

$$\mathcal{G}(P_{k+1}) = \text{mex}\{\mathcal{G}(P_1) \oplus \mathcal{G}(P_{k+1-l}), \mathcal{G}(P_2) \oplus \mathcal{G}(P_{k-l}), \mathcal{G}(P_3) \oplus \mathcal{G}(P_{k-l-1}), \ldots,$$
$$\mathcal{G}(P_{k+1-l}) \oplus \mathcal{G}(P_1) : 1 \leq l \leq k\}.$$

As $\mathcal{G}(P_1) = 0$, we note that $\{\mathcal{G}(P_1), \mathcal{G}(P_2), \mathcal{G}(P_3), \ldots, \mathcal{G}(P_k)\}$ is a subset of

$$\{\mathcal{G}(P_1) \oplus \mathcal{G}(P_{k+1-l}), \mathcal{G}(P_2) \oplus \mathcal{G}(P_{k-l}), \mathcal{G}(P_3) \oplus \mathcal{G}(P_{k-l-1}), \ldots,$$
$$\mathcal{G}(P_{k+1-l}) \oplus \mathcal{G}(P_1) : 1 \leq l \leq k\}.$$

By the inductive hypothesis, we have that $\mathcal{G}(P_{k+1}) \geq \text{mex}\{0, 1, 2, \ldots, k - 1\} = k$.

Now, we recall the following fact, which will help us finish the induction proof:

- Let $n, r, s \geq 1$, where $r + s = n$. Then, $r \oplus s \leq n$.

This fact is informally given in [3], Volume 1, without a proof. However, this is a true fact. To see this, write r and s in binary and observe that $r \dotplus s = (r \oplus s) \dotplus (r \text{ AND } s) \dotplus (r \text{ AND } s)$. Here, $\dotplus$ denotes binary addition, $\oplus$ denotes BitXor addition and 'AND' denotes the bitwise AND operator.

Returning to the induction proof, we see (using this fact, along with the inductive hypothesis,) that each element of

$$\{\mathcal{G}(P_1) \oplus \mathcal{G}(P_{k+1-l}), \mathcal{G}(P_2) \oplus \mathcal{G}(P_{k-l}), \mathcal{G}(P_3) \oplus \mathcal{G}(P_{k-l-1}), \ldots,$$

$$\mathcal{G}(P_{k+1-l}) \oplus \mathcal{G}(P_1) : 1 \leq l \leq k\}$$

is less than or equal to $k-l$, for some l ($1 \leq l \leq k$). In particular, each element is less than or equal to $k-1$. Thus, $\mathcal{G}(P_{k+1}) = k$. By induction, the claim is established. □

A *caterpillar* consists of a path, along with pendant edges connected to some (possibly all or none) of the vertices of the path. The *spinal* vertices of a caterpillar are the vertices of degree two or more. The *spine* of the caterpillar is the unique path containing only the spinal vertices of the caterpillar. For example, $K_{1,n}$ ($n \geq 2$) can be viewed as a caterpillar with one spinal vertex (and spine length zero).

Notation. Let $m, n \geq 1$. Then, $C = \langle [m,n] \rangle = \langle [n,m] \rangle$ denotes the caterpillar C with spine length 1 and m and n pendant edges (at the two spinal vertices, respectively). For example, $\langle [1,1] \rangle \cong P_4$ (the path on four vertices). Figure 2 gives another illustration of this notation.

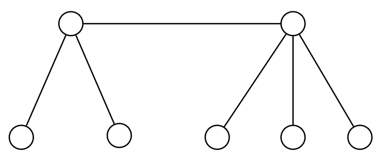

Figure 2: The caterpillar $C = \langle [2,3] \rangle$. Deleting the spine of C yields $\langle [2],[3] \rangle$.

Also note that throughout this paper, we use the notation $\mathcal{G}\langle [1,n] \rangle$ to mean $\mathcal{G}(\langle [1,n] \rangle)$. We do this for the sake of readability.

Lemma 1. *Let $n \geq 1$. Then,*

$$\mathcal{G}\langle [1,n] \rangle = \begin{cases} 3, & \textit{if } n \in \overline{1}; \\ 4, & \textit{if } n \in \overline{2}; \\ 5, & \textit{if } n \in \overline{0}. \end{cases}$$

Proof. We induct on n. For $n = 1, 2$ and 3, it is easy to check that $\mathcal{G}\langle [1,n] \rangle = 3, 4$ and 5, respectively. Now, assume that the claim is true, for all $n \leq k$. We examine $\mathcal{G}\langle [1, k+1] \rangle$. If a player removes a path of length one, the resulting position will have Grundy-value $\mathcal{G}(K_{1,k+2})$, $\mathcal{G}(K_{1,1}) \oplus \mathcal{G}(K_{1,k+1})$ or $\mathcal{G}\langle [1,k] \rangle$. If a player removes a path of length two, the resulting position will have Grundy-value $\mathcal{G}(K_{1,k+1})$, $\mathcal{G}(K_{1,1}) \oplus \mathcal{G}(K_{1,k})$ or $\mathcal{G}\langle [1,k-1] \rangle$. Lastly, if a player removes a path of length three, the resulting position will have Grundy-value $\mathcal{G}(K_{1,k})$. Thus, $\mathcal{G}\langle [1,k+1] \rangle = \text{mex}\{k+2 \pmod 3, 1 \oplus (k+1 \pmod 3)), \mathcal{G}\langle [1,k] \rangle, k+1 \pmod 3, 1 \oplus$

$(k \pmod 3)), \mathcal{G}\langle[1,k-1]\rangle, k \pmod 3)\}$.

`Case 1`. If $k+1 \in \overline{4}$, then $\mathcal{G}\langle[1,k+1]\rangle = \text{mex}\{2,0,5,1,1,4,0\} = 3$.
`Case 2`. If $k+1 \in \overline{5}$, then $\mathcal{G}\langle[1,k+1]\rangle = \text{mex}\{0,3,3,2,0,5,1\} = 4$.
`Case 3`. If $k+1 \in \overline{6}$, then $\mathcal{G}\langle[1,k+1]\rangle = \text{mex}\{1,1,4,0,3,3,2\} = 5$.

By induction, the claim is established. □

Lemma 2. *Let $n \geq 1$. Then,*

$$\mathcal{G}\langle[2,n]\rangle = \begin{cases} 4, & \text{if } n \in \overline{1}; \\ 1, & \text{if } n \in \overline{2}; \\ 6, & \text{if } n \in \overline{0}. \end{cases}$$

Proof. We induct on n. For $n = 1, 2$ and 3, it is easy to check that $\mathcal{G}\langle[2,n]\rangle = 4, 1$ and 6, respectively. Now, assume that the claim is true, for all $n \leq k$. We examine $\mathcal{G}\langle[2,k+1]\rangle$. Here, $\mathcal{G}\langle[2,k+1]\rangle = \text{mex}\{\mathcal{G}\langle[1,k+1]\rangle, 2 \oplus (k+1 \pmod 3)), \mathcal{G}\langle[2,k]\rangle, k+2 \pmod 3), 1 \oplus (k+1 \pmod 3)), 2 \oplus (k \pmod 3)), \mathcal{G}\langle[2,k-1]\rangle, 1 \oplus (k \pmod 3))\}$.

`Case 1`. If $k+1 \in \overline{4}$, then $\mathcal{G}\langle[2,k+1]\rangle = \text{mex}\{3,3,6,2,0,2,1,1\} = 4$.
`Case 2`. If $k+1 \in \overline{5}$, then $\mathcal{G}\langle[2,k+1]\rangle = \text{mex}\{4,0,4,0,3,3,6,0\} = 1$.
`Case 3`. If $k+1 \in \overline{6}$, then $\mathcal{G}\langle[2,k+1]\rangle = \text{mex}\{5,2,1,1,1,0,4,3\} = 6$.

By induction, the claim is established. □

Lemma 3. *Let $n \geq 1$. Then,*

$$\mathcal{G}\langle[3,n]\rangle = \begin{cases} 5, & \text{if } n \in \overline{1}; \\ 6, & \text{if } n \in \overline{2}; \\ 1, & \text{if } n \in \overline{0}. \end{cases}$$

Proof. We induct on n. For $n = 1, 2$ and 3, it is easy to check that $\mathcal{G}\langle[3,n]\rangle = 5, 6$ and 1, respectively. Now, assume that the claim is true, for all $n \leq k$. We examine $\mathcal{G}\langle[3,k+1]\rangle$. Here, $\mathcal{G}\langle[3,k+1]\rangle = \text{mex}\{\mathcal{G}\langle[2,k+1]\rangle, k+1 \pmod 3), \mathcal{G}\langle[3,k]\rangle, \mathcal{G}\langle[1,k+1]\rangle, 2 \oplus (k+1 \pmod 3)), k \pmod 3), \mathcal{G}\langle[3,k-1]\rangle, 2 \oplus (k \pmod 3))\}$.

`Case 1`. If $k+1 \in \overline{4}$, then $\mathcal{G}\langle[3,k+1]\rangle = \text{mex}\{4,1,1,3,3,0,6,2\} = 5$.
`Case 2`. If $k+1 \in \overline{5}$, then $\mathcal{G}\langle[3,k+1]\rangle = \text{mex}\{1,2,5,4,0,1,1,3\} = 6$.
`Case 3`. If $k+1 \in \overline{6}$, then $\mathcal{G}\langle[3,k+1]\rangle = \text{mex}\{6,0,6,5,2,2,5,0\} = 1$.

By induction, the claim is established. □

Lemma 4. *Let $n \geq 2$. Then,*

$$\mathcal{G}\langle[4,n]\rangle = \begin{cases} 2, & \text{if } n \in \overline{1}; \\ 4, & \text{if } n \in \overline{2}; \\ 5, & \text{if } n \in \overline{0}. \end{cases}$$

Proof. First, note that $\mathcal{G}\langle[4,1]\rangle = 3$, and is not equal to two. We induct on n. For $n = 2, 3$ and 4, it is easy to check that $\mathcal{G}\langle[4,n]\rangle = 4, 5$ and 2, respectively. Now, assume that the claim is true, for all $n \leq k$. We examine $\mathcal{G}\langle[4,k+1]\rangle$. Here, $\mathcal{G}\langle[4,k+1]\rangle = \text{mex}\{\mathcal{G}\langle[3,k+1]\rangle, 1 \oplus (k+1 \pmod 3), \mathcal{G}\langle[4,k]\rangle, \mathcal{G}\langle[2,k+1]\rangle, k+1 \pmod 3, 1 \oplus (k \pmod 3), \mathcal{G}\langle[4,k-1]\rangle, k \pmod 3\}$.

Case 1. If $k+1 \in \overline{7}$, then $\mathcal{G}\langle[4,k+1]\rangle = \text{mex}\{5,0,5,4,1,1,4,0\} = 2$.
Case 2. If $k+1 \in \overline{5}$, then $\mathcal{G}\langle[4,k+1]\rangle = \text{mex}\{6,3,2,1,2,0,5,1\} = 4$.
Case 3. If $k+1 \in \overline{6}$, then $\mathcal{G}\langle[4,k+1]\rangle = \text{mex}\{1,1,4,6,0,3,2,2\} = 5$.

By induction, the claim is established. □

In the study of Harary's *Caterpillar Game*, we wrote a computer program in Python to calculate Grundy-values. We observed a repeating pattern of Grundy-values from Lemmas 2, 3 and 4. To formally prove that this is the case, we need Lemmas 5 and 6. They will be used in the base case of an induction proof of Theorem 3.

Lemma 5. *Let $n \geq 1$. Then,*

$$\mathcal{G}\langle[5,n]\rangle = \begin{cases} 4, & \text{if } n \in \overline{1}; \\ 1, & \text{if } n \in \overline{2}; \\ 6, & \text{if } n \in \overline{0}. \end{cases}$$

Proof. We induct on n. For $n = 1, 2$ and 3, it is easy to check that $\mathcal{G}\langle[5,n]\rangle = 4, 1$ and 6, respectively. Now, assume that the claim is true, for all $n \leq k$. We examine $\mathcal{G}\langle[5,k+1]\rangle$. Here, $\mathcal{G}\langle[5,k+1]\rangle = \text{mex}\{\mathcal{G}\langle[4,k+1]\rangle, 2 \oplus (k+1 \pmod 3), \mathcal{G}\langle[5,k]\rangle, \mathcal{G}\langle[3,k+1]\rangle, 1 \oplus (k+1 \pmod 3), 2 \oplus (k \pmod 3), \mathcal{G}\langle[5,k-1]\rangle, 1 \oplus k \pmod 3\}$.

Case 1. If $k+1 \in \overline{4}$, then $\mathcal{G}\langle[5,k+1]\rangle = \text{mex}\{2,3,6,5,0,2,1,1\} = 4$.
Case 2. If $k+1 \in \overline{5}$, then $\mathcal{G}\langle[5,k+1]\rangle = \text{mex}\{4,0,4,6,3,3,6,0\} = 1$.
Case 3. If $k+1 \in \overline{6}$, then $\mathcal{G}\langle[5,k+1]\rangle = \text{mex}\{5,2,1,1,1,0,4,3\} = 6$.

By induction, the claim is established. □

Lemma 6. *Let $n \geq 1$. Then,*

$$\mathcal{G}\langle[6,n]\rangle = \begin{cases} 5, & \text{if } n \in \overline{1}; \\ 6, & \text{if } n \in \overline{2}; \\ 1, & \text{if } n \in \overline{0}. \end{cases}$$

Proof. We induct on n. For $n = 1, 2$ and 3, it is easy to check that $\mathcal{G}\langle[6,n]\rangle = 5, 6$ and 1, respectively. Now, assume that the claim is true, for all $n \leq k$. We examine

$\mathcal{G}\langle[6,k{+}1]\rangle$. Here, $\mathcal{G}\langle[6,k{+}1]\rangle = \text{mex}\{\mathcal{G}\langle[5,k{+}1]\rangle, (k{+}1 \pmod 3)), \mathcal{G}\langle[6,k]\rangle, \mathcal{G}\langle[4,k+1]\rangle, 2 \oplus (k+1 \pmod 3)), k \pmod 3)), \mathcal{G}\langle[6,k-1]\rangle, 2 \oplus (k \pmod 3))\}$.

`Case 1`. If $k+1 \in \overline{4}$, then $\mathcal{G}\langle[6,k+1]\rangle = \text{mex}\{4,1,1,2,3,0,6,2\} = 5$.
`Case 2`. If $k+1 \in \overline{5}$, then $\mathcal{G}\langle[6,k+1]\rangle = \text{mex}\{1,2,5,4,0,1,1,3\} = 6$.
`Case 3`. If $k+1 \in \overline{6}$, then $\mathcal{G}\langle[6,k+1]\rangle = \text{mex}\{6,0,6,5,2,2,5,0\} = 1$.

By induction, the claim is established. □

Theorem 3. *Let $m,n \geq 2$. Then,*

$$\mathcal{G}\langle[m,n]\rangle = \begin{cases} 2, & \text{if } m \in \overline{1},\ n \in \overline{1}; \\ 4, & \text{if } m \in \overline{1},\ n \in \overline{2}; \\ 5, & \text{if } m \in \overline{1},\ n \in \overline{0}; \\ 4, & \text{if } m \in \overline{2},\ n \in \overline{1}; \\ 1, & \text{if } m \in \overline{2},\ n \in \overline{2}; \\ 6, & \text{if } m \in \overline{2},\ n \in \overline{0}; \\ 5, & \text{if } m \in \overline{0},\ n \in \overline{1}; \\ 6, & \text{if } m \in \overline{0},\ n \in \overline{2}; \\ 1, & \text{if } m \in \overline{0},\ n \in \overline{0}. \end{cases}$$

Proof. First, note that our claim holds when at least one of the m and n is equal to $2, 3$ or 4. This is because of Lemmas 2, 3 and 4. Now, let $m,n \geq 4$. We use double induction to prove the rest of the claim.

Let $S(m,n)$ be the asserted claim. To establish the base case, we examine the cases where $a = 4, 5$ and 6. When $a = 4$, $b \geq 4$; $S(a,b)$ is true by Lemma 4. When $a = 5$, $b \geq 4$; $S(a,b)$ is true by Lemma 5. When $a = 6$, $b \geq 4$; $S(a,b)$ is true by Lemma 6. Now, we induct over m. Assume $S(k,b)$ is true, for some positive integer $k \geq a$. For $S(k+1,b)$, we see that $\mathcal{G}\langle[k+1,b]\rangle = \text{mex}\{\mathcal{G}\langle[k,b]\rangle, \mathcal{G}\langle[k{+}1],[b]\rangle, \mathcal{G}\langle[k{+}1,b{-}1]\rangle, \mathcal{G}\langle[k{+}1{-}2,b]\rangle, \mathcal{G}\langle[k{+}1{-}1],[b]\rangle, \mathcal{G}\langle[k+1],[b-1]\rangle, \mathcal{G}\langle[k+1,b-2]\rangle, \mathcal{G}\langle[k+1-1],[b-1]\rangle\}$.

`Case 1.` If $k+1 \in \overline{5}$ and $b \in \overline{7} : \text{mex}\{2,3,6,5,0,2,1,1\} = 4$.
$b \in \overline{5} : \text{mex}\{4,0,4,6,3,3,6,0\} = 1$.
$b \in \overline{6} : \text{mex}\{5,2,1,1,1,0,4,3\} = 6$.

`Case 2.` If $k+1 \in \overline{6}$ and $b \in \overline{7} : \text{mex}\{4,1,1,2,3,0,6,2\} = 5$.
$b \in \overline{5} : \text{mex}\{1,2,5,4,0,1,1,3\} = 6$.
$b \in \overline{6} : \text{mex}\{6,0,6,5,2,2,5,0\} = 1$.

`Case 3.` If $k+1 \in \overline{7}$ and $b \in \overline{7} : \text{mex}\{5,0,5,4,1,1,4,0\} = 2$.
$b \in \overline{5} : \text{mex}\{6,3,2,1,2,0,5,1\} = 4$.
$b \in \overline{6} : \text{mex}\{1,1,4,6,0,3,2,2\} = 5$.

So, $S(k+1,b)$ is true. Now, we induct over n. Assume S(h,k) is true, for some positive integers h,k, where $h \geq a$ and $k \geq b$. For $S(h,k+1)$, we see that $\mathcal{G}\langle[h,k+1]\rangle = \text{mex}\{\mathcal{G}\langle[h-1,k+1]\rangle, \mathcal{G}\langle[h],[k+1]\rangle, \mathcal{G}\langle[h,k+1-1]\rangle, \mathcal{G}\langle[h-2,k+1]\rangle, \mathcal{G}\langle[h-1],[k+1]\rangle, \mathcal{G}\langle[h],[k+1-1]\rangle, \mathcal{G}\langle[h,k+1-2]\rangle, \mathcal{G}\langle[h-1],[k+1-1]\rangle\}$.

`Case 1.` If $k+1 \in \overline{5}$ and $h \in \overline{7} : \text{mex}\{6,3,2,1,2,0,5,1\} = 4.$
$h \in \overline{5} : \text{mex}\{4,0,4,6,3,3,6,0\} = 1.$
$h \in \overline{6} : \text{mex}\{1,2,5,4,0,1,1,3\} = 6.$

`Case 2.` If $k+1 \in \overline{6}$ and $h \in \overline{7} : \text{mex}\{1,1,4,6,0,3,2,2\} = 5.$
$h \in \overline{5} : \text{mex}\{5,2,1,1,1,0,4,3\} = 6.$
$h \in \overline{6} : \text{mex}\{6,0,6,5,2,2,5,0\} = 1.$

`Case 3.` If $k+1 \in \overline{7}$ and $h \in \overline{7} : \text{mex}\{5,0,5,4,1,1,4,0\} = 2.$
$h \in \overline{5} : \text{mex}\{2,3,6,5,0,2,1,1\} = 4.$
$h \in \overline{6} : \text{mex}\{4,1,1,2,3,0,6,2\} = 5.$

So, $S(h,k+1)$ is true. Thus by double induction, the claim is established. □

Corollary 1. *Let $m,n \geq 2$ and $s \geq 0$. Then, $\mathcal{G}\langle[m+3s,n]\rangle = \mathcal{G}\langle[m,n]\rangle$.*

Proof. This follows immediately from Theorem 3. □

Remark. If $m,n \geq 2$ and $s,t \geq 0$, then $\mathcal{G}\langle[m+3s,n+3t]\rangle = \mathcal{G}\langle[m,n]\rangle$.

Corollary 2. *Let $m,n \geq 1$. Then, $\langle[m,n]\rangle$ is an $\mathcal{N}$-position.*

Proof. The claim follows immediately from Lemmas 1, 2, 3, 4, and Corollary 1. □

3. Some General Results on Caterpillars

Notation. Let $n \geq 3$ and $x_1, x_n \neq 0$. Then, $C = \langle[x_1,x_2,\ldots,x_n]\rangle$ denotes the caterpillar C with n spinal vertices (i.e. spine length equal to $n-1$), where the k^{th} spinal vertex has x_k pendant edges. Note that $C = \langle[x_1,x_2,\ldots,x_n]\rangle = \langle[x_n,\ldots,x_2,x_1]\rangle$. Also, while x_1 and/or x_n can be 0 (for example, $\langle[0,2,1,0]\rangle = \langle[3,2]\rangle$), we will typically restrict our analysis to positions where $x_1, x_n \neq 0$. We define the *internal value* of C to be $IV(C) = \bigoplus_{i=2}^{n-1}[x_i \pmod 3]$, written in binary. The *external value* of C is defined to be $EV(C) = [x_1 \pmod 3] \oplus [x_n \pmod 3]$, written in binary.

Example 4. In Figure 1, the top caterpillar is $C = \langle [3, 1, 2] \rangle$, $IV(C) = 01$ and $EV(C) = [3 \pmod 3] \oplus [2 \pmod 3] = 00 \oplus 10 = 10$. Note that the second graph in Figure 1 can be described in several ways, namely, $\langle [3], [0, 2] \rangle = \langle [3], [0], [0, 2] \rangle = \langle [3], [0], [3] \rangle$. When convenient, isolated vertices can be omitted in the notation. Thus, the second graph in Figure 1 can also be described by $\langle [3], [3] \rangle$. ♢

The following two game strategies will be used in the proofs of subsequent results.

- The universal strategy is defined to be the following: Remove the entire spine of C, along with zero or one leg from each of the two "end-leg" groups. (Moves (a)-(d) in the list below).
- The modified universal strategy is defined to be the following: Remove the entire spine of C except for the segment connecting x_1 and x_2, along with zero or one leg from both x_2 and x_n (moves (e)-(l) in the list below).

a) $\langle [x_1, x_2, \ldots, x_n] \rangle \to \langle [x_1], [x_2], \ldots, [x_n] \rangle$.

b) $\langle [x_1, x_2, \ldots, x_n] \rangle \to \langle [x_1 - 1], [x_2], \ldots, [x_n] \rangle$.

c) $\langle [x_1, x_2, \ldots, x_n] \rangle \to \langle [x_1], [x_2], \ldots, [x_n - 1] \rangle$.

d) $\langle [x_1, x_2, \ldots, x_n] \rangle \to \langle [x_1 - 1], [x_2], \ldots, [x_n - 1] \rangle$.

e) $\langle [x_1, x_2, x_3, \ldots, x_n] \rangle \to \langle [x_1, x_2], [x_3], \ldots, [x_n] \rangle$.

f) $\langle [x_1, x_2, x_3, \ldots, x_n] \rangle \to \langle [x_1, x_2 - 1], [x_3], \ldots, [x_n] \rangle$.

g) $\langle [x_1, x_2, x_3, \ldots, x_n] \rangle \to \langle [x_1, x_2], [x_3], \ldots, [x_n - 1] \rangle$.

h) $\langle [x_1, x_2, x_3, \ldots, x_n] \rangle \to \langle [x_1, x_2 - 1], [x_3], \ldots, [x_n - 1] \rangle$.

In the cases where $x_2 = 0$:

i) $\langle [x_1, x_2, x_3, \ldots, x_n] \rangle \to \langle [x_1 + 1], [x_3], \ldots, [x_n] \rangle$.

j) $\langle [x_1, x_2, x_3, \ldots, x_n] \rangle \to \langle [x_1], [x_3], \ldots, [x_n] \rangle$.

k) $\langle [x_1, x_2, x_3, \ldots, x_n] \rangle \to \langle [x_1 + 1], [x_3], \ldots, [x_n - 1] \rangle$.

l) $\langle [x_1, x_2, x_3, \ldots, x_n] \rangle \to \langle [x_1], [x_3], \ldots, [x_n - 1] \rangle$.

Notation. Let C be a caterpillar. Relative to the above list of moves, let C^μ denote the game position after move μ has been applied to C.

Theorem 4. *Suppose that* $C = \langle [x_1, x_2, \ldots, x_n] \rangle$*, where* $n \geq 3$ *and* $x_1 \neq x_n \pmod 3$*. Then,* C *is an* $\mathcal{N}$*-position.*

Proof. There are three cases to consider:

1. $x_1 \in \overline{1}$, $x_n \in \overline{2}$.
2. $x_1 \in \overline{1}$, $x_n \in \overline{3}$.
3. $x_1 \in \overline{2}$, $x_n \in \overline{3}$.

Note that the other three cases are covered by symmetry. For example, $x_1 \in \overline{1}$, $x_n \in \overline{2}$ and $x_1 \in \overline{2}$, $x_n \in \overline{1}$ would have the same winning strategy for P1, just mirrored.

`Case 1.`

- If $IV(C) = 00$, P1 invokes strategy (c). This leaves a position C^c where $\mathcal{G}(C^c) = x_1 \oplus \mathrm{IV}(C^c) \oplus x_n = 1 \oplus 0 \oplus 1 = 0$.
- If $IV(C) = 01$, P1 invokes strategy (d).
- If $IV(C) = 10$, P1 invokes strategy (b).
- If $IV(C) = 11$, P1 invokes strategy (a).

`Case 2.`

- If $IV(C) = 00$, P1 invokes strategy (b).
- If $IV(C) = 01$, P1 invokes strategy (a).
- If $IV(C) = 10$, P1 invokes strategy (d).
- If $IV(C) = 11$, P1 invokes strategy (c).

`Case 3.`

- If $IV(C) = 00$, P1 invokes strategy (c).
- If $IV(C) = 01$, P1 invokes strategy (b).
- If $IV(C) = 10$, P1 invokes strategy (a).
- If $IV(C) = 11$, P1 invokes strategy (d).

□

Table 1 summarizes Theorem 4.

	x_1		
x_n	$\overline{1}$	$\overline{2}$	$\overline{3}$
$\overline{1}$		$\mathcal{N}$	$\mathcal{N}$
$\overline{2}$	$\mathcal{N}$		$\mathcal{N}$
$\overline{3}$	$\mathcal{N}$	$\mathcal{N}$	

Table 1: The $\mathcal{N}$-positions established by Theorem 4.

Remark. Theorems 7 and 11-14 (corresponding to Table 2) will fill in portions of the first empty box on the diagonal of Table 1. Theorems 5, 6 and 10 (corresponding to Table 3) will fill in portions of the second empty box on the diagonal of Table 1. Lastly, Theorems 8 and 9 (corresponding to Table 4) will fill in portions of the last empty box on the diagonal of Table 1.

Theorem 5. *Suppose that* $C = \langle[x_1, x_2, \ldots, x_n]\rangle$, *where* $n \geq 3$ *and* $x_1, x_n \in \overline{2}$. *If* $IV(C) = 00$ *or* 11, *then* C *is an* $\mathcal{N}$*-position.*

Proof. There are two cases to consider.

`Case 1`. Let C be the position where x_1, $x_n \in \overline{2}$, and $IV(C) = 00$. Then, P1 uses strategy (a) which yields $EV(C^a) = 10 \oplus 10 = 00$. Furthermore, $IV(C^a) \oplus EV(C^a) = 00$. Since P1 can reduce the position to one with Grundy-value 0, P2 loses.

`Case 2`. Let C be the position where x_1, $x_n \in \overline{2}$, and $IV(C) = 11$. Then P1 can use either strategy (b) or (c) (due to symmetry). Suppose P1 uses (b), then $EV(C^b) = 01 \oplus 10 = 11$, and $EV(C^b) \oplus IV(C^b) = 11 \oplus 11 = 00$. So, (b) is a winning move for P1. Similarly, (c) is also a winning move for P1. □

Notation. The following notation is used in the proofs of Theorems 6, 9, 13 and 14.

- $\alpha(C) = \mathcal{G}\langle[\, x_1,\, x_2\,]\rangle \oplus x_n (\text{mod } 3)$.
- $IV_\alpha(C) = \bigoplus_{i=3}^{n-1} [x_i (\text{mod } 3)]$.

Theorem 6. *Suppose that* $C = \langle[x_1, x_2, \ldots, x_n]\rangle$, *where* $n \geq 3$, $x_1, x_n \in \overline{2}$ *and* $x_1 = x_2 \pmod 3$. *If* $IV(C) = 01$ *or* 10, *then* C *is an* $\mathcal{N}$*-position.*

Proof. There are two cases to consider.

Case 1. Let C be the position where x_1, x_2, $x_n \in \overline{2}$ and $IV(C) = 01$. Since $IV(C) = 01$ and $x_2 \in \overline{2}$, then $IV_\alpha(C) = 11$ (since $IV(C) = x_2 \pmod 3 \oplus IV_\alpha(C)$). Here P1 would use strategy (e) which would yield $\alpha(C^e) = \mathcal{G}\langle[2, 2]\rangle \oplus 2$ and $IV_\alpha(C^e) = 11$. Using Theorem 3, we have $\alpha(C^e) = 1 \oplus 2 = 01 \oplus 10 = 11$. Since $\alpha(C^e) \oplus IV_\alpha(C^e) = 11 \oplus 11 = 00$, P1 can reduce C to a position with Grundy-value 0.

Case 2. Let C be the position where x_1, x_2, $x_n \in \overline{2}$ and $IV(C) = 10$. Since $IV(C) = 10$ and $x_2 \in \overline{2}$, then $IV_\alpha(C) = 00$. Here P1 would use strategy (g) which yields $\alpha(C^g) = \mathcal{G}\langle[2,2]\rangle \oplus 1$ and $IV_\alpha(C^g) = 00$. Using Theorem 3, we have $\alpha(C^g) = 1 \oplus 1 = 01 \oplus 01 = 00$. Since $\alpha(C^g) \oplus IV_\alpha(C^g) = 00 \oplus 00 = 00$, P1 can reduce C to a position with Grundy-value 0. □

Theorem 7. *Suppose that $C = \langle[x_1, x_2, \ldots, x_n]\rangle$, where $n \geq 3$ and $x_1, x_n \in \overline{1}$. If $IV(C) = 00$ or 01, then C is an $\mathcal{N}$-position.*

Proof. There are two cases to consider.

Case 1. Let C be the position where x_1, $x_n \in \overline{1}$ and $IV(C) = 00$. Similar to the proof of Theorem 5 (Case 1), P1 can win by using strategy (a), as $EV(C^a) = 01 \oplus 01 = 00$ and $IV(C^a) = 00$. $EV(C^a) \oplus IV(C^a) = 00$, and thereby C is an $\mathcal{N}$-position.

Case 2. Let C be the position where x_1, $x_n \in \overline{1}$ and $IV(C) = 01$. Similar to Theorem 5 (Case 2), P1 can use stategy (b) or (c). If (b), then $EV(C^b) = 00 \oplus 01 = 01$ and $IV(C^b) = 01$. $EV(C^b) \oplus IV(C^b) = 01 \oplus 01 = 00$. Using (c) would lead to the same result; therefore C is an $\mathcal{N}$-position. □

Theorem 8. *Suppose that $C = \langle[x_1, x_2, \ldots, x_n]\rangle$, where $n \geq 3$ and $x_1, x_n \in \overline{3}$. If $IV(C) = 00$ or 10, then C is an $\mathcal{N}$-position.*

Proof. There are two cases to consider.

Case 1. Let C be the position where x_1, $x_n \in \overline{3}$ and $IV(C) = 00$. Similar to the proof of Theorem 5 (Case 1), P1 can win by using strategy (a), as $EV(C^a) = 00 \oplus 00 = 00$ and $IV(C^a) = 00$. $EV(C^a) \oplus IV(C^a) = 00$, and thereby C is an $\mathcal{N}$-position.

Case 2. Let C be the position where x_1, $x_n \in \overline{3}$ and $IV(C) = 10$. Similar to Theorem 5 (Case 2), P1 can use strategy (b) or (c). If (b), then $EV(C^b) = 10 \oplus 00 = 10$ and $IV(C^b) = 10$. $EV(C^b) \oplus IV(C^b) = 10 \oplus 10 = 00$. Using (c) would lead to the same result, therefore C is an $\mathcal{N}$-position. □

Notation. The following notation is used in the proofs of Theorems 9-12.

- $\beta(C) = \mathcal{G}(\, K_{1,\,(x_1+1)}\,) \oplus x_n \pmod 3$.

- $IV_\beta(C) = \bigoplus_{i=3}^{n-1}[x_i(\text{mod } 3)]$.

Theorem 9. *Suppose that* $C = \langle[x_1, x_2, \ldots, x_n]\rangle$*, where* $n \geq 3$*,* $x_1, x_n \in \overline{3}$ *and* $x_2 \in \overline{0}$*. If* $IV(C) = 01$ *or* 11*, then* C *is an* $\mathcal{N}$*-position.*

Proof. There are four cases to consider.

Case 1. Let C be the position where x_1, $x_n \in \overline{3}$, $x_2 \in \overline{3}$, and $IV(C) = 01$. P1 can use (e) to reduce the position to get $\alpha(C^e) = \mathcal{G}\langle[3,3]\rangle \oplus 0 = 1 \oplus 0 = 01 \oplus 00 = 01$, and $IV_\alpha(C^e) = 01$. Thus, $\alpha(C^e) \oplus IV_\alpha(C^e) = 01 \oplus 01 = 00$. So, C is an $\mathcal{N}$-position.

Case 2. Let C be the position where x_1, $x_n \in \overline{3}$, $x_2 = 0$, and $IV(C) = 01$. Here, P1 can use (i) to reduce C to get $\beta(C^i) = \mathcal{G}(K_{1,3+1}) \oplus 0 = 1 \oplus 0 = 01$, and $IV_\beta(C^i) = 01$, since $IV_\beta(C^i) \oplus x_2(\text{mod } 3) = IV(C)$. Thus, $\beta(C^i) \oplus IV_\beta(C^i) = 01 \oplus 01 = 00$. So, (i) is a winning move for P1.

Case 3. Let C be the position where x_1, $x_n \in \overline{3}$, $x_2 \in \overline{3}$, and $IV(C) = 11$. P1 can use (g) to get $\alpha(C^g) = \mathcal{G}\langle[3,3]\rangle \oplus 2 = 1 \oplus 2 = 01 \oplus 10 = 11$, and $IV_\alpha(C^g) = 11$. So, $\alpha(C^g) \oplus IV_\alpha(C^g) = 11 \oplus 11 = 00$. Thus, (g) is a winning move for P1.

Case 4. Let C be the position where x_1, $x_n \in \overline{3}$, $x_2 = 0$, and $IV(C) = 11$. Here, P1 can use (k) to get $\beta(C^k) = \mathcal{G}(K_{1,3+1}) \oplus 2 = 1 \oplus 2 = 01 \oplus 10 = 11$, and $IV_\beta(C^k) = 11$. Thus, $\beta(C^k) \oplus IV_\beta(C^k) = 11 \oplus 11 = 00$. So, C is an $\mathcal{N}$-position. □

Theorem 10. *Suppose that* $C = \langle[x_1, x_2, \ldots, x_n]\rangle$*, where* $n \geq 3$*,* $x_1, x_n \in \overline{2}$*,* $x_2 = 0$*. If* $IV(C) = 01$ *or* 10*, then* C *is an* $\mathcal{N}$*-position.*

Proof. There are two cases to consider.

Case 1. Let C be the position where x_1, $x_n \in \overline{2}$, $x_2 = 0$, and $IV(C) = 01$. P1 can use strategy (k) to get $\beta(C^k) = \mathcal{G}(K_{1,2+1}) \oplus 1 = 0 \oplus 1 = 00 \oplus 01 = 01$, and $IV_\beta(C^k) = 01$. So, $\beta(C^k) \oplus IV_\beta(C^k) = 01 \oplus 01 = 00$. So, (k) is a winning move for P1.

Case 2. Let C be the position where x_1, $x_n \in \overline{2}$, $x_2 = 0$, and $IV(C) = 10$. If P1 uses (i), then $\beta(C^i) = \mathcal{G}(K_{1,2+1}) \oplus 2 = 0 \oplus 2 = 10$, and $IV_\beta(C^i) = 10$. So, $\beta(C^i) \oplus IV_\beta(C^i) = 10 \oplus 10 = 00$. Hence, (i) is a winning move for P1. □

Theorem 11. *Suppose that* $C = \langle[x_1, x_2, \ldots, x_n]\rangle$*, where* $n \geq 3$*,* $x_1, x_n \in \overline{1}$*,* $x_2 = 1$*. If* $IV(C) = 10$ *or* 11*, then* C *is an* $\mathcal{N}$*-position.*

Proof. There are two cases to consider.

Case 1. Let C be the position where x_1, $x_n \in \overline{1}$, $x_2 = 1$, and $IV(C) = 10$. P1 can use strategy (f) to get $\beta(C^f) = \mathcal{G}(K_{1,1+1}) \oplus 1 = 2 \oplus 1 = 10 \oplus 01 = 11$, and $IV_\beta(C^f) = 11$ (this is because $IV(C) = 10 = IV_\beta(C^f) \oplus 01 = IV_\beta(C^f) \oplus$

$x_2 (\text{mod } 3) \implies IV_\beta(C^f) = 11$). Thus, $\beta(C^f) \oplus IV_\beta(C^f) = 11 \oplus 11 = 00$. So, C is an $\mathcal{N}$-position.

`Case 2`. Let C be the position where $x_1, x_n \in \overline{1}$, $x_2 = 1$, and $IV(C) = 11$. P1 can use (h) to get $\beta(C^h) = \mathcal{G}(K_{1,1+1}) \oplus 0 = 2 \oplus 0 = 10 \oplus 00 = 10$, and $IV_\beta(C^h) = 10$ (since $IV(C) = 11 = IV_\beta(C^h) \oplus x_2 (\text{mod } 3) \implies IV_\beta(C^h) = 10$). Thus, $\beta(C^h) \oplus IV_\beta(C^h) = 10 \oplus 10 = 00$. So, C is an $\mathcal{N}$-position. □

Theorem 12. *Suppose that* $C = \langle [x_1, x_2, \ldots, x_n] \rangle$, *where* $n \geq 3$, $x_1, x_n \in \overline{1}$, $x_2 = 0$. *If* $IV(C) = 10$ *or* 11, *then* C *is an* $\mathcal{N}$*-position.*

Proof. There are two cases to consider.

`Case 1`. Let C be the position where $x_1, x_n \in \overline{1}$, $x_2 = 0$, and $IV(C) = 10$. P1 can use (k) to get $\beta(C^k) = \mathcal{G}(K_{1,1+1}) \oplus 0 = 2 \oplus 0 = 10 \oplus 00 = 10$, and $IV_\beta(C^k) = 10$. So, $\beta(C^k) \oplus IV_\beta(C^k) = 10 \oplus 10 = 00$. Thus, C is an $\mathcal{N}$-position.

`Case 2`. Let C be the position where $x_1, x_n \in \overline{1}$, $x_2 = 0$, and $IV(C) = 11$. Strategy (i) gives $\beta(C^i) = \mathcal{G}(K_{1,1+1}) \oplus 1 = 2 \oplus 1 = 10 \oplus 01 = 11$, and $IV_\beta(C^i) = 11$. Also, $\beta(C^i) \oplus IV_\beta(C^i) = 11 \oplus 11 = 00$. So, (i) is a winning move for P1. □

Theorem 13. *Suppose that* $C = \langle [x_1, x_2, \ldots, x_n] \rangle$, *where* $n \geq 3$, $x_n \in \overline{1}$, $x_1 = 1$, $x_2 \in \overline{4}$. *If* $IV(C) = 10$ *or* 11, *then* C *is an* $\mathcal{N}$*-position.*

Proof. There are two cases to consider.

`Case 1`. Let C be the position where $x_1 = 1$, $x_2 \in \overline{4}$, $x_n \in \overline{1}$, and $IV(C) = 10$. From Lemma 1, $\mathcal{G}\langle [1, 1] \rangle = 3$, so P1 can use (g) to get $\alpha(C^g) = \mathcal{G}\langle [1, 4] \rangle \oplus 0 = 3 \oplus 0 = 11 \oplus 00 = 11$, and since $10 = IV(C) = IV_\alpha(C^g) \oplus 01 \implies IV_\alpha(C^g) = 11$. So, $\alpha(C^g) \oplus IV_\alpha(C^g) = 11 \oplus 11 = 00$, and thereby (g) is a winning move for P1.

`Case 2`. Let C be the position where $x_1 = 1$, $x_2 \in \overline{4}$, $x_n \in \overline{1}$, and $IV(C) = 11$. P1 can use (e) to get $\alpha(C^e) = \mathcal{G}\langle [1, 4] \rangle \oplus 1 = 3 \oplus 1 = 11 \oplus 01 = 10$, and since $11 = IV(C) = IV_\alpha(C^e) \oplus 01 \implies IV_\alpha(C^e) = 10$. So, $\alpha(C^e) \oplus IV_\alpha(C^e) = 10 \oplus 10 = 00$. Thus, (e) is a winning move for P1. □

Theorem 14. *Suppose that* $C = \langle [x_1, x_2, \ldots, x_n] \rangle$, *where* $n \geq 3$, $x_1, x_n \in \overline{1}$, $x_1 \neq 1$, *and* $x_2 \in \overline{4}$. *If* $IV(C) = 10$ *or* 11, *then* C *is an* $\mathcal{N}$*-position.*

Proof. There are two cases to consider.

`Case 1`. Let C be the position where $x_1, x_2 \in \overline{4}$, $x_n \in \overline{1}$, and $IV(C) = 10$. From Theorem 3, $\mathcal{G}\langle [4, 4] \rangle = 2$, so P1 can use (e) to get $\alpha(C^e) = \mathcal{G}\langle [4, 4] \rangle \oplus 1 = 2 \oplus 1 = 10 \oplus 01 = 11$, and since $10 = IV(C) = IV_\alpha(C^e) \oplus 01 \implies IV_\alpha(C^e) = 11$. So, $\alpha(C^e) \oplus IV_\alpha(C^e) = 11 \oplus 11 = 00$. Thus, (e) is a winning move for P1.

`Case 2`. Let C be the position where $x_1,\, x_2 \in \overline{4}$, $x_n \in \overline{1}$, and $IV(C) = 11$. P1 can use (g) to get $\alpha(C^g) = \mathcal{G}\langle[4,4]\rangle \oplus 0 = 2 \oplus 0 = 10 \oplus 00 = 10$, and since $11 = IV(C) = IV_\alpha(C^g) \oplus 01 \implies IV_\alpha(C^g) = 10$. So, $\alpha(C^g) \oplus IV_\alpha(C^g) = 10 \oplus 10 = 00$, and thereby (g) is a winning move for P1. ☐

Tables 2, 3 and 4 summarize the results of Theorems 5 – 14.

$x_1,\, x_n \in \overline{1}$

		x_2					
		0	1	2	$\overline{3}$	$\overline{4}$	$\overline{5}$
$IV(C)$	00	$\mathcal{N}$ 7	$\mathcal{N}$ 7	$\mathcal{N}$ 7	$\mathcal{N}$ 7	$\mathcal{N}$ 7	$\mathcal{N}$ 7
	01	$\mathcal{N}$ 7	$\mathcal{N}$ 7	$\mathcal{N}$ 7	$\mathcal{N}$ 7	$\mathcal{N}$ 7	$\mathcal{N}$ 7
	10	$\mathcal{N}$ 12	$\mathcal{N}$ 11			$\mathcal{N}$ 13/14	
	11	$\mathcal{N}$ 12	$\mathcal{N}$ 11			$\mathcal{N}$ 13/14	

Table 2: $\mathcal{N}$-positions established by Theorems 7 and 11-14.

$x_1,\, x_n \in \overline{2}$

		x_2					
		0	1	2	$\overline{3}$	$\overline{4}$	$\overline{5}$
$IV(C)$	00	$\mathcal{N}$ 5	$\mathcal{N}$ 5	$\mathcal{N}$ 5	$\mathcal{N}$ 5	$\mathcal{N}$ 5	$\mathcal{N}$ 5
	01	$\mathcal{N}$ 10		$\mathcal{N}$ 6			$\mathcal{N}$ 6
	10	$\mathcal{N}$ 10		$\mathcal{N}$ 6			$\mathcal{N}$ 6
	11	$\mathcal{N}$ 5	$\mathcal{N}$ 5	$\mathcal{N}$ 5	$\mathcal{N}$ 5	$\mathcal{N}$ 5	$\mathcal{N}$ 5

Table 3: $\mathcal{N}$-positions established by Theorems 5, 6 and 10.

$x_1,\, x_n \in \overline{3}$

x_2

$IV(C)$	0	1	2	$\overline{3}$	$\overline{4}$	$\overline{5}$
00	$\mathcal{N}$ 8	$\mathcal{N}$ 8	$\mathcal{N}$ 8	$\mathcal{N}$ 8	$\mathcal{N}$ 8	$\mathcal{N}$ 8
01	$\mathcal{N}$ 9			$\mathcal{N}$ 9		
10	$\mathcal{N}$ 8	$\mathcal{N}$ 8	$\mathcal{N}$ 8	$\mathcal{N}$ 8	$\mathcal{N}$ 8	$\mathcal{N}$ 8
11	$\mathcal{N}$ 9			$\mathcal{N}$ 9		

Table 4: $\mathcal{N}$-positions established by Theorems 8 and 9.

To further analyze $\mathcal{N}$- and $\mathcal{P}$-positions, we introduce a <u>variation</u> of the modified universal strategy. Remove the entire spine of C except for the segments connecting x_1 and x_2, and x_{n-1} and x_n, along with zero or one leg from x_2. The added restriction is needed as it is possible that for fixed x_1, x_2, x_n, and for some x_{n-1}, the position might be already established as an $\mathcal{N}$-position by the preceding theorems due to symmetry.

Using this variation of the modified universal strategy, we can make further refinements to Tables 2, 3 and 4. In particular, Tables 5, 6 and 7 give additional N-positions.

$x_1,\, x_n \in \overline{1}$

x_2

$IV(C)$	2	$\overline{3}$	$\overline{4}$	$\overline{5}$
10		$\mathcal{N}$ $x_{n-1} \in \overline{2}$	$\mathcal{N}$ 13/14	
10	$\mathcal{N}$ $x_{n-1} \in \overline{3}$			$\mathcal{N}$ $x_{n-1} \in \overline{3}$
11		$\mathcal{N}$ $x_{n-1} \in \overline{2}$	$\mathcal{N}$ 13/14	
11	$\mathcal{N}$ $x_{n-1} \in \overline{3}$			$\mathcal{N}$ $x_{n-1} \in \overline{3}$

Table 5: Refinement of Table 2.

$x_1,\, x_n \in \overline{2}$

x_2

<table>
<tr><th></th><th></th><th>1</th><th>2</th><th>$\overline{3}$</th><th>$\overline{4}$</th></tr>
<tr><td rowspan="6">$IV(C)$</td><td rowspan="3">01</td><td></td><td rowspan="3">$\mathcal{N}$
6</td><td></td><td></td></tr>
<tr><td></td><td></td><td>$\mathcal{N}$
$x_{n-1} \in \overline{3}$</td></tr>
<tr><td></td><td>$\mathcal{N}$
$x_{n-1} \in \overline{4}$</td><td></td></tr>
<tr><td rowspan="3">10</td><td></td><td rowspan="3">$\mathcal{N}$
6</td><td></td><td>$\mathcal{N}$
$x_{n-1} = 1$</td></tr>
<tr><td></td><td></td><td></td></tr>
<tr><td>$\mathcal{N}$
$x_{n-1} \in \overline{4}$</td><td></td><td>$\mathcal{N}$
$x_{n-1} \in \overline{4}$</td></tr>
</table>

Table 6: Refinement of Table 3.

$x_1,\, x_n \in \overline{3}$

x_2

<table>
<tr><th></th><th></th><th>1</th><th>2</th><th></th><th>$\overline{4}$</th><th>$\overline{5}$</th></tr>
<tr><td rowspan="2">$IV(C)$</td><td rowspan="2">11</td><td></td><td>$\mathcal{N}$
$x_{n-1} \in \overline{1}$</td><td>$\cdots$</td><td></td><td>$\mathcal{N}$
$x_{n-1} \in \overline{1}$</td></tr>
<tr><td>$\mathcal{N}$
$x_{n-1} \in \overline{2}$</td><td>$\mathcal{N}$
$x_{n-1} \in \overline{2}$</td><td></td><td>$\mathcal{N}$
$x_{n-1} \in \overline{2}$</td><td>$\mathcal{N}$
$x_{n-1} \in \overline{2}$</td></tr>
</table>

Table 7: Refinement of Table 4.

Remark. The remaining unresolved positions are summarized in Table 8. The reader should note that the entries in Table 8, as well as the proofs of Theorems 15 and 16, abuse the notation used earlier in this paper. More specifically, $\langle[\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_n]\rangle$ represents $\langle[x_1, x_2, \ldots, x_n]\rangle$, where $x_i \in \overline{y}_i$, for $1 \leq i \leq n$.

$x_1, x_n \in \overline{1}$	$IV(C)$	$x_1, x_n \in \overline{2}$	$IV(C)$	$x_1, x_n \in \overline{3}$	$IV(C)$
$\langle[\overline{1}, \overline{2}, \ldots, \overline{2}, \overline{1}]\rangle$	10	$\langle[\overline{2}, \overline{1}, \ldots, \overline{1}, \overline{2}]\rangle$	01	$\langle[\overline{3}, \overline{1}, \ldots, \overline{1}, \overline{3}]\rangle$	01
$\langle[\overline{1}, \overline{3}, \ldots, \overline{3}, \overline{1}]\rangle$	10	$\langle[\overline{2}, 1, \ldots, \overline{3}, \overline{2}]\rangle$	01	$\langle[\overline{3}, \overline{1}, \ldots, \overline{2}, \overline{3}]\rangle$	01
$\langle[\overline{1}, \overline{2}, \ldots, \overline{2}, \overline{1}]\rangle$	11	$\langle[\overline{2}, \overline{3}, \ldots, \overline{3}, \overline{2}]\rangle$	01	$\langle[\overline{3}, \overline{2}, \ldots, \overline{2}, \overline{3}]\rangle$	01
$\langle[\overline{1}, \overline{3}, \ldots, \overline{3}, \overline{1}]\rangle$	11	$\langle[\overline{2}, 1, \ldots, 1, \overline{2}]\rangle$	10	$\langle[\overline{3}, \overline{1}, \ldots, \overline{1}, \overline{3}]\rangle$	11
		$\langle[\overline{2}, \overline{1}, \ldots, \overline{3}, \overline{2}]\rangle$	10		
		$\langle[\overline{2}, \overline{3}, \ldots, \overline{3}, \overline{2}]\rangle$	10		

Table 8: The remaining unresolved positions.

4. Caterpillars With Spine Length Two and Three

In this section, we completely characterize the $\mathcal{P}$-positions for caterpillars with spine length two and three.

Theorem 15. *The caterpillar* $C = \langle[x_1, x_2, x_3]\rangle$ *is a* $\mathcal{P}$*-position* $\Leftrightarrow$ *one of the following hold:*

1. $x_1 \in \overline{1}$, $x_2 \in \overline{2}$, *and* $x_3 \in \overline{1}$.
2. $x_1 \in \overline{3}$, $x_2 \in \overline{1}$, *and* $x_3 \in \overline{3}$.
3. $x_1 \in \overline{2}$, $x_2 \in \overline{1}$, *and* $x_3 \in \overline{2}$.

Proof. Let $C = \langle[x_1, x_2, x_3]\rangle$. We use symmetry to reduce the number of positions that need to be considered. From Theorems 4-14, all possible caterpillars of spine length two are $\mathcal{N}$-positions with the exception of three positions, namely $\langle[\overline{1}, \overline{2}, \overline{1}]\rangle$, $\langle[\overline{2}, \overline{1}, \overline{2}]\rangle$ and $\langle[\overline{3}, \overline{1}, \overline{3}]\rangle$. We will show that these three positions are $\mathcal{P}$-positions. In the cases below, the first arrow indicates a move by P1, and the second arrow indicates a move by P2. The value $-n$ (above an arrow) indicates that n edges of a path are removed.

`Case 1`. Let $C = \langle[\overline{1},\overline{2},\overline{1}]\rangle$.

- $\langle[\overline{4},\overline{2},\overline{1}]\rangle \xrightarrow{-1} \langle[\overline{3},\overline{2},\overline{1}]\rangle \xrightarrow{-4} \langle[\overline{2}],[\overline{2}],[\overline{0}]\rangle$.
- $\langle[1,\overline{2},\overline{1}]\rangle \xrightarrow{-1} \langle[0,\overline{2},\overline{1}]\rangle \xrightarrow{-2} \langle[0,\overline{2}],[\overline{0}]\rangle$.
- $C \xrightarrow{-1} \langle[\overline{1},\overline{1},\overline{1}]\rangle \xrightarrow{-3} \langle[\overline{0}],[\overline{1}],[\overline{1}]\rangle$.
- $C \xrightarrow{-1} \langle[\overline{1}],[\overline{2},\overline{1}]\rangle \xrightarrow{-3} \langle[\overline{1}],[\overline{1}],[\overline{0}]\rangle$.
- $\langle[\overline{4},\overline{2},\overline{1}]\rangle \xrightarrow{-2} \langle[\overline{2},\overline{2},\overline{1}]\rangle \xrightarrow{-3} \langle[\overline{2}],[\overline{2}],[\overline{0}]\rangle$.
- $C \xrightarrow{-2} \langle[\overline{1},\overline{0},\overline{1}]\rangle \xrightarrow{-2} \langle[\overline{1}],[\overline{0}],[\overline{1}]\rangle$.
- $C \xrightarrow{-2} \langle[\overline{0}],[\overline{2},\overline{1}]\rangle \xrightarrow{-2} \langle[\overline{0}],[\overline{1}],[\overline{1}]\rangle$.
- $C \xrightarrow{-2} \langle[\overline{1}],[\overline{1},\overline{1}]\rangle \xrightarrow{-2} \langle[\overline{1}],[\overline{1}],[\overline{0}]\rangle$.
- $C \xrightarrow{-2} \langle[\overline{1}],[\overline{2}],[\overline{1}]\rangle \xrightarrow{-2} \langle[\overline{1}],[\overline{0}],[\overline{1}]\rangle$.
- $C \xrightarrow{-3} \langle[\overline{0}],[\overline{1},\overline{1}]\rangle \xrightarrow{-1} \langle[\overline{0}],[\overline{1}],[\overline{1}]\rangle$.
- $C \xrightarrow{-3} \langle[\overline{0}],[\overline{2}],[\overline{1}]\rangle \xrightarrow{-1} \langle[\overline{0}],[\overline{1}],[\overline{1}]\rangle$.
- $C \xrightarrow{-4} \langle[\overline{0}],[\overline{2}],[\overline{0}]\rangle \xrightarrow{-2} \langle[\overline{0}],[\overline{0}],[\overline{0}]\rangle$.

`Case 2`. Let $C = \langle[\overline{2},\overline{1},\overline{2}]\rangle$.

- $C \xrightarrow{-1} \langle[\overline{1},\overline{1},\overline{2}]\rangle \xrightarrow{-4} \langle[\overline{0}],[\overline{1}],[\overline{1}]\rangle$.
- $C \xrightarrow{-1} \langle[\overline{2},\overline{0},\overline{2}]\rangle \xrightarrow{-2} \langle[\overline{2}],[\overline{0}],[\overline{2}]\rangle$.
- $C \xrightarrow{-1} \langle[\overline{2}],[\overline{1},\overline{2}]\rangle \xrightarrow{-2} \langle[\overline{2}],[\overline{0}],[\overline{2}]\rangle$.
- $\langle[\overline{5},\overline{1},\overline{2}]\rangle \xrightarrow{-2} \langle[\overline{3},\overline{1},\overline{2}]\rangle \xrightarrow{-3} \langle[\overline{3}],[\overline{1}],[\overline{1}]\rangle$.
- $\langle[2,\overline{1},\overline{2}]\rangle \xrightarrow{-2} \langle[0,\overline{1},\overline{2}]\rangle \xrightarrow{-1} \langle[0,\overline{1}],[\overline{2}]\rangle$.
- $\langle[\overline{2},\overline{4},\overline{2}]\rangle \xrightarrow{-2} \langle[\overline{2},\overline{2},\overline{2}]\rangle \xrightarrow{-2} \langle[\overline{2},\overline{2}],[\overline{1}]\rangle$ (Lemma 2 and Theorem 3).
- $C \xrightarrow{-2} \langle[\overline{1}],[\overline{1},\overline{2}]\rangle \xrightarrow{-3} \langle[\overline{1}],[\overline{0}],[\overline{1}]\rangle$.
- $C \xrightarrow{-2} \langle[\overline{2}],[\overline{0},\overline{2}]\rangle \xrightarrow{-1} \langle[\overline{2}],[\overline{0}],[\overline{2}]\rangle$.
- $C \xrightarrow{-2} \langle[\overline{2}],[\overline{1}],[\overline{2}]\rangle \xrightarrow{-1} \langle[\overline{2}],[\overline{0}],[\overline{2}]\rangle$.

- $C \xrightarrow{-3} \langle[\overline{1}], [\overline{0}, \overline{2}]\rangle \xrightarrow{-2} \langle[\overline{1}], [\overline{0}], [\overline{1}]\rangle$.
- $C \xrightarrow{-3} \langle[\overline{1}], [\overline{1}], [\overline{2}]\rangle \xrightarrow{-2} \langle[\overline{1}], [\overline{1}], [\overline{0}]\rangle$.
- $C \xrightarrow{-4} \langle[\overline{1}], [\overline{1}], [\overline{1}]\rangle \xrightarrow{-1} \langle[\overline{1}], [\overline{1}], [\overline{0}]\rangle$.

`Case 3`. Let $C = \langle[\overline{3}, \overline{1}, \overline{3}]\rangle$.

- $C \xrightarrow{-1} \langle[\overline{2}, \overline{1}, \overline{3}]\rangle \xrightarrow{-3} \langle[\overline{1}], [\overline{1}], [\overline{3}]\rangle$.
- $C \xrightarrow{-1} \langle[\overline{3}, \overline{0}, \overline{3}]\rangle \xrightarrow{-2} \langle[\overline{3}], [\overline{0}], [\overline{3}]\rangle$.
- $C \xrightarrow{-1} \langle[\overline{3}], [\overline{1}, \overline{3}]\rangle \xrightarrow{-2} \langle[\overline{3}], [\overline{0}], [\overline{3}]\rangle$.
- $C \xrightarrow{-2} \langle[\overline{1}, \overline{1}, \overline{3}]\rangle \xrightarrow{-2} \langle[\overline{1}], [\overline{1}], [\overline{3}]\rangle$.
- $\langle[\overline{3}, \overline{4}, \overline{3}]\rangle \xrightarrow{-2} \langle[\overline{3}, \overline{2}, \overline{3}]\rangle \xrightarrow{-3} \langle[\overline{2}], [\overline{2}], [\overline{3}]\rangle$.
- $C \xrightarrow{-2} \langle[\overline{2}], [\overline{1}, \overline{3}]\rangle \xrightarrow{-3} \langle[\overline{2}], [\overline{0}], [\overline{2}]\rangle$.
- $C \xrightarrow{-2} \langle[\overline{3}], [\overline{1}], [\overline{3}]\rangle \xrightarrow{-1} \langle[\overline{3}], [\overline{0}], [\overline{3}]\rangle$.
- $C \xrightarrow{-2} \langle[\overline{3}], [\overline{0}, \overline{3}]\rangle \xrightarrow{-1} \langle[\overline{3}], [\overline{0}], [\overline{3}]\rangle$.
- $C \xrightarrow{-3} \langle[\overline{2}], [\overline{0}, \overline{3}]\rangle \xrightarrow{-2} \langle[\overline{2}], [\overline{0}], [\overline{2}]\rangle$.
- $C \xrightarrow{-3} \langle[\overline{2}], [\overline{1}], [\overline{3}]\rangle \xrightarrow{-1} \langle[\overline{1}], [\overline{1}], [\overline{3}]\rangle$.
- $C \xrightarrow{-4} \langle[\overline{2}], [\overline{1}], [\overline{2}]\rangle \xrightarrow{-1} \langle[\overline{2}], [\overline{0}], [\overline{2}]\rangle$.

In all of these cases, P2 wins since they are able to reduce to a position with Grundy-value 0. □

Theorem 16. *The caterpillar $C = \langle[x_1, x_2, x_3, x_4]\rangle$ is a $\mathcal{P}$-position $\Leftrightarrow$ $x_1 \in \overline{2}$, $x_2 = 1$, $x_3 \in \overline{3}$, and $x_4 \in \overline{2}$.*

Proof. We use symmetry to reduce the number of positions that need to be considered. From Theorems 4-14, the only unresolved caterpillars of spine length 3 are $\langle[\overline{1}, \overline{2}, \overline{3}, \overline{1}]\rangle$, $\langle[\overline{2}, \overline{1}, \overline{3}, \overline{2}]\rangle$ and $\langle[\overline{3}, \overline{1}, \overline{2}, \overline{3}]\rangle$. Of these positions, we want to show that the only $\mathcal{P}$-positions are of the form $\langle[\overline{2}, 1, \overline{3}, \overline{2}]\rangle$, a subset of $\langle[\overline{2}, \overline{1}, \overline{3}, \overline{2}]\rangle$.

For the $\mathcal{N}$-positions, there are three cases to consider.

`Case 1`. Let $C = \langle[\overline{1}, \overline{2}, \overline{3}, \overline{1}]\rangle$. P1 can reduce the position to $\langle[\overline{1}, \overline{2}], [\overline{2}, \overline{1}]\rangle$, which has Grundy-value 0 by Lemmas 1, 2 and Theorem 3.

`Case 2`. Let $C = \langle[\overline{2},\overline{4},\overline{3},\overline{2}]\rangle$. P1 can reduce the position to $\langle[\overline{2},\overline{3}],[\overline{3},\overline{2}]\rangle$, which has Grundy-value 0 by Lemmas 2, 3 and Theorem 3.

`Case 3`. Let $C = \langle[\overline{3},\overline{1},\overline{2},\overline{3}]\rangle$. P1 can reduce the position to $\langle[\overline{3},\overline{1}],[\overline{1},\overline{3}]\rangle$, which has Grundy-value 0 by Lemmas 1, 3, and Theorem 3.

For the $\mathcal{P}$-positions, we need to show that for any move played by P1, P2 can counter by reducing to a position with Grundy-value 0. We denote moves by arrows, with the first arrow used for P1's move and the second arrow for P2's move. The value $-n$ (above an arrow) indicates that n edges of a path are removed.

So, let $C = \langle[\overline{2},1,\overline{3},\overline{2}]\rangle$.

- $C \xrightarrow{-1} \langle[\overline{1},1,\overline{3},\overline{2}]\rangle \xrightarrow{-5} \langle[\overline{0}],[1],[\overline{3}],[\overline{1}]\rangle$.
- $C \xrightarrow{-1} \langle[\overline{2}],[1,\overline{3},\overline{2}]\rangle \xrightarrow{-3} \langle[\overline{2}],[0],[\overline{3}],[\overline{2}]\rangle$.
- $C \xrightarrow{-1} \langle[\overline{2},0,\overline{3},\overline{2}]\rangle \xrightarrow{-3} \langle[\overline{2}],[0],[\overline{3}],[\overline{2}]\rangle$.
- $C \xrightarrow{-1} \langle[\overline{2},1],[\overline{3},\overline{2}]\rangle \xrightarrow{-2} \langle[\overline{2},1],[\overline{1},\overline{2}]\rangle$ (Lemmas 1, 2 and Theorem 3).
- $C \xrightarrow{-1} \langle[\overline{2},1,\overline{2},\overline{2}]\rangle \xrightarrow{-4} \langle[\overline{2}],[1],[\overline{2}],[\overline{1}]\rangle$.
- $C \xrightarrow{-1} \langle[\overline{2},1,\overline{3}],[\overline{2}]\rangle \xrightarrow{-4} \langle[\overline{1}],[1],[\overline{2}],[\overline{2}]\rangle$.
- $C \xrightarrow{-1} \langle[\overline{2},1,\overline{3},\overline{1}]\rangle \xrightarrow{-5} \langle[\overline{1}],[1],[\overline{3}],[\overline{0}]\rangle$.
- $C \xrightarrow{-2} \langle[\overline{0},1,\overline{3},\overline{2}]\rangle \xrightarrow{-4} \langle[\overline{0}],[1],[\overline{3}],[\overline{1}]\rangle$.
- $C \xrightarrow{-2} \langle[\overline{1}],[1,\overline{3},\overline{2}]\rangle \xrightarrow{-4} \langle[\overline{1}],[0],[\overline{3}],[\overline{1}]\rangle$.
- $C \xrightarrow{-2} \langle[\overline{2}],[0,\overline{3},\overline{2}]\rangle \xrightarrow{-2} \langle[\overline{2}],[0],[\overline{3}],[\overline{2}]\rangle$.
- $C \xrightarrow{-2} \langle[\overline{2},0],[\overline{3},\overline{2}]\rangle \xrightarrow{-2} \langle[\overline{2},0],[\overline{2}],[\overline{2}]\rangle$.
- $C \xrightarrow{-2} \langle[\overline{2}],[1],[\overline{3},\overline{2}]\rangle \xrightarrow{-3} \langle[\overline{2}],[1],[\overline{2}],[\overline{1}]\rangle$.
- $C \xrightarrow{-2} \langle[\overline{2},1],[\overline{2},\overline{2}]\rangle \xrightarrow{-1} \langle[\overline{2},1],[\overline{1},\overline{2}]\rangle$ (Lemmas 1, 2 and Theorem 3).
- $C \xrightarrow{-2} \langle[\overline{2},1,\overline{1},\overline{2}]\rangle \xrightarrow{-3} \langle[\overline{2}],[1],[\overline{1}],[\overline{2}]\rangle$.
- $C \xrightarrow{-2} \langle[\overline{2},1,\overline{2}],[\overline{2}]\rangle \xrightarrow{-2} \langle[\overline{2},1,\overline{2}],[\overline{0}]\rangle$ (Theorem 15).
- $C \xrightarrow{-2} \langle[\overline{2},1],[\overline{3}],[\overline{2}]\rangle \xrightarrow{-2} \langle[\overline{2}],[0],[\overline{3}],[\overline{2}]\rangle$.
- $C \xrightarrow{-2} \langle[\overline{2},1,\overline{3}],[\overline{1}]\rangle \xrightarrow{-3} \langle[\overline{2}],[1],[\overline{2}],[\overline{1}]\rangle$.

- $C \xrightarrow{-2} \langle[\overline{2},1,\overline{3},\overline{0}]\rangle \xrightarrow{-4} \langle[\overline{1}],[1],[\overline{3}],[\overline{0}]\rangle$.
- $C \xrightarrow{-3} \langle[\overline{1}],[0,\overline{3},\overline{2}]\rangle \xrightarrow{-3} \langle[\overline{1}],[0],[\overline{3}],[\overline{1}]\rangle$.
- $C \xrightarrow{-3} \langle[\overline{1}],[1],[\overline{3},\overline{2}]\rangle \xrightarrow{-2} \langle[\overline{1}],[1],[\overline{2}],[\overline{2}]\rangle$.
- $C \xrightarrow{-3} \langle[\overline{2},0],[\overline{2},\overline{2}]\rangle \xrightarrow{-1} \langle[\overline{2},0],[\overline{2}],[\overline{2}]\rangle$.
- $C \xrightarrow{-3} \langle[\overline{2},0],[\overline{3}],[\overline{2}]\rangle \xrightarrow{-1} \langle[\overline{2}],[0],[\overline{3}],[\overline{2}]\rangle$.
- $C \xrightarrow{-3} \langle[\overline{2}],[1],[\overline{2},\overline{2}]\rangle \xrightarrow{-2} \langle[\overline{2}],[1],[\overline{1}],[\overline{2}]\rangle$.
- $C \xrightarrow{-3} \langle[\overline{2},1,\overline{2}],[\overline{1}]\rangle \xrightarrow{-1} \langle[\overline{2},1,\overline{2}],[\overline{0}]\rangle$ (Theorem 15).
- $C \xrightarrow{-3} \langle[\overline{2},1],[\overline{3}],[\overline{1}]\rangle \xrightarrow{-3} \langle[\overline{1}],[0],[\overline{3}],[\overline{1}]\rangle$.
- $C \xrightarrow{-3} \langle[\overline{2}],[1],[\overline{3}],[\overline{2}]\rangle \xrightarrow{-1} \langle[\overline{2}],[0],[\overline{3}],[\overline{2}]\rangle$.
- $C \xrightarrow{-4} \langle[\overline{1}],[1],[\overline{3}],[\overline{2}]\rangle \xrightarrow{-2} \langle[\overline{1}],[1],[\overline{3}],[\overline{0}]\rangle$.
- $C \xrightarrow{-4} \langle[\overline{1}],[1],[\overline{2},\overline{2}]\rangle \xrightarrow{-1} \langle[\overline{1}],[1],[\overline{2}],[\overline{2}]\rangle$.
- $C \xrightarrow{-4} \langle[\overline{2},0],[\overline{3}],[\overline{1}]\rangle \xrightarrow{-1} \langle[\overline{2},0],[\overline{3}],[\overline{0}]\rangle$.
- $C \xrightarrow{-4} \langle[\overline{2}],[1],[\overline{3}],[\overline{1}]\rangle \xrightarrow{-2} \langle[\overline{0}],[1],[\overline{3}],[\overline{1}]\rangle$.
- $C \xrightarrow{-5} \langle[\overline{1}],[1],[\overline{3}],[\overline{1}]\rangle \xrightarrow{-1} \langle[\overline{1}],[1],[\overline{3}],[\overline{0}]\rangle$.

In all of these cases, P2 wins since they are able to reduce to a position with Grundy-value 0. □

5. Concluding Remarks

In summary, we have completely determined the $\mathcal{N}$- and $\mathcal{P}$-positions for caterpillars with spine length zero, one, two and three. Furthermore, for caterpillars with spine length four or greater, we analyzed $\frac{1}{9}(\frac{36+6}{48} + \frac{54+5}{72} + \frac{32+6}{48}) + \frac{6}{9} = 94.3\%$ of them. Here, all of them turned out to be $\mathcal{N}$-positions. To complete the analysis of the *Caterpillar Game* (for caterpillars with spine length four or greater), the positions in Table 8 need to be resolved.

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