#G2 INTEGERS 21 (2021)

AMALGAMATION NIM

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Abstract

We discuss a version of Nim in which players are allowed to use a move from the traditional form of Nim or to amalgamate two heaps. We provide winning strategies for games which start with all heaps of height three or less. We provide a list of P-positions for the three heap game with one heap of height at most seven, and observations about the period of the three heap game with one heap of height at most eleven.

1. Introduction and Definitions

The winning strategy for the game of Nim was determined in 1902 by Bouton [4]. Since that seminal paper, there have been more than 100 scholarly articles on Nim and variants of Nim, and on the use of Nim and Sprague-Grundy numbers to solve other combinatorial games. For a general introduction to combinatorial games, see [1], [2], or [5].

One variant, commonly known as Lasker's Nim [11], allows a player to either move as if playing regular Nim, or select a heap or height two or more, and split the heap into two non-empty heaps. In [9] and [10], Grundy numbers for variants of Lasker's Nim were useful, since each heap could be considered as a separate game. Grundy's game (see [2] or [5]) is a variant of Nim which begins with one heap. At each turn, a player splits a heap into two heaps of different sizes. Berlekamp, Conway, and Guy conjecture that the Sprague-Grundy numbers for this game are eventually periodic. The reader may check values at the on-line encyclopedia of integer sequences, A0022188 [14]. There are other take and break games. One example is Kayles (see [5] or [7]. A detailed study of take and break games and octal games is presented in [6].

For Amalgamation Nim, the variant discussed in this article, the individual heaps cannot be considered as separate games. In [8], a 3-heap Nim variant, LIM, is discussed and solved. In this variant, a player removes some number m of coins from two of the heaps and adds m coins to the third heap.

A combinatorial 2-person game is a game of complete information. Two players P1 and P2 alternate moves, with P1 making the first move. Both players are aware of the game position at any time, and there is no element of chance. The variant we consider is a normal game, meaning that the player who makes the last move wins.

In Amalgamation Nim, the two players begin with several heaps of coins and then alternate moves, with the player who makes the last legal move declared the winner. The moves are of two types:

(i) A player can remove any non-zero number of coins from any heap; or

(ii) A player can select any two non-empty heaps and replace them by a single heap containing the total number of coins in the original two heaps.

One origin of Amalgamation Nim was a query [12] from a student after one of the authors (Locke) mentioned Lasker's Nim in class.

We write $[a_1, a_2, \ldots, a_k]$ for the position with k heaps, with a_j coins in the j^{th} heap, and $\langle b_1, b_2, \ldots, b_k \rangle$ for the position with b_j heaps of height j, $1 \leqslant j \leqslant k$. Each a_j and each b_j is a non-negative integer. Any a_j which is zero could be deleted from the list, but it is convenient to write a move from $[1, 2, 3, 4]$ to $[1, 2, 0, 4]$, making it immediately obvious what the move was. Let $\varepsilon([a_1, a_2, \ldots, a_k]) = \sum_{j=1}^{k} (1 + a_j)$. For any move from a position Q to a position R, $\varepsilon(Q) > \varepsilon(R)$. Hence, the game must terminate since we start with a finite number of coins and a finite number of heaps. This decrease in the function ε after each move naturally leads to proofs by induction. We note that Amalgamation Nim is an impartial game.

A position is a P-position if the player who has just moved can force a win, and an N position if the player who will move next can force a win. A terminal position is one from which there are no moves. In normal play or in misère play, the terminal positions are N-positions, from every P-position, a player can only move to N-positions, and from every non-terminal N-position, a player can move to at least one P-position. In both normal and misère forms, no draws can occur. We make use of the following theorem from [1] (Theorem 2.15 on page 43), without further explicit reference to it.

Partition Theorem for Impartial Games. *Suppose that the positions of a finite impartial game can be partitioned into mutually exclusive sets A and B, such that every move from a position in A leaves a position in B and from every position in B, some move leaves a position in A. Then, A is the set of P-positions and B is the set of N-positions.*

Note that if $\sigma \in S_k$ is a permutation of $\{1, 2, \ldots, k\}$, the games $[a_1, a_2, \ldots, a_k]$ and $\left[a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(k)}\right]$ are essentially the same. We write $[a_1, a_2, \ldots, a_k] \cong \left[a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(k)}\right]$. In addition, when writing a set of games S, if $R \in S$, then for any $Q \cong R$, we write $Q \in S$. This is a slight abuse of notation, but is more convenient than listing all equivalent versions of each game in S.

We will call a position $\langle 2b_1, 2b_2, \ldots, 2b_k\rangle$ a *mirror position*. As will be seen in Lemma 6, P2 can mirror P1's move or remove one heap to create another position of this same type and thus a mirror position is a P-position. Our data does suggest that if the total number of coins is even, then the P-positions are exactly the mirror positions.

If position Q can be reached in one move from position R, we write $R \to Q$. If we know Q to be a P-position, we emphasize this by writing $R \rightsquigarrow Q$. If $R \rightsquigarrow Q$, then R is an N-position. For example, in Section 4, we will see that $\langle 2b_1, 2b_2, 2b_3, 2b_4\rangle$ is a P-position and thus $\langle 2b_1, 2b_2, 2b_3, 2b_4 + 1\rangle \rightsquigarrow \langle 2b_1, 2b_2, 2b_3, 2b_4\rangle$ and $\langle 2b_1, 2b_2, 2b_3 + 1, 2b_4\rangle \rightsquigarrow \langle 2b_1, 2b_2, 2b_3, 2b_4\rangle$. Hence, $\langle 2b_1, 2b_2, 2b_3, 2b_4 + 1\rangle$ and $\langle 2b_1, 2b_2, 2b_3 + 1, 2b_4\rangle$ are N-positions.

As we will see in Section 2, $\langle 1, 1, 0, 1\rangle$ is a P-position, and thus $\langle 0, 1, 1, 1\rangle \rightsquigarrow \langle 1, 1, 0, 1\rangle$. One might have hoped that one could split the game $\langle 1, 1, 2, 1\rangle$ into the two games $\langle 1, 1, 0, 1\rangle$ and $\langle 0, 0, 2, 0\rangle$, both of which are P-positions. However, $\langle 1, 1, 2, 1\rangle$ is an N-position. From these examples, P-positions $\langle 0, 1, 1, 1\rangle$, $\langle 1, 1, 2, 1\rangle$ and N-position $\langle 1, 1, 0, 1\rangle$, we see that the number of odd entries in $\langle b_1, b_2, b_3, b_4\rangle$ is not enough to determine whether or not this position is a P-position.

Theorem 1. *Let $[a_1, a_2, \ldots, a_k]$ be a game position. There is exactly one non-negative integer a_{k+1} such that $[a_1, a_2, \ldots, a_{k+1}]$ is a P-position.*

Proof. Suppose that $[a_1, a_2, \ldots, a_{k+1}]$ is a P-position for some particular a_{k+1}. Then, $[a_1, a_2, \ldots, a_k, a_{k+1} + j]$ is an N-position for any positive integer j, since $[a_1, a_2, \ldots, a_k, a_{k+1} + j] \rightsquigarrow [a_1, a_2, \ldots, a_{k+1}]$. Hence, there is at most one value of a_{k+1} such that $[a_1, a_2, \ldots, a_{k+1}]$ is a losing position. We must now show that there is a non-negative integer a_{k+1} such that $[a_1, a_2, \ldots, a_{k+1}]$ is a losing position.

Assume that there is no non-negative integer a_{k+1} such that $[a_1, a_2, \ldots, a_{k+1}]$ is a P-position. For any N-position $[a_1, a_2, \ldots, a_{k+1}]$ there is a move to a P-position $[c_1, c_2, \ldots, c_k, a_{k+1}]$ or an amalgamation move to a position $[d_1, d_2, \ldots, d_k]$, or perhaps both. Note that there are no P-positions reached by reducing the $(k+1)^{\text{st}}$ heap.

For fixed $[a_1, a_2, \ldots, a_k]$, let $\mathbb{N} = S_1 \cup S_2$, where $a_{k+1} \in S_1$ if $[a_1, a_2, \ldots, a_{k+1}]$ has a move to a P-position $[c_1, c_2, \ldots, c_k, a_{k+1}]$ and $a_{k+1} \in S_2$ if $[a_1, a_2, \ldots, a_{k+1}]$ has is a move to a P-position $[d_1, d_2, \ldots, d_k]$ using amalgamation. Note that if $a_{k+1}, a'_{k+1} \in$

S_1, with $a_{k+1} \neq a'_{k+1}$, then $[c_1, c_2, \ldots, c_k, a_{k+1}]$ and $\left[c_1, c_2, \ldots, c_k, a'_{k+1}\right]$ cannot both be P-positions. Hence, $|S_1| \leqslant a_1 + a_2 + \cdots + a_k$, since each $[c_1, c_2, \ldots, c_k]$ differs from $[a_1, a_2, \ldots, a_k]$ in one position.

We now consider $a_{k+1} \in S_2$. There are k possible amalgamations involving the $(k+1)^{\text{st}}$ heap. If we merge the $(k+1)^{\text{st}}$ heap with the j^{th} heap, the result is $P' = [a_1, \ldots, a_{j-1}, a_j + a_{k+1}, a_{j+1}, \ldots, a_k]$. But, removing the j^{th} heap from this position leaves $[a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k]$ and there is at most one value for $a_j + a_{k+1}$ such that P' is a P-position. Similarly, if we merge the i^{th} heap and the j^{th} heap, for $1 < i < j < k+1$, there is at most one value of a_{k+1} for which the result is a P-position. Thus, $|S_2| \leqslant k + \binom{k}{2}$. Hence, $|\mathbb{N}| = |S_1 \cup S_2| \leqslant |S_1| + |S_2|$, which is finite. Since this is impossible, it contradicts the assumption that there is no a_{k+1} such that $[a_1, a_2, \ldots, a_{k+1}]$ is a P-position. Therefore, there is exactly one choice of a_{k+1} such that $[a_1, a_2, \ldots, a_{k+1}]$ is a P-position. □

2. Three heaps, Part (i)

We now consider games $[a_1, a_2, a_3]$, which start with three heaps. We assume that $a_1 \leqslant a_2 \leqslant a_3$. In this section, we consider the cases in which $a_1 \leqslant 3$. These results are useful in §3. We continue the discussion in Section 5 for larger values of a_1.

For a set of pairs S, let $S + p\mathbb{N} = \{[a + kp, b + kp] : [a, b] \in S, k \geqslant 0\}$. We say that $S + p\mathbb{N}$ is *periodic with period* p if there is no $p' < p$ and set T such that $S + p\mathbb{N} = T + p'\mathbb{N}$. We use $L(j) = \{[a_2, a_3] : [j, a_2, a_3] \in \mathcal{P}, a_2 \leqslant a_3\}$, where $\mathcal{P}$ is the set of all P-positions, to record the set of three-heap P-positions with a heap of height j. That is, for a given a_1, the three-heap P-positions are $\{[a_1, a_2, a_3] : [a_2, a_3] \in L(a_1)\}$. It is convenient to display the non-periodic part $L_1(j)$ and the periodic part $L_2(j)$ of $L(j)$ separately.

In Theorem 2, we present results for positions $[a_1, a_2, a_3]$ with $0 \leqslant a_1 \leqslant a_2 \leqslant a_3$ and $a_1 \leqslant 7$. The proofs are presented later as a series of theorems. Brief descriptions of the cases $8 \leqslant a_1 \leqslant 11$ are displayed in Empirical Observations 16 to 19.

Theorem 2. *Given a position with at most three non-empty heaps, the set of P-positions is as follows:*

(i) One non-empty heap: no P-positions;

(ii) Two non-empty heaps: the only P-positions are $[a, a]$ *for* $a = 1, 2, \ldots$*;*

(iii) For three heaps and smallest heap of size a_1, $0 < a_1 \leqslant 7$, *the P-positions are given by:*

$L_1(1) = \{\}$ *and* $L_2(1) = \{[2,4], [3,5]\} + 4\mathbb{N}$*;*

$L_1(2) = \{\}$ *and* $L_2(2) = \{[1,4]\} + 2\mathbb{N}$*;*

$L_1(3) = \{[3,4,8], [3,9,10], [3,7,11]\}$ *and* $L_2(1) = \{[12,13]\} + 2\mathbb{N}$*;*

$L_1(4) = \{[1,2], [3,8], [5,6], [7,12], [9,11]\}$ *and* $L_2(4) = \{[10,13], [15,16]\} + 4\mathbb{N}$.

$L_1(5) = \{[1,3], [2,8], [4,6], [7,13], [9,15], [10,11]\}$ *and*
$L_2(5) = \{[12,14], [17,19]\} + 4\mathbb{N}$.

$L_1(6) = \{[1,8], [2,3], [4,5] [7,14], [9,13], [10,15], [11,12]\}$ *and*
$L_2(6) = \{[16,19], [17,18]\} + 4\mathbb{N}$.

$L_1(7) = \{[1,9], [2,10], [3,11], [4,12], [5,13], [6,14], [8,16]\}$ *and*
$L_2(7) = \{[15,19], [17,21], [18,22], [20,24]\} + 8\mathbb{N}$.

Proof. We discuss parts (i) and (ii) here. Part (iii) will be proven in separate theorems later.

From a position $[0,0,a_3]$ with $0 < a_3$, play to $[0,0,0]$. From a position $[0,a_2,a_3]$ with $a_2 < a_3$, play to $[0,a_2,a_2]$, and then mirror the opponent's moves. That is, if P1 takes j coins from one of the heaps, then P2 takes j coins from the other heap. If P1 amalgamates the two non-empty heaps, than P2 takes that new heap. □

Theorem 3. *Positions* $[1, 4k+2, 4k+4]$ *and* $[1, 4k+3, 4k+5]$ *are P-positions, for* $k \geqslant 0$*, and all other positions* $[1, a_2, a_3]$*, with* $1 \leqslant a_2 \leqslant a_3$*, are N-positions.*

Proof. Our basic approach for this proof, and for some of the later proofs, is to proceed by induction. In this case, we consider triples $[1,a_2,a_3]$, and we assume that we have proven the result for all triples $[1,a_2',a_3']$ with $a_2'+a_3' < a_2+a_3$.

Let S be the set of positions $\{[1,4k+2,4k+4], [1,4k+3,4k+5], [0,k,k] : k \geqslant 0\}$. We need to show that if P1 is playing from a position $[1,a_2,a_3] \notin S$, then P1 has move to a position $[a_1',a_2',a_3'] \in S$, and if P1 is playing from a position $[1,a_2,a_3] \in S$, then P1 has no move to a position in S. We have already classified the positions $[0,a_2',a_3']$ in Theorem 2.

Consider the position $P = [1,a_2,a_3]$ with $a_2 \leqslant a_3$, and P1 about to move. If $a_2 = a_3$, P1 moves: $[1,a_2,a_2] \rightsquigarrow [0,a_2,a_2]$. If $a_2+1 = a_3$, P1 moves: $[1,a_2,a_2+1] \rightsquigarrow [a_2+1, a_2+1]$ using amalgamation. Also, if $a_2 = 1$, P1 moves $[1,1,a_3] \rightsquigarrow [1,1,0]$. Hence, we only need to consider positions with $1 < a_2 \leqslant a_3 - 2$.

For each position $P = [1,a_2,a_3]$ with $1 < a_2 \leqslant a_3 - 2$ and $\{a_2,a_3\} \notin M = \{\{4k+2,4k+4\}, \{2k+3,4k+5\} : k \geqslant 0\}$, we display a move to a position $[1,a_2',a_3']$

with $\{a_2', a_3'\} \in M$.

If $a_2 = 4k+2$, for $k \geqslant 0$, $[1, a_2, a_3] \rightsquigarrow [1, 4k+2, 4k+4]$.

If $a_2 = 4k+3$, for $k \geqslant 0$, $[1, a_2, a_3] \rightsquigarrow [1, 4k+3, 4k+5]$.

If $a_2 = 4k+4$, for $k \geqslant 0$, $[1, a_2, a_3] \rightsquigarrow [1, 4k+4, 4k+2] \cong [1, 4k+2, 4k+4]$.

If $a_2 = 4k+5$, for $k \geqslant 0$, $[1, a_2, a_3] \rightsquigarrow [1, 4k+5, 4k+3] \cong [1, 4k+3, 4k+5]$.

We need consider the positions $P_\alpha = [1, 4k+2, 4k+4]$ or $P_\beta = [1, 4k+3, 4k+5]$. No amalgamation move or removing the heap of height 1 leaves a position of the form $[a, a]$. By removing from either of the other two heaps, no position of the form $[1, 4k'+2, 4k'+4]$ and from $[1, 4k'+3, 4k'+5]$ can be reached.

Therefore, if P1 is playing from a position $[1, a_2, a_3] \notin S$, then P1 has move to a position $[a_1', a_2', a_3'] \in S$, and if P1 is playing from a position $[1, a_2, a_3] \in S$, then P1 has no move to a position in S. Thus the positions in S are P-positions and the positions of the type $[1, a_2, a_3] \notin S$ with $1 \leqslant a_2 \leqslant a_3$ are N-positions. □

Theorem 4. *If $a_1 = 2$, positions $[2, 2k+1, 2k+4]$ are P-positions, for $k \geqslant 0$, and all other positions are N-positions.*

Proof. Let $T = \{[2, 2k+1, 2k+4] : k \geqslant 0\}$. A move $[2, 2k+1, 2k+4] \to [2, a_2', a_3']$ will have $|a_2' - a_3'| \neq 3$ and $[2, a_2', a_3'] \notin T$ or $[2, a_2', a_3'] = [2, 2k+1, 2k-2] \notin T$. Moves involving reducing the first heap are $[2, 2k+1, 2k+4] \to [1, 2k+1, 2k+4]$ and $[2, 2k+1, 2k+4] \to [0, 2k+1, 2k+4]$. The possible amalgamation moves are $[2, 2k+1, 2k+4] \to [2k+3, 2k+4]$, $[2, 2k+1, 2k+4] \to [2k+1, 2k+6]$ and $[2, 2k+1, 2k+4] \to [2, 4k+5]$. Hence, all of P1's moves from T result in an N-position or a position $[2, a_2', a_3'] \notin T$.

Now, suppose $R = [2, a_2, a_3] \notin T$, with $2 \leqslant a_2 \leqslant a_3$. If $a_2 = a_3$, $R \rightsquigarrow [0, a_2, a_2]$. If $a_2 + 2 = a_3$, $R \rightsquigarrow [a_3, a_3]$ using amalgamation. If $a_2 = 2$, $R \rightsquigarrow [2, 2, 0]$. If a_2 is odd and $a_3 > a_2 + 3$, $R \to [2, a_2, a_2 + 3] \in T$. For $a_2 = 3$ and $a_2 < a_3 \leqslant a_2 + 3$, the two possibilities are $R = [2, 3, 4] \rightsquigarrow [2, 1, 4]$ and $R = [2, 3, 5] \rightsquigarrow [1, 3, 5]$. We may now assume that $a_2 \geqslant 4$. If a_2 is even, then $R \to [2, a_2, a_2 - 3] \cong [2, a_2 - 3, a_2] \in T$. Hence, we may now assume that a_2 is odd and $a_2 < a_3 \leqslant a_2 + 3$. But, if $a_3 = a_2 + 3$, that contradicts the choice of $R \notin T$. Thus, we may assume that $a_3 = a_2 + 1 \geqslant 4$. But now, $R \to [2, a_2 - 2, a_2 + 1] \in T$.

Therefore, in all cases if $R \notin T$, then P1 has a winning move or a move to a position in T. If $R \in T$, all of P1's moves from T result in an N-position or a position $[2, a_2', a_3'] \notin T$. Thus the positions in T are P-positions and the positions of the type $[2, a_2, a_3] \notin T$ are N-positions. □

Theorem 5. *If $a_1 = 3$ and $a_1 \leqslant a_2 \leqslant a_3$, positions $[3, 4, 8]$, $[3, 9, 10]$, $[3, 7, 11]$, and $[3, 2k+12, 2k+13]$, for $k \geqslant 0$, are P-positions, and all other positions are N-positions.*

Proof. Let $W = \{[3, 4, 8], [3, 9, 10], [3, 7, 11]\} \cup \{[3, 2k+12, 2k+13] : k \geqslant 0\}$. Note that for any $P \in W$, with P1 about to move, all amalgamation moves result in a win for P2. Any move from $P = [3, a_2, a_3] \to [3, a_2', a_3'] \notin W$. Any move from $P = [3, a_2, a_3] \to [0, a_2, a_3]$ leaves a win for P2, since $a_2' \neq a_3'$. It is not difficult to check that moves $P = [3, a_2, a_3] \to [1, a_2, a_3]$ and $P = [3, a_2, a_3] \to [2, a_2, a_3]$ leave wins for P2. Hence any move from a position in W leaves a position not in W, of results in a win for P2.

Now, suppose that $P = [3, a_2, a_3] \notin W$, with $3 \leqslant a_2 \leqslant a_3$. If $a_2 \in \{3, a_3\}$ or if $a_3 = a_2 + 3$, there is a move of the form $P \to [k, k]$. If $a_2 = 4$ and $a_3 > 8$, $P \to [3, 4, 8] \in W$. Since, $[3, 4, 6] \rightsquigarrow [3, 2, 6]$, $[3, 4, 5] \rightsquigarrow [3, 1, 5]$, $[3, 5, k + 6] \rightsquigarrow [3, 5, 1]$ and $[3, 6, k + 7] \rightsquigarrow [3, 6, 2]$, we may assume that $a_2 \geqslant 7$. Now, $[3, 7, k + 12] \to [3, 7, 11] \in W$, $[3, 7, 10] \rightsquigarrow [10, 10]$ using amalgamation, $[3, 7, 9] \rightsquigarrow [1, 7, 9]$, $[3, 7, 8] \to [3, 4, 8] \in W$, $[3, 8, 9 + k] \to [3, 8, 4] \in W$, and we may assume that $a_2 \geqslant 9$. But, $[3, 9, 11 + k] \to [3, 9, 10] \in W$, $[3, 10, 11 + k] \to [3, 10, 9] \in W$, $[3, 11, 12 + k] \to [3, 11, 7] \in W$, and we may assume that $a_2 \geqslant 12$. Now, if a_2 is even, $[3, a_2, a_3] \to [3, a_2, a_2 + 1] \in W$ and if a_2 is odd, $[3, a_2, a_3] \to [3, a_2, a_2 - 1] \in W$.

Therefore, in all cases if $P \notin W$, then P1 has a winning move or a move to a position in W. If $P \in W$, all of P1's moves from W result in a winning position for P2 or a position $[3, a_2', a_3'] \notin W$. Thus the positions in W are P-positions and the positions of the type $[3, a_2, a_3] \notin W$ are N-positions. □

3. Restricted Heights

The following lemma provides a nice class of P-positions. However, one should note that these are not all of the P-positions. In each lemma, we assume that P1 is the next player to move.

Lemma 6. *The position $\langle b_1, b_2, \ldots, b_k \rangle$ is a P-position if b_j is even for each j, $1 \leqslant j \leqslant k$.*

Proof. Let $P = \langle b_1, b_2, \ldots, b_k \rangle$ be a position with b_j even for each j, $1 \leqslant j \leqslant k$. If $b_j = 0$ for each j, $1 \leqslant j \leqslant k$, then P1 has lost. Hence, we assume that some $b_j > 0$. The proof is by induction on $\varepsilon(P)$.

If P1 removes x coins from a heap of height j, then P2 can remove x coins from a remaining heap of height j. If P1 merges a heap of height i and a heap of height j, with $i \neq j$, then P2 makes the same move since there is still some heap of height i and some heap of height j. If P1 merges two heaps of height j, then P2 removes the new heap of height $2j$. In each case, the remaining configuration $Q = \langle b_1', b_2', \ldots, b_k' \rangle$ has b_j' is even for each j, $1 \leqslant j \leqslant k$,, and $\varepsilon(Q) < \varepsilon(P)$. □

We are now prepared to consider the case in which no heap has height exceeding three.

Theorem 7. *The position $\langle b_1, b_2, b_3 \rangle$ is a P-position if and only if b_j is even for each j, $1 \leqslant j \leqslant 3$.*

Proof. We need only show that if some b_j is odd, then there is a move to a position $\langle b'_1, b'_2, b'_3 \rangle$ with b'_j even for each j, $1 \leqslant j \leqslant 3$. If exactly one b_j is odd, remove a heap of height j. If b_1, b_2, b_3 are all odd, amalgamate a heap of height one and a heap of height two. If exactly two of b_1, b_2, b_3 are odd, say b_i, b_j are odd, with $i < j$. Remove $j - i$ coins from a heap of height j. In each case, the remaining configuration $\langle b'_1, b'_2, b'_k \rangle$ has b'_j is even for each j, $1 \leqslant j \leqslant 3$. □

Theorem 7 completely identifies the winning and losing position when no heap has height exceeding three. The next few lemmas lead up to Theorem 9 which identifies many winning positions if no heap has height exceeding four. A few winning positions are easy to identify. We identify the following three cases. In each case, there is a move which leaves a mirror position.

Lemma 8(a). *A position $P = \langle b_1, b_2, \ldots, b_k \rangle$ is an N-position if b_j is odd for exactly one value of j.*

Proof. If b_j is odd and b_i is even, for all $i \neq j$, then remove one heap of height j. □

Lemma 8(b). *A position $P = \langle b_1, b_2, \ldots, b_k \rangle$ is an N-position if there are exactly two values of j such that b_j is odd.*

Proof. If b_j and b_i are odd, $i < j$, and b_s is even, for all $s \notin \{i, j\}$, then remove $j - i$ coins from a heap of height j. □

Lemma 8(c). *A position $P = \langle b_1, b_2, \ldots, b_k \rangle$ is an N-position if b_i ,b_j and b_t are odd, with $i + j = t$, and b_s is even for all $s \notin \{i, j, t\}$.*

Proof. Merge a heap of height i with a heap of height j creating a new heap of height t. □

4. Heaps of Size at Most Four

In this section, we restrict our attention to games with all heaps of height at most four.

Theorem 9. *A position $P = \langle b_1, b_2, b_3, b_4 \rangle$ is an N-position if $b_1 + b_3$ is even and not all of b_1, b_2, b_3, b_4 have the same parity.*

Proof. Theorem 7 completely solves the game if no heap has height exceeding three. Lemmas 8(a), 8(b), 8(c), together with Theorem 3, solve games $\langle b_1, b_2, b_3, b_4\rangle$ if there are at most two indices j such that b_j is odd, or if
$\langle b_1, b_2, b_3, b_4\rangle = \langle 2k_1+1, 2k_2+1, 2k_3+1, 2k_4\rangle$ or
$\langle b_1, b_2, b_3, b_4\rangle = \langle 2k_1+1, 2k_2, 2k_3+1, 2k_4+1\rangle$. □

For games with all heaps of height at most four and with an even number of coins, that is b_1+b_3 is even, there remains only one case not considered: $\langle b_1, b_2, b_3, b_4\rangle = \langle 2k_1+1, 2k_2+1, 2k_3+1, 2k_4+1\rangle$. This appears to be a win, leaving the only P-positions with an even number of coins those for which there are an even number of heaps of each height. Thus, we have the following conjecture.

Conjecture 10. *The position $P = \langle b_1, b_2, b_3, b_4\rangle$ is an N-position if b_1+b_3 is even and at least one of b_1, b_2, b_3, b_4 is odd.*

5. Three Heaps, Part (ii)

We now consider games $[a_1, a_2, a_3]$, which start with three heaps, with $4 \leqslant a_1 \leqslant 11$ and $a_1 \leqslant a_2 \leqslant a_3$. For values of a_1 with $4 \leqslant a_1 \leqslant 7$, proofs are provided, and for values of a_1 with $8 \leqslant a_1 \leqslant 11$, we present the observed data without proofs, although the proofs in each case would be straightforward and could be constructed computationally.

Theorem 11. *When $a_1 = 4$, the P-positions are given by*
$L_1(4) = \{[1,2], [3,8], [5,6], [7,12], [9,11]\}$ *and* $L_2(4) = \{[10,13], [15,16]\} + 4\mathbb{N}$.

Proof. Let $T = \{[4,1,2], [4,3,8], [4,5,6], [4,7,12], [4,9,11]\} \cup \{[4, 10+4k, 13+4k], [4, 15+4k, 16+4k] : k \geqslant 0\}$. Let $[4, a_2, a_3] \notin T$, with $4 \leqslant a_2 \leqslant a_3$. Note $[4, a_2, a_2] \rightsquigarrow [0, a_2, a_2]$, $[4,4,a_3] \rightsquigarrow [4,4,0]$, $[4,5,7+k] \to [4,5,6] \in T$, $[4,6,a_3] \to [4,6,5] \in T$. Thus, $a_3 > a_2 \geqslant 7$. Now, $[4,7,8] \rightsquigarrow [4,3,8]$, $[4,7,9] \rightsquigarrow [1,7,9]$, $[4,7,10] \rightsquigarrow [2,7,10]$, $[4,7,11] \rightsquigarrow [3,7,11]$, and $[4,7,13+k] \to [4,7,12] \in T$. Hence, $a_2 \geqslant 8$. Then, $[4,8,9+k] \to [4,8,3] \in T$, $[4,9,10] \rightsquigarrow [4,9,10]$, $[4,9,12+k] \to [4,9,11] \in T$, and $a_2 \geqslant 10$. Next, $[4,10,11] \to [4,9,11] \in T$, $[4,10,12] \rightsquigarrow [1,10,12]$, $[4,10,13+k] \to [4,10,13] \in T$, $[4,11,12+k] \to [4,11,9] \in T$, $[4,12,13+k] \to [4,12,7] \in T$, $[4,13,14+k] \to [4,13,10] \in T$, and $a_2 \geqslant 14$. Now, $[4, 4j+1, 4j+2+k] \to [4, 4j+1, 4j-2] \in T$, $[4, 4j+2, 4j+3] \rightsquigarrow [3, 4j+2, 4j+3]$, $[4, 4j+2, 4j+4] \rightsquigarrow [1, 4j+2, 4j+4]$, $[4, 4j+2, 4j+5+k] \to [4, 4j+2, 4j+5] \in T$, $[4, 4j+3, 4j+5+k] \to [4, 4j+3, 4j+4] \in T$, and
$[4, 4j+4, 4j+5] \rightsquigarrow [3, 4j+4, 4j+5]$. Thus, in all cases, if $[4, a_2, a_3] \notin T$, with $4 \leqslant a_2 \leqslant a_3$, there is either a move to $[4, a_2', a_3'] \in T$, or a move to a losing position $[a_1, a_2, a_3]$ with $a_1 < 4$.

Now suppose that $P = [4, a_2, a_3] \in T$. If $a_2 < 4$ or $a_3 < 4$, then these cases have been covered by previous theorems. Hence, we may assume that $4 \leqslant a_2 \leqslant a_3$. From the list of possibilities for P, it is obvious that no amalgamation move from P results in a losing position, and any move $[4, a_2, a_3] \to [4, a_2', a_3'] \notin T$. Any move $[4, a_2, a_3] \to [0, a_2, a_3]$ leaves a win for P2, since $a_2 \neq a_3$. Any move from some $[4, a_2, a_3] \to [1, a_2, a_3]$ has $[a_2, a_3] \notin L(1)$ since $a_3 - a_2 \neq 2$ or $[a_2, a_3] = [9, 11]$. Similarly, $[4, a_2, a_3] \to [2, a_2, a_3]_W$ since $[a_2, a_3] \notin L(2)$ and $[4, a_2, a_3] \to [3, a_2, a_3]_W$ since $[a_2, a_3] \notin L(3)$. Hence, any move from P results in a win for P2. Thus, T is the set of losing position with $a_1 = 4$. □

Theorem 12. *When $a_1 = 5$, the P-positions are given by*
$L_1(5) = \{[1,3], [2,8], [4,6], [7,13], [9,15], [10,11]\}$ *and*
$L_2(5) = \{[12,14], [17,19]\} + 4\mathbb{N}$.

Proof. Let $Y_1 = \{[5,1,3], [5,2,8], [5,4,6], [5,7,13], [5,9,15], [5,10,11]\}$,
$Y_2 = \{[5, 12+4m, 14+4m], [5, 17+4m, 19+4m] : m \geqslant 0\}$, $Y = Y_1 \cup Y_2$ and $P = [5, a_2, a_3] \in Y$. Note that $[a_2, a_3] \notin L(j)$, for $j \leqslant 4$. Thus, there is no winning move $P \to [j, a_2, a_3]$, for $j < 5$. It is obvious that no amalgamation move from P leaves a losing position and any move $[5, a_2, a_3] \to [5, a_2', a_3'] \notin Y$. Hence, all moves from P leave an N-position or leave a position $[5, a_2', a_3'] \notin Y$.

Now, suppose $P = [5, a_2, a_3] \notin Y$, with $a_2 \leqslant a_3$. If $a_2 < 5$, previous theorems apply. If $5 = a_2$ or $a_2 = a_3$, move to $[5,5,0]$ or $[0, a_2, a_2]$. Hence, we may assume that $5 < a_2 < a_3$. Since $[5,6,a_3] \rightsquigarrow [5,6,4]$, $a_2 \geqslant 7$, $[5,7,8] \rightsquigarrow [5,2,8]$, $[5,7,9] \rightsquigarrow [1,7,9]$, $[5,7,10] \rightsquigarrow [2,7,10]$, $[5,7,11] \rightsquigarrow [3,7,11]$, $[5,7,12] \rightsquigarrow [0,12,12]$, $[5,7,14+k] \to [5,7,13] \in Y$, and $[5,8,9+k] \rightsquigarrow [5,8,2]$, we may assume that, $a_2 \geqslant 9$. Next, $[5,9,10] \rightsquigarrow [3,9,10]$, $[5,9,11] \rightsquigarrow [4,9,11]$, $[5,9,12] \rightsquigarrow [2,9,12]$, $[5,9,14+k] \to [5,9,13] \in Y$, $[5,10,12+k] \to [5,10,11] \in Y$, and $[5,11,12+k] \to [5,11,10] \in Y$. Hence $a_2 \geqslant 12$. Now, $[5,12,13] \rightsquigarrow [3,12,13]$, $[5,12,14+k] \to [5,12,13] \in Y$, $[5,13,14+k] \to [5,13,7] \in Y$, $[5,14,15+k] \to [5,14,1] \in Y$, $[5,15,16+k] \to [5,15,9] \in Y$, and $a_2 \geqslant 16$. $[5,16,17] \rightsquigarrow [3,16,17]$, $[5,16,19+k] \to [5,16,18] \in Y$, $[5,17,18] \to [5,16,18] \in Y$, $[5,17,20+k] \to [5,17,19] \in Y$. Thus, $a_2 \geqslant 18$.

Now, $[5, 4m+18, 4m+19+k] \to [5, 4m+18, 4m+16] \in Y$, $[5, 4m+19, 4m+20+k] \to [5, 4m+19, 4m+17] \in Y$, $[5, 4m+20, 4m+21] \rightsquigarrow [3, 4m+20, 4m+21]$, $[5, 4m+20, 4m+23+k] \to [5, 4m+20, 4m+22] \in Y$, $[5, 4m+21, 4m+22] \to [5, 4m+20, 4m+22] \in Y$, $[5, 4m+21, 4m+24+k] \to [5, 4m+21, 4m+23] \in Y$. Hence, any move from P results in a win for P2. Thus, Y is the set of losing position with $a_1 = 5$. □

Theorem 13. *When $a_1 = 6$, the P-positions are given by*
$L_1(6) = \{[1,8], [2,3], [4,5]\,[7,14], [9,13], [10,15], [11,12]\}$*, and*
$L_2(6) = \{[16,19], [17,18]\} + 4\mathbb{N}$.

Proof. Let $W = \{[6,1,8], [6,2,3], [6,4,5]\, [6,7,14], [6,9,13], [6,10,15], [6,11,12]\} \cup \{[6,4m+16,4m+19], [6,4m+17,4m+18] : m \geqslant 0\}$, and $P = [6,a_2,a_3] \in W$. Again, it is not hard to check that $[a_2,a_3] \notin L(j)$, for $j \leqslant 5$, there is no amalgamation move from P which leaves a losing position and any move $[6,a_2,a_3] \to [6,a_2',a_3'] \notin W$.

Now, suppose that $P = [6,a_2,a_3] \notin W$, with $6 < a_2 < a_3$. The possible cases and a winning move for each are listed. $[6,7,8] \to [6,1,8] \in W$, $[6,7,9] \rightsquigarrow [1,7,9]$, $[6,7,10] \rightsquigarrow [2,7,10]$, $[6,7,11] \rightsquigarrow [3,7,11]$, $[6,7,12] \rightsquigarrow [4,7,12]$, $[6,7,13] \rightsquigarrow [5,7,13]$, $[6,7,15+k] \to [6,7,14] \in W$, $[6,8,9+k] \to [6,8,1] \in W$, $[6,9,10] \rightsquigarrow [3,9,10]$, $[6,9,11] \rightsquigarrow [4,9,11]$, $[6,9,12] \rightsquigarrow [2,9,12]$, $[6,9,14+k] \to [6,9,13] \in W$, $[6,10,11] \rightsquigarrow [5,10,11]$, $[6,10,12] \rightsquigarrow [1,10,12]$, $[6,10,13] \to [4,10,13]$, $[6,10,14] \to [6,7,14] \in W$, $[6,10,16+k] \to [6,10,15] \in W$, $[6,11,13+k] \to [6,11,12] \in W$, $[6,12,13+k] \to [6,12,11] \in W$, $[6,13,14+k] \to [6,13,9] \in W$, $[6,14,15+k] \to [6,14,7] \in W$, $[6,15,16+k] \to [6,15,10] \in W$, $[6,4m+16,4m+17] \rightsquigarrow [3,4m+16,4m+17]$, $[6,4m+16,4m+18] \rightsquigarrow [5,4m+16,4m+18]$, $[6,4m+16,4m+20+k] \to [6,4m+16,4m+19] \in W$, $[6,4m+17,4m+19+k] \to [6,4m+17,4m+18] \in W$, $[6,4m+18,4m+19+k] \to [6,4m+18,4m+17] \in W$, $[6,4m+19,4m+20+k] \to [6,4m+19,4m+16] \in W$. □

Theorem 14. *When $a_1 = 7$, the P-positions are given by*
$L_1(7) = \{[1,9], [2,10], [3,11], [4,12], [5,13], [6,14], [8,16]\}$ *and*
$L_2(7) = \{[15,19], [17,21], [18,22], [20,24]\} + 8\mathbb{N}$.

Proof. Let $Y = \{[7,1,9], [7,2,10], [7,3,11], [7,4,12], [7,5,13], [7,6,14], [7,8,16]\} \cup \{[7,8m+15,8m+19], [7,8m+17,8m+21], m \geqslant 0\} \cup \{[7,8m+18,8m+22], [7,8m+20,8m+24], m \geqslant 0\}$, and $P = [7,a_2,a_3] \in Y$. Again, it is not hard to check that $[a_2,a_3] \notin L(j)$, for $j \leqslant 6$, there is no amalgamation move from P which leaves a P-position and any move $[7,a_2,a_3] \to [7,a_2',a_3'] \notin Y$.

Now, suppose that $P = [7,a_2,a_3] \notin Y$, with $7 < a_2 < a_3$. The possible cases are listed.

For $a_2 \geqslant 8$ and $a_3 \in \{8,9,10,11,12,13,14\}$, $P \to [7,a_3-8,a_3] \in Y$.
$[7,8,15] \rightsquigarrow [0,15,15]$ (amalgamation), $[7,8,17+k] \to [7,8,16] \in Y$.
For $a_2 \in \{9,10,11,12,13,14\}$, $[7,a_2,a_2+1+k] \to [7,a_2,a_2-8] \in Y$,
$[7,8m+15,8m+16] \rightsquigarrow [4,8m+15,8m+16]$,
$[7,8m+15,8m+17] \rightsquigarrow [1,8m+15,8m+17]$,
$[7,8m+15,8m+18] \rightsquigarrow [2,8m+15,8m+18]$,
$[7,8m+15,8m+20+k] \to [7,8m+15,8m+19] \in Y$,
$[7,16,17+k] \to [7,16,8] \in Y$,
$[7,8m+24,8m+25+k] \to [7,8m+24,8m+20] \in Y$,
$[7,8m+17,8m+18] \rightsquigarrow [4,8m+17,8m+18]$,
$[7,8m+17,8m+19] \rightsquigarrow [5,8m+17,8m+19]$,
$[7,8m+17,8m+20] \rightsquigarrow [2,8m+17,8m+20]$,

$[7, 8m + 17, 8m + 22 + k] \to [7, 8m + 17, 8m + 21] \in Y,$
$[7, 8m + 18, 8m + 19] \to [7, 8m + 15, 8m + 19] \in Y,$
$[7, 8m + 18, 8m + 20] \rightsquigarrow [1, 8m + 18, 8m + 20],$
$[7, 8m + 18, 8m + 21] \to [7, 8m + 17, 8m + 21] \in Y,$
$[7, 8m + 18, 8m + 23 + k] \to [7, 8m + 18, 8m + 22] \in Y,$
$[7, 8m + 19, 8m + 20 + k] \to [7, 8m + 19, 8m + 15] \in Y,$
$[7, 8m + 20, 8m + 21] \rightsquigarrow [3, 8m + 20, 8m + 21],$
$[7, 8m + 20, 8m + 22] \rightsquigarrow [5, 8m + 20, 8m + 22],$
$[7, 8m + 20, 8m + 23] \rightsquigarrow [6, 8m + 20, 8m + 23],$
$[7, 8m + 20, 8m + 25 + k] \to [7, 8m + 20, 8m + 24] \in Y,$
$[7, 8m + 21, 8m + 22 + k] \to [7, 8m + 21, 8m + 17] \in Y,$
$[7, 8m + 22, 8m + 23 + k] \to [7, 8m + 22, 8m + 18] \in Y.$ □

A jump in the length of the period occurs at $a_1 \in \{8, 10, 11\}$. We comment on observed data for $8 \leqslant a_1 \leqslant 11$. The reader can easily generate the P-positions for heaps $[a_1, a_2, a_3]$, $8 \leqslant a_1 \leqslant 11$, with a short program, or via A. Siegel's Combinatorics Game Suite [13].

Empirical Observation 15.
$L_1(8) = \{[1, 6], [2, 5], [3, 4], [7, 16], [9, 14], [11, 15], [10, 17], [12, 18], [13, 19]\}$ *and* $L_2(8)$ *starts with* $[20, 25]$ *and has period 40.*

Empirical Observation 16.
$L_1(9) = \{[1, 7], [2, 12], [3, 10], [4, 11], [5, 15], [6, 13], [8, 14]\}$ *and* $L_2(9)$ *starts at* $[16, 20]$ *and has period 40.*

Empirical Observation 17.
$L_1(10) = \{[1, 12], [2, 7], [3, 9], [4, 13], [5, 11], [6, 15], [8, 17]\}$ *and* $L_2(10)$ *starts with* $[14, 18]$ *and has period 160.*

Empirical Observation 18.
$L_1(11) = \{[1, 13], [2, 14], [3, 7], [4, 9], [5, 10], [6, 12], [8, 15], [18, 24], [20, 27], [21, 28]\}$ *and* $L_2(11)$ *starts with* $[16, 22]$ *and has period 960.*

Comment. We briefly discuss a graph-theoretic interpretation of winning moves. For graph theoretic terms, we suggest [3]. We begin with a general idea of how one might play the 3-heap game when a_1 is small, $a_1 > 0$. We Write $[j, k, \ell]_P$ to indicate that $[j, k, \ell]$ is a P-position. Here, we are not requiring that $j \leqslant k \leqslant \ell$.

Consider the graph G with vertex set $V(G) = \{k \in \mathbb{Z} : k \geqslant 1\}$ and edge set $\{\{j, k\} : j, k \in V(G) \text{ and } \exists \ell \geqslant 1 \text{ such that } [j, k, \ell]_P\}$. For a given value of j, let G_j denote the subgraph of G with vertex set $V(G) - \{j\}$ and edge set $M =$

$\{\{k,\ell\} : k,\ell \in V(G) \text{ and } [j,k,\ell]_P\}$. Note that every vertex of G_j has degree one. A *perfect matching* of a graph H is a subset X of the edge set of H such that every vertex of H is incident with exactly one edge of X. That is, M is a perfect matching of G_j.

Consider the case $a_1 = 1$. Faced with a position $[1, a_2, a_3]$, with $1 < a_2 \leqslant a_3$, P1 can quickly check whether there is a winning amalgamation move, or if $[1, a_2, a_3] \to [0, a_2, a_3]$ is a winning move. Otherwise, P1 identifies the pair $\{a_2, \ell\} \in M$ and reduces the heap of height a_3 to height ℓ. $L(j)$ is also the set of edges in the graph G_j, for $j > 0$.

6. Misère Amalgamation Nim

The misère version of Amalgamation Nim, mAN, is played by the same rules as the normal version of Amalgamation Nim, AN, with the exception that the player who takes the last coin loses. The position $\langle 2k+2\rangle$ is an N-position and the position $\langle 2k+1\rangle$ is a P-position, for $k \geqslant 0$. A position $R \neq \langle k+1\rangle$ but which could move to position $\langle k\rangle$ must have exactly one heap of height greater than one, and either k or $k-1$ heaps of height one. This position R is an N-position in AN and in mAN. The P-positions for mAN are therefore those P-positions for AN which have some heap of height greater than one, as well as those positions which have an odd number of heaps, all of which are height one. The N-positions for mAN are therefore those N-positions for AN which have some heap of height greater than one, as well as those positions which have an even number of heaps, all of which are height one. In other words, one plays both games in the same fashion, until there is a move which could leave all of the heaps of height one. Then, select the correct parity for the version of the game being played. Hence, the misère version of Amalgamation Nim is a tame game.

7. Conclusion

We hope these results and the data presented inspire other researchers to investigate this problem. Amalgamation Nim is a game with very simple rules but with an apparently complex analysis.

Even for 3-heap Amalgamation Nim, it may be difficult to give a fast method, similar to that of [4], for determining which positions are losing and which are winning. However, it should be possible to prove that $L(j)$ is eventually periodic, similar to the work of Steven Byrnes (see [15]), showing that 3-row Chomp is eventually periodic.

We have not completed the listing for losing positions with no heap of height greater than four. Examining the data appears to demonstrate that there are cases in which adding two heaps of the same height to a losing position results in a winning position. This makes it difficult to analyze via parity arguments.

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