ON THE ADDITIVE PERIOD LENGTH OF THE SPRAGUE-GRUNDY FUNCTION OF CERTAIN NIM-LIKE GAMES

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Abstract

We examine the structure of the additive period of the Sprague-Grundy function of Nim-like games, among them Wythoff's Game, and deduce a bound for the length of the period and preperiod.

1. Introduction

Consider a sequence $(\mathcal{Y}_x)_{x=0}^{\infty}$ of finite subsets of $\mathbb{Z}$. This sequence defines a function $G : \mathbb{N}_0 \to \mathbb{N}_0$ by

$$G(x) = \operatorname{mex}\left(\{G(x')|x' < x\} \cup \mathcal{Y}_x\right)$$

where $\operatorname{mex}(\mathcal{Y}) = \min\left(\mathbb{N}_0 \setminus \mathcal{Y}\right)$ is the minimal excluded operator, defined for any finite subset $\mathcal{Y}$ of $\mathbb{N}_0$ or $\mathbb{Z}$. Such functions appear as Sprague-Grundy functions in the study of certain impartial combinatorial games, such as Nim [1]. We will impose the condition of additive periodicity on $(\mathcal{Y}_x)_{x=0}^{\infty}$, as defined below.

Definition 1. Let $(\mathcal{Y}_x)_{x=0}^{\infty}$ be a sequence of finite subsets of $\mathbb{Z}$. Then $(\mathcal{Y}_x)_{x=0}^{\infty}$ is *additively periodic*, if there exists $P' \in \mathbb{N}_0$ and $p \in \mathbb{N}_1$ such that for all $x \geq P'$, we have $\mathcal{Y}_{x+p} = \mathcal{Y}_x + p$. The uniquely determined smallest numbers P', p for which this condition holds are called the *preperiod length* and the *period length* of $(\mathcal{Y}_x)_{x=0}^{\infty}$.

A function $G : \mathbb{N}_0 \to \mathbb{N}_0$ is additively periodic, if there exists $\tilde{P} \in \mathbb{N}_0$ and $\overline{p} \in \mathbb{N}_1$ such that for all $x \geq \tilde{P}$, we have $G(x + \overline{p}) = G(x) + \overline{p}$. Again, the smallest numbers $\tilde{P}, \overline{p}$ for which this condition holds are called the preperiod length and the period length of G.

Originally shown in [2], additive periodicity of $(\mathcal{Y}_x)_{x=0}^{\infty}$ implies that G will be additively periodic as well. The motivation for this paper was to find the optimal bound for the period length of G, given only the period length of $(\mathcal{Y}_x)_{x=0}^{\infty}$, and the upper and lower bounds of the elements in the sequence $(\mathcal{Y}_x - x)_{x=0}^{\infty}$.

To cover some practical cases from game theory where a finite number of the values of the Sprague-Grundy function are not defined by $(\mathcal{Y}_x)$, we will introduce a seed in the definition of G, inspired by a similar definition in [6].

Definition 2. Let $(\mathcal{Y}_x)_{x=0}^{\infty}$ be a sequence of additively periodic finite subsets of $\mathbb{Z}$. Let $L \in \mathbb{N}_0$, and let $[g_0, \dots, g_{L-1}]$ be an L-tuple, where for all $x, x' \in \{0, \dots, L-1\}$, we have $g_x \in \mathbb{N}_0$, and $g_x = g_{x'}$ implies $x = x'$. Let $G : \mathbb{N}_0 \to \mathbb{N}_0$ be defined by

$$G(x) = \begin{cases} g_x & \text{for } x < L, \\ \operatorname{mex}(\{G(x') | x' < x\} \cup \mathcal{Y}_x) & \text{for } x \geq L. \end{cases}$$

Then we call G for a *Nim sequence over* $(\mathcal{Y}_x)_{x=0}^{\infty}$, and $[g_0, \dots, g_{L-1}]$ is its *seed*.

Remark 1. In any example, we may redefine L as $\max(L, P')$. If previously $L > P'$, then for all $\tilde{x} \in \{L+1, \dots, P'\}$, we now have $G(\tilde{x}) = g_{\tilde{x}}$ as a part of the seed. As the mex operator excludes values of $G(x)$ for all $x < \tilde{x}$, we have $g_{\tilde{x}} \neq g_x$, so the requirement that $g_x = g_{x'}$ implies $x = x'$ is fulfilled for the extended seed. Since the sets $(\mathcal{Y}_x)_{x=0}^{L-1}$ play no role in the definition of G, we can pretend that the sequence $(\mathcal{Y}_x)$ is additively periodic for all indices. Thus, we will silently assume that $P' = 0$ in this paper.

We study Nim sequences using their difference functions, the definition of which is given in [5].

Definition 3. Given a function $G : \mathbb{N}_0 \to \mathbb{N}_0$, we define its *difference function* $d : \mathbb{N}_0 \to \mathbb{Z}$ by $d(x) = G(x) - x$. If G is additively periodic, we define its *difference period* starting at $x \geq \tilde{P}$ as the $\overline{p}$-tuple $[d(x), \dots, d(x + \overline{p} - 1)]$.

Given $\mathcal{Y}_x \subseteq \mathbb{Z}$, we use the notation $d(\mathcal{Y}_x) = \mathcal{Y}_x - x$.

It is clear that G is additively periodic if and only if its difference function is periodic, so $d(x + \overline{p}) = d(x)$ for all $x \geq \tilde{P}$. Similarly, if $(\mathcal{Y}_x)$ is additively periodic, then $d(\mathcal{Y}_{x+p}) = d(\mathcal{Y}_x)$, so $\max_{x \in \mathbb{N}_0} d(\mathcal{Y}_x), \min_{x \in \mathbb{N}_0} d(\mathcal{Y}_x)$ will be well-defined.

Example 1. Let $\overline{M} \in \mathbb{N}_1$ and $\underline{M} \in -\mathbb{N}_1$. For all $x \in \mathbb{N}_0$, define

$$\mathcal{Y}_x = \left\{\underline{M} + 1, \dots, \overline{M} - 1\right\} + x,$$

so $p = 1$ and $\max d(\mathcal{Y}_x) = \overline{M} - 1$, $\min d(\mathcal{Y}_x) = \underline{M} + 1$. Let the seed be empty. Set $M = |\underline{M}| + \overline{M}$. For $x \in \{0, \dots, M-1\}$, we calculate

$$G(x) = \begin{cases} \operatorname{mex}\left(\left\{\overline{M}, \dots, \overline{M} + x - 1\right\} \right. \\ \left. \cup \left\{\underline{M} + x + 1, \dots, \overline{M} + x - 1\right\}\right) & = & \overline{M} + x & \text{for } x < |\underline{M}|, \\ \operatorname{mex}\left(\left\{\overline{M}, \dots, M - 1\right\} \right. \\ \cup \left\{0, \dots, \underline{M} + x - 1\right\} \\ \left. \cup \left\{\underline{M} + x + 1, \dots, \overline{M} + x - 1\right\}\right) & = & \underline{M} + x & \text{for } x \geq |\underline{M}|. \end{cases}$$

Now $\{G(x) \mid x = 0, \ldots, M-1\} = \{0, \ldots, M-1\}$. By induction, we find for all $r \in \mathbb{N}_0$ that

$$G(x + rM) = \begin{cases} \overline{M} + x + rM & \text{for } (x \bmod M) < |\underline{M}|, \\ \underline{M} + x + rM & \text{for } (x \bmod M) \geq |\underline{M}|. \end{cases}$$

Then $\overline{p} = |\underline{M}| + \overline{M} = M$ with $\tilde{P} = 0$, as the difference period becomes:

$$[\,\underbrace{\overline{M}, \overline{M}, \ldots, \overline{M}}_{|\underline{M}| \text{ times}}, \underbrace{\underline{M}, \underline{M}, \ldots, \underline{M}}_{\overline{M} \text{ times}}\,].$$

$\diamondsuit$

Inspired by this example, we formally define $\overline{M}$ and $\underline{M}$ as constants determined by $(\mathcal{Y}_x)_{x=0}^{\infty}$.

Definition 4. Let $(\mathcal{Y}_x)_{x=0}^{\infty}$ be a sequence of additively periodic finite subsets of $\mathbb{Z}$. Then, we define the *difference bounds* $\overline{M} = \max_{x \in \mathbb{N}_0} d(\mathcal{Y}_x) + 1$, $\underline{M} = \min_{x \in \mathbb{N}_0} d(\mathcal{Y}_x) - 1$, and $M = |\underline{M}| + \overline{M}$.

The reason why we offset these maximal and minimal values by 1 is that they will become the bounds for the difference function d of G, as we will prove in this section. Now we can formulate the main result of this paper:

Theorem 1. *Let G be a Nim sequence over $(\mathcal{Y}_x)_{x=0}^{\infty}$ that has period length p. Then G is additively periodic, and the length $\overline{p}$ of its period is bounded by:*

$$\overline{p} \leq K_{\underline{M}, \overline{M}}\, p$$

where $K_{\underline{M}, \overline{M}} \in \mathbb{N}_1$ is a constant, approximately defined by

$$K_{\underline{M}, \overline{M}} \sim \exp\left(\sqrt{\mathrm{Li}^{-1}\left(|\underline{M}| + \overline{M} - 1\right)}\right).$$

Here, $\mathrm{Li}(x) = \int_2^x \frac{1}{\log t} \mathrm{d}t$ is the offset logarithmic integral, and the symbol $\sim$ signifies that $K_{\underline{M}, \overline{M}}$ has the same divergence speed as the expression.

The first part of this theorem, that G is additively periodic, was proven by Pink in his diploma thesis. His proof was simplified by Dress and Flammenkamp in [2], which was followed by an even simpler proof by Landman in [3]. These proofs all rely on pigeonhole methods, which lead to larger bounds for $\overline{p}$.

The canonical example of additive periodicity in game theory is Wythoff's Game, which we will briefly discuss.

Example 2. For all $y \in \mathbb{N}_0$, define a Nim sequence G_y by

$$G_y(x) = \operatorname{mex}\left(\{G_y(x')|x' < x\} \cup \{G_{y'}(x)|y' < y\} \cup \{G_{y-k}(x-k)|k \in \mathbb{N}_1, y-k \geq 0, x-k \geq 0\}\right). \quad (1)$$

Then $G_y(x)$ is the Sprague-Grundy value of the position (x, y) in Wythoff's Game, as defined in [1] and [3].

We give a matrix showing the first values of $G_y(x)$.

(x, y)	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	2	0	4	5	3	7	8	6	10	11	9	13	14
2	2	0	1	5	3	4	8	6	7	11	9	10	14	12
3	3	4	5	6	2	0	1	9	10	12	8	7	15	11

Wythoff showed that the positions, where $G_y(x) = 0$, are defined by

$$(x, y) = \left(\lfloor k\varphi \rfloor, \lfloor k\varphi^2 \rfloor\right) \text{ for all } k \in \mathbb{N}_0,$$

where φ is the golden ratio. That is, the zeroes approximately lie on the two diagonals emanating from the corner whose slopes equal φ (see [7]). However, the remaining values were considered to be chaotic (see [1]), until the proof by Pink was published. Indeed, let $\tilde{P}_y, \overline{p}_y$ be the preperiod length and the additive period length of G_y. Set

$$\mathcal{Y}_x = \{G_{y'}(x)|y' < y\} \cup \{G_{y-k}(x-k)|k \in \mathbb{N}_1, y-k \geq 0, x-k \geq 0\}.$$

With $p = \prod_{y'<y} \overline{p}_{y'}$, assuming $x \geq \max_{y'<y} \tilde{P}_{y'} + y$ we find $\mathcal{Y}_{x+p} = \mathcal{Y}_x + p$. We can now use expression (1) plus Theorem 1 to create an inductive proof which shows that each G_y is additively periodic, as in [2] or [3].

The following table shows the preperiod length and the period length for the first few rows.

y	$\tilde{P}_y$	$\overline{p}_y$	Difference period
0	0	1	[0]
1	0	3	[+1, +1, −2]
2	0	3	[+2, −1, −1]
3	8	6	[+2, +3, −2, −4, +3, −2]

◊

The lemmas in this introduction and their methods of proof are directly derived from Landman. As he uses a more narrow framework which only uses d implicitly, we give the proofs in full; the reader familiar with any of the papers cited here is free to read only the lemmas in this section and skip their proofs.

Lemma 1. *Let G be a Nim sequence over $(\mathcal{Y}_x)_{x=0}^{\infty}$. Then G is a bijective function, meaning that G is a permutation of $\mathbb{N}_0$.*

Proof. Injectivity of G follows from the definition of the mex operator and the requirement of the seed that $g_x \neq g_{x'}$ for $x \neq x'$.

Surjectivity of G: With $x, y \in \mathbb{N}_0$, there might be four causes why $G(x) \neq y$:

(i) $G(x) < y$; this can at most occur y times.

(ii) $y \in \mathcal{Y}_x$; this can at most occur $\sum_{j=0}^{p-1} \#\mathcal{Y}_j$ times.

(iii) With $x < L$, so $G(x)$ lies in the seed; this can at most occur L times.

(iv) $G(x') = y$ for some $x' < x$. As (i), (ii) and (iii) occurs only finitely many times, (iv) must happen.

□

Thus, we can imagine G as a greedy permutation that always chooses the smallest number not contained in $\mathcal{Y}_x$, as in [5].

Lemma 2. *Let G be a Nim sequence over $(\mathcal{Y}_x)_{x=0}^{\infty}$. Then there exists some $C \in \mathbb{N}_0$ such that*

$$\max_{x \geq C} d(x) \leq \max(0, \overline{M}).$$

Proof. Set $\hat{M} = \max(0, \overline{M})$. First, assume $d(x) \leq \hat{M}$ for all indices $x < L$ in the seed. If we assume there exists $x \in \mathbb{N}_0$ with $d(x) > \hat{M}$, there must exist $x_0 < x$ such that $G(x_0) = x + \hat{M}$, because $\hat{M} \notin d(\mathcal{Y}_x)$, and G is a greedy permutation. As G is bijective, it would be impossible that $G(x') > x + \hat{M}$ for all $x' > x$. So when we set $a = \min \left\{ a' \in \mathbb{N}_0 \mid G^{-1}(x + \hat{M} - a') > x \right\} > 0$, then $a > 0$ and well-defined.

For all $a' \in \{1, \ldots, a-1\}$, set $x_{a'} = G^{-1}(x + \hat{M} - a') < x$. A counting argument shows that there exists $a'' \in \{0, \ldots, a-1\}$ with $x_{a''} \leq x - a$. Also, as $G(x_{a'}) > x + \hat{M} - a \notin \{G(0), \ldots, G(x_{a'} - 1)\}$, as $G^{-1}(x + \hat{M} - a) > x$, we must have $x + \hat{M} - a \in \mathcal{Y}_{x_0} \cup \ldots \cup \mathcal{Y}_{x_{a-1}}$. Then

$$\begin{aligned} \max_{x \in \mathbb{N}_0}(d(\mathcal{Y}_x)) &< \hat{M} \\ &\leq x + \hat{M} - a - x_{a''} \\ &\in d(\mathcal{Y}_{x_{a''}}) \end{aligned}$$

which is a contradiction.

If $d(x) > \hat{M}$ for some $x < L$, we first sort the seed so $g_0 < g_1 < \cdots < g_{L-1}$; as the mex-operator depends on sets, this does not change the other values of G.

If $G(L) > g_{L-1}$, we must have $G(L) = g_{L-1} + 1$ due to the bounds on $(\mathcal{Y}_x)$; if $G(L) < g_{L-1}$, we have $d(L) < d(L-1) - 1$. In both cases, we add $G(L)$ to the seed, offset L by 1 and sort the seed again. Then, either $d(L-1)$ will remain the same, or $d(L-1)$ will decrease by 1. We continue this process of adding more elements to the seed, until $d(L-1) = \hat{M}$, which must happen because G is surjective. Now set $C = L$, and continue as in the first part of the proof. $\square$

Lemma 3. *Let G be a Nim sequence over $(\mathcal{Y}_x)_{x=0}^{\infty}$. Then there exists some $C \in \mathbb{N}_0$ with $C \leq \max_{x<L} g_x + 1$ such that*

$$\min_{x \geq C} d(x) \geq \min(0, \underline{M}).$$

Proof. Set $\hat{M} = \min(0, \underline{M})$. Assume that $x \in \mathbb{N}_0$ is the first index where $d(x) < \hat{M}$. As in the proof of Lemma 2, we can assume that the seed is sorted by size, so for all $x' < L$ we have $g_{x'} \geq x'$, so $x \geq L$. We have assumed $G(x-1) \geq x - 1 + \hat{M}$, so $G(x-1) > G(x)$. If $x > L$, then $G(x) \in \mathcal{Y}_{x-1}$ and

$$\begin{aligned}
\min_{x \in \mathbb{N}_0}(d(\mathcal{Y}_x)) &> \hat{M} \\
&\geq d(x) + 1 \\
&= x + d(x) - (x-1) \\
&\in d(\mathcal{Y}_{x-1})
\end{aligned}$$

which is a contradiction.

If $x = L$, we add $G(L)$ to the seed, offset L by 1 and sort the seed again. Then, $d(L-1)$ will decrease by 1. We continue this process until $G(L-1) < G(L)$, when we set $C = L$ and continue as in the first part of this proof.

This process could at most be repeated $\max_{x<L} g_x - L$ times, where L is the original size of the seed. So the maximal size of the increased seed is $L + \max_{x<L} g_x - L = C - 1$. $\square$

Corollary 1. *Let G be a Nim sequence over $(\mathcal{Y}_x)_{x=0}^{\infty}$. If G is additively periodic with preperiod length $\tilde{P}$, we have for all $x \geq \tilde{P}$*

$$\min(0, \underline{M}) \leq d(x) \leq \max(0, \overline{M}).$$

Proof. This follows, as the difference values in the period repeat themselves. $\square$

As we want d to be bounded by a negative $\underline{M}$ and a positive $\overline{M}$, we need to deal with the degenerate case when it is not.

Theorem 2. *Let G be a Nim sequence over $(\mathcal{Y}_x)_{x=0}^{\infty}$. If $\underline{M} \geq 0$ or $\overline{M} \leq 0$, then G is additively periodic with period length $\overline{p} = 1$.*

Proof. Assume $\overline{M} \leq 0$. By Lemma 2 we find $C \in \mathbb{N}_0$, so for all $x \geq C$ we have $G(x) \leq x$. As in the proofs of Lemma 2 and 3, we keep extending the seed by adding $G(x)$ and sorting it, until we reach an index L with $G(L) = L$. Then the seed will be equal to $\{0, \ldots, L-1\}$. By induction, we find that $G(x) = \text{mex}(\{0, \ldots, x-1\}) = x$ for all $x \geq L$, so the difference period becomes $[0]$.

The proof for $\underline{M} \geq 0$ is similar. □

In the next section, we will use the global bounds for d to prove the additive periodicity of G.

2. Periodicity Conditions

Throughout this section, $(\mathcal{Y}_x)_{x=0}^{\infty}$ stands for a sequence of additively periodic finite subsets of $\mathbb{Z}$ with period length p that defines the difference bounds $\underline{M} \in -\mathbb{N}_1, \overline{M} \in \mathbb{N}_1$. Also, G is a Nim sequence over $(\mathcal{Y}_x)$ with a seed of length L. We note that while p always stands for the smallest number that satisfies $\mathcal{Y}_x + p = \mathcal{Y}_{x+p}$, when we find $p' \in \mathbb{N}_1$ such that $G(x + p') = G(x) + p'$ for all $x \geq \tilde{P}'$ for some $\tilde{P}' \in \mathbb{N}_0$, then $p', \tilde{P}'$ may not be the true period / preperiod length. However, the true period length $\overline{p}$ of G will divide p'.

We begin with a simple lemma that contains an important definition.

Lemma 4 (Exclusion Lemma)**.** *Let (x, y) be an inversion of G; that is, $x < y$ with $G(x) > G(y)$. Then for all $a \in \mathbb{Z}$ with $x + ap \geq L$, we have*

$$d(x + ap) \neq G(y) - x.$$

We say that (the difference value) $G(y) - x$ is excluded *at the index x (mod p).*

Proof. As G is a greedy permutation, we must have $G(y) \in \mathcal{Y}_x$. Then $d(x + ap) \notin d(\mathcal{Y}_{x+ap}) = d(\mathcal{Y}_x) \ni G(y) - x$. □

We mention two simple facts. First, the excluded value is bounded by $\underline{M} < G(y) - x < \overline{M}$. Second, the excluded value can be calculated without knowing the values of y and $G(x)$.

We now define for any $x \in \mathbb{N}_0$ the two sets of indices where the permutation G ascends above x and where it descends below x.

Definition 5. Let $x \in \mathbb{N}_0$. We define the *cut after x* as the pair of sets (S_x, T_x), where

$$\begin{aligned} S_x &= \left\{ x' \in \mathbb{N}_0 \mid x' < x + \frac{1}{2}, G(x') > x + \frac{1}{2} \right\}, \\ T_x &= \left\{ x' \in \mathbb{N}_0 \mid x' > x + \frac{1}{2}, G(x') < x + \frac{1}{2} \right\}. \end{aligned}$$

With each cut, we associate the sets $S_x^* = G(T_x)$ and $T_x^* = G(S_x)$.

For $x, y \in \mathbb{N}_0$, we call the sets S_x, S_y *equivalent*, written $S_x \sim S_y$ if and only if $d(S_x) = d(S_y)$.

Remark 2. If $x' \in S_x$, then $x' + d(x') > x + \frac{1}{2}$. As $d(x') \leq \overline{M}$ (for x sufficiently large), we have $x' > x - \overline{M}$, so $x' \in \{x - \overline{M} + 1, \ldots, x\}$. So $\#S_x \leq \overline{M}$, and similarly $\#T_x \leq |\underline{M}|$. Since G is bijective, we see that

$$\#S_x = \#T_x = \#S_x^* = \#T_x^* \leq \min(|\underline{M}|, \overline{M}) < \infty.$$

Lemma 5. *Let $x, y \in \mathbb{Z}$. Define a relation $\sim$ on subsets S_x, S_y of Z, indexed by x and y, by $S_x \sim S_y$ if and only if $d(S_x) = d(S_y)$. Then $\sim$ is an equivalence relation.*

Proof. This follows, as $=$ is an equivalence relation. □

We can of course use $\sim$ to compare two sets with indices $x, y \in \mathbb{N}_0$.

Lemma 6. *Let $x \in \mathbb{N}_0$ with $x \geq L$. Then it is possible to calculate $G(x+1)$, S_{x+1}^* and T_{x+1}^* given only S_x^*, T_x^* and $\mathcal{Y}_{x+1}$.*

Proof. From the definitions of S_x^* and T_x^* and the global bounds of d, we know that

$$G(x+1) \in S_x^* \cup \left(\left\{x+1, \ldots, x+\overline{M}\right\} \setminus T_x^*\right) \cup \left\{x+1+\overline{M}\right\}.$$

So if $G(x+1) < x+1$, then $G(x+1) \in S_x^*$ with $G(x+1) \geq x+1+\underline{M}$; and if $G(x+1) \geq x+1$, then $G(x+1) \notin T_x^*$ with $G(x+1) \leq x+1+\overline{M}$. From the definition of the mex operator, $G(x+1)$ is the smallest element in this union of sets not excluded by $\mathcal{Y}_{x+1}$, that is,

$$G(x+1) = \operatorname{mex}\left(\{0, 1, \ldots, x+\underline{M}\} \cup (\{x+\underline{M}+1, \ldots, x\} \setminus S_x^*) \cup T_x^* \cup \mathcal{Y}_{x+1}\right).$$

We now find S_{x+1}^*. Initially, we set $S_{x+1}^* = S_x^*$. If $d(x+1) < 0$, then $G(x+1) \in S_x^*$. Then we must remove $G(x+1)$ from S_{x+1}^*, but all other elements of S_x^* will be contained in S_{x+1}^*. If $x+1 \notin T_x^*$ and $d(x+1) \neq 0$, we must have $G^{-1}(x+1) > x+1$, and in this case we adjoin $x+1$ to S_{x+1}^*.

Also, we initially set $T_{x+1}^* = T_x^*$. If $x+1 \in T_x^*$, then $G^{-1}(x+1) < x+1$. So we remove $x+1$, and all other elements of T_x^* will be contained in T_{x+1}^*. If $d(x+1) > 0$, we adjoin $G(x+1)$ to T_{x+1}^*. □

Lemma 7. *Let $R \in \mathbb{N}_1$. Then G is additively periodic with period length $\overline{p}$ that divides Rp if and only if there exists $x \geq L$ such that $S_x^* \sim S_{x+Rp}^*$ and $T_x^* \sim T_{x+Rp}^*$.*

Proof. If G is additively periodic, it is trivial that $S_x^* \sim S_{x+k\overline{p}}^*$ and $T_x^* \sim T_{x+k\overline{p}}^*$ for all $x \geq \tilde{P}$ and $k \in \mathbb{N}_0$. So we choose $k \in \mathbb{N}_0$ such that $k\overline{p} = Rp$.

Assume instead the equivalence conditions hold for the four sets for some $x \geq L$. For simplicity, set $R = 1$; if $R > 1$, you can replace p with Rp in the following argument.

First, we use the expression from the proof of Lemma 6 to calculate

$$\begin{aligned}
G(x+1+p) &= \operatorname{mex}\big(\{0,\ldots,x+p+\underline{M}\} \\
&\quad \cup \big(\{x+p+\underline{M}+1,\ldots,x+p\} \setminus S^*_{x+p}\big) \\
&\quad \cup T^*_{x+p} \cup \mathcal{Y}_{x+p+1}\big) \\
&= \operatorname{mex}\big(\{0,\ldots,x+p+\underline{M}\} \cup (\{x+\underline{M}+1,\ldots,x\} \setminus S^*_x) + p \\
&\quad \cup (T^*_x + p) \cup (\mathcal{Y}_{x+1} + p)\big) \\
&= \operatorname{mex}\big(\{0,\ldots,x+\underline{M}\} \cup (\{x+\underline{M}+1,\ldots,x\} \setminus S^*_x) \\
&\quad \cup T^*_x \cup \mathcal{Y}_{x+1}\big) + p \\
&= G(x+1) + p,
\end{aligned}$$

which means that $d(x+1+p) = d(x+1)$.

Again from Lemma 6, we know how S^*_{x+p+1} and T^*_{x+p+1} are created by removing and adjoining elements from respectively S^*_{x+p} and T^*_{x+p}. So when $d(x+1+p) < 0$, we remove $G(x+1+p) = G(x+1)+p$ from S^*_{x+p}; but then $d(x+1) < 0$, and $G(x+1)$ will be removed from S^*_x to create S^*_{x+1}. When $x+1+p \notin T^*_{x+p}$ and $d(x+1) \neq 0$, we adjoin $x+1+p$ to S^*_{x+p+1}; but then $x+1 \notin T^*_x$, so $x+1$ is adjoined to S^*_{x+1}. It follows that $S^*_{x+p+1} \sim S^*_{x+1}$.

Similarly, when $x+p+1 \in T^*_{x+p}$, then $x+1 \in T^*_x$, and we remove $x+p+1$ from T^*_{x+p} to create T^*_{x+p+1}, just as we remove $x+1$ from T^*_x to create T^*_{x+1}. When $d(x+p+1) = d(x+p) > 0$, we adjoin $G(x+p+1)$ to T^*_{x+p+1}, and $G(x+1)$ to T^*_{x+1}. So $T^*_{x+1+p} \sim T^*_{x+1}$.

We continue by induction, and show for $k \in \mathbb{N}_0$ that $d(x+p+k) = d(x+k)$ with $S^*_{x+p+k} \sim S^*_{x+k}$, $T^*_{x+p+k} \sim T^*_{x+k}$. It follows that G is additively periodic. □

Example 3. We demonstrate an algorithm outlined in [3] to efficiently calculate the difference values and find the value of R from Lemma 7. The algorithm uses a Boolean array with M elements to represent the values of $(S^*_x \cup T^*_x) - x$. These elements are indexed by $i \in \{\underline{M}+1,\ldots,\overline{M}\}$. The element at index i is `True` if and only if $x+i \in S^*_x \cup \big(\{x+1,\ldots,x+\overline{M}\} \setminus T^*_x\big)$. Using this array and the elements of $(d(\mathcal{Y}_x))_{x=0}^{p-1}$, we can now calculate $d(x+1)$ using Lemma 7.

As an example, set $\underline{M} = -4$, $\overline{M} = 3$, and let the Boolean array have the values [`T` for `True`, `F` for `False`]

Indices:	-3	-2	-1	0	1	2	3
Values:	F	F	T	T	F	T	F

.

We read from the array that $d(S_x^*) = \{-1, 0\}$ and $d(T_x^*) = \{1, 3\}$. It follows that $d(x+1) \in \{-2, -1, 1, 3\}$.

To calculate $d(x+1)$, we need to know $d(\mathcal{Y}_{x+1}) = d\left(\mathcal{Y}_{(x+1) \bmod p}\right)$. Suppose $d\left(\mathcal{Y}_{(x+1) \bmod p}\right) = \{-2, -1\}$. Then $d(x+1) = 1$. We adjoin 2 to $d(T_x^*)$, and remove 1 from $d(T_x^*)$. Then we left-shift the array, accounting for the index change from S_x^*, T_x^* to S_{x+1}^*, T_{x+1}^*. The new element adjoined to the end of the array will be `True`, as $3 \notin d\left(T_{x+1}^*\right)$. In our example, the new values of the array will be

Values:	F	T	T	F	F	F	T

.

Suppose instead $d\left(\mathcal{Y}_{(x+1) \bmod p}\right) = \{-3, -2\}$. Then $d(x+1) = -1$. We must then flip the elements at indices 0 and 1, and the new values after the left shift will be

Values:	F	T	F	F	T	F	T

.

We store the previous values of the array. When we find $x \geq L$, $R \in \mathbb{N}_1$ where the stored values are identical after x and $x' = x + Rp$, we know that $S_x^* \sim S_{x+Rp}^*$ and $T_x^* \sim T_{x+Rp}^*$, so the condition of Lemma 6 is fulfilled; since there are 2^M possible values for the bits in the array, this eventually occurs by the pigeonhole principle. So 2^M is an upper bound for $K_{\underline{M},\overline{M}}$, as given by Landman.

As $\#S_x^* = \#T_x^*$, we have a constraint on the values of the array, so this bound can certainly be improved. We defer the proof until the end of this section. $\diamondsuit$

Lemma 8. *Let $x \in \mathbb{N}_0$ with $x \geq L$. Then it is possible to calculate T_{x+1} and T_{x+1}^* given only T_x, T_x^* and $(\mathcal{Y}_{x'})_{x'=x+1}^{\infty}$. Also:*

(a) If $x+1 \notin T_x$, we are able to calculate $G(x+1)$;

(b) If $x+1 \notin T_x^*$, we are able to find $x' \in T_{x+1} \cup \{x+1\}$ with $G(x') = x+1$.

Proof. In case (a), we must have $G(x+1) \geq x+1$. So we calculate

$$G(x+1) = \operatorname{mex}\left(\{0, \ldots, x\} \cup T_x^* \cup \mathcal{Y}_{x+1}\right).$$

In case (b), we must have $G^{-1}(x+1) \geq x+1$. We set

$$x' = \min\left\{y \geq x+1 \mid y \notin T_x, x+1 \notin \mathcal{Y}_y\right\}.$$

As $x' \notin T_x$, we have $G(x') \geq x+1$, so we need to show $G(x') \leq x+1$.

Assume that $G(x') > x+1$. As $x+1 \notin \mathcal{Y}_{x'}$, there must exist $x'' < x'$ with $G(x'') = x+1$; but then $x'' \notin T_x$ with $x+1 \notin \mathcal{Y}_{x''}$, which is impossible due to the definition of x' as a minimum. Thus, $G(x') = x+1$.

We now find T_{x+1}. Initially, we set $T_{x+1} = T_x$. If $x+1 \in T_x$, we must remove $x+1$ from T_{x+1}, but all other elements remain. If $x+1 \notin T^*_x$, and we have found $x' \neq x+1$, we adjoin x' to T_{x+1}.

Also, we initially set $T^*_{x+1} = T^*_x$. If $x+1 \in T^*_x$, we must remove $x+1$ from T^*_{x+1}, but all other elements remain. If $x+1 \notin T_x$, and we have found $d(x+1) > 0$, we adjoin $G(x+1)$ to T^*_{x+1}. □

Lemma 9. *Let $R \in \mathbb{N}_1$. Then G is additively periodic with period length $\overline{p}$ that divides Rp if and only if there exists $x \geq L$ such that $T_x \sim T_{x+Rp}$ and $T^*_x \sim T^*_{x+Rp}$.*

Proof. If G is additively periodic, it is trivial that $T_x \sim T_{x+k\overline{p}}$ and $T^*_x \sim T^*_{x+k\overline{p}}$ for all $x \geq \tilde{P}$ and $k \in \mathbb{N}_0$. Assume instead the equivalence conditions hold for some $x \geq L$. Set $R = 1$; for $R > 1$, replace p with Rp in the following argument.

First, if $x+1 \notin T_x$, by assumption we have $x+1+p \notin T_{x+p}$. Using Lemma 8, we calculate

$$\begin{aligned} G(x+1+p) &= \operatorname{mex}\left(\{0,\ldots,x+p\} \cup T^*_{x+p} + \mathcal{Y}_{x+p}\right) \\ &= \operatorname{mex}\left(\{0,\ldots,x\} \cup T^*_x + \mathcal{Y}_x\right) + p \\ &= G(x+1) + p \end{aligned}$$

Second, if $x+1 \notin T^*_x$, we use Lemma 8 to find $x' \in T_{x+1}$ with $G(x') = x+1$. Using our expression for x' from the proof of the lemma, we find that

$$\begin{aligned} x' + p &= \min\{y \geq x+1 \mid y \notin T_x, x+1 \notin \mathcal{Y}_y\} + p \\ &= \min\{y \geq x+1+p \mid y \notin T_x + p, x+1 \notin \mathcal{Y}_y + p\} \\ &= \min\{y \geq x+1+p \mid y \notin T_{x+p}, x+1 \notin \mathcal{Y}_{y+p}\}. \end{aligned}$$

As $x+1+p \notin T^*_{x+p}$ by assumption, we realize that

$$G(x'+p) = x+1+p = G(x') + p.$$

By Lemma 8, we know how T_{x+1+p} and T^*_{x+1+p} are created by removing and adjoining elements from respectively T_{x+p} and T^*_{x+p}. So when $x+1+p \in T_{x+p}$, we remove $G(x+1+p) = G(x+1)+p$ from T_{x+p}; but then $x+1 \in T_x$, and $G(x+1)$ will be removed from T_x to create T_{x+1}. When $x+1+p \notin T^*_{x+p}$, and we have found $x'+p \neq x+1+p$, we adjoin $x'+p$ to T_{x+p+1}; but then $x+1 \notin T^*_x$ as well as $x' \neq x+1$, so x' is adjoined to T_{x+1}. It follows that $T_{x+1+p} \sim T_{x+1}$.

Similarly to the proof of Lemma 7, we show that $T^*_{x+1+p} \sim T^*_{x+1}$.

We continue by induction, noting that for all $x' > x + |\underline{M}|$ we will calculate $G(x') = G(x'-p) + p$ either while examining $T_{x'-1}, T^*_{x'-1}$ if $d(x') \geq 0$, or while examining $T_{G(x')-1}, T^*_{G(x')-1}$ if $d(x') < 0$. It follows that G is additively periodic. □

We could use Lemma 9 to create an algorithm similar to Example 3 to find the difference values. There is no clear benefit, as the difference values would be calculated out of order. In Section 5, we will come back to this idea of using a binary representation of (T_x, T^*_x).

Lemma 10. *Let $R \in \mathbb{N}_1$ and $x \geq L$. Then:*

(a) If $S_x \sim S_{x+Rp}$, then $T^*_x \sim T^*_{x+Rp}$; in fact, we have $G(x') + Rp = G(x' + Rp)$ for all $x' \in S_x$.

(b) If $S^*_x \sim S^*_{x+Rp}$, then $T_x \sim T_{x+Rp}$; in fact, we have $G^{-1}(y') + Rp = G^{-1}(y' + Rp)$ for all $y' \in S^*_x$.

Proof. Set $R = 1$; for $R > 1$, replace p with Rp in the following argument.

To prove (a), let $N = \#S_x$, and let $\{x_1, \ldots, x_N\} = S_x$ be in ascending order. As $G(x_1) > x + 1$, we have $G(x_1) + p > x_1 + p$. First assume $G(x_1) + p < G(x_1 + p)$, so $G(x_1) + p > x + p$. If $G(x_1) + p \notin T^*_{x+p}$, this means $G^{-1}(G(x_1) + p) > x_1 + p$. If $G(x_1) + p \in T^*_{x+p}$, its inverse must belong to $S_x \setminus \{x_1\}$. So $G^{-1}(G(x_1) + p) \in \{x_2 + p, \ldots, x_N + p\} \cup \{x + p + 1, \ldots\}$, which means we have an inversion $(x_1 + p, G^{-1}(G(x_1) + p))$ that excludes $G(x_1) + p - (x_1 + p) = d(x_1)$ at the index $x_1 + p$. Subtract p from the index to see that $d(x_1)$ is excluded at its own index, which is a contradiction by Lemma 4.

If $G(x_1) + p > G(x_1 + p)$, we find an inversion $(x_1, G^{-1}(G(x_1 + p) - p))$ in the same manner. Here, $G(x_1 + p) - p - x_1 = d(x_1 + p)$ is excluded at the index x_1, also a contradiction.

As we now have $G(x_1)+p = G(x_1+p)$, let us look at x_2. If $G(x_2)+p < G(x_2+p)$, we would have $G^{-1}(G(x_2)+p) \in \{x_3+p, \ldots, x_N+p\} \cup \{x+p+1, \ldots\}$, and we would find an impossible exclusion. The same thing happens if $G(x_2) + p > G(x_2 + p)$. So $G(x_2) + p = G(x_2 + p)$, and we continue inductively for each $n \in \{3, \ldots, N\}$ to show $G(x_n) + p = G(x_n + p)$, which implies $T^*_x \sim T^*_{x+p}$.

To prove (b), let $\{y_1, \ldots, y_N\} = S^*_x$ be in ascending order and continue using similar ideas. For instance, if $G^{-1}(y_1)+p < G^{-1}(y_1+p)$, then $(G^{-1}(y_1)+p, G^{-1}(y_1+p))$ would be an inversion that excludes $G(G^{-1}(y_1 + p)) - (G^{-1}(y_1) + p) = y_1 - G^{-1}(y_1) = d(G^{-1}(y_1))$ at the index $G^{-1}(y_1) + p$. □

Note that the reverse implications of (a) and (b) are not true in general.

Corollary 2. *Let $R \in \mathbb{N}_1$. Then G is additively periodic with period length $\overline{p}$ that divides Rp if and only if there exists $x \geq L$ such that $S_x \sim S_{x+Rp}$ and one of the following statements holds:*

(a) $T_x \sim T_{x+Rp}$*; or*

(b) $S^*_x \sim S^*_{x+Rp}$*.*

Proof. Combine Lemma 10 with Lemma 9 for (a), and with Lemma 7 for (b). □

We now use the idea behind the algorithm in Example 3 to give a formal proof of the additive periodicity of G. This result was originally shown by Dress, Flammenkamp and Pink in [2], which used a different proof technique.

Theorem 3. *Any Nim sequence G is additively periodic, and the length $\overline{p}$ of its period is bounded by:*

$$\overline{p} \leq \binom{M}{\min\left(|\underline{M}|, \overline{M}\right)} p.$$

Proof. Let $x \in \mathbb{N}_0$ be sufficiently large so the bounds from Lemma 2 and 3 hold. Then look at the sets $(d(S_{x+rp}), d(T_{x+rp}))_{r=0}^{\infty}$. By the pigeonhole principle, there exists $r, r' \in \mathbb{N}_0$, $r < r'$ such that $d(S_{x+rp}) = d(S_{x+r'p})$ and $d(T_{x+rp}) = d(T_{x+r'p})$, when G becomes additively periodic with period length less or equal to $(r' - r)p$ by Corollary 2.

The maximal value of $r' - r$ can be calculated by counting all possibilites for $(d(S_x), d(T_x))$. By taking a sum over $i = \#S_x = \#T_x$ and using the bounds from Remark 2, we get

$$r' - r \leq \sum_{i=0}^{\min\left(|\underline{M}|, \overline{M}\right)} \binom{\overline{M}}{i} \binom{|\underline{M}|}{i} = \binom{M}{\min\left(|\underline{M}|, \overline{M}\right)}$$

using the Chu-Vandermonde identity. □

Note that we could replace all references to (S_x, T_x) in this proof with any one of the pairs $(S^*_x, T_x), (S_x, T^*_x), (S^*_x, T^*_x)$ and still reach the same result.

3. Cut Sets

To move beyond the pigeonhole proofs, we need to create a framework where cuts exist without reference to the permutation G or the sets $(\mathcal{Y}_x)$, so $x \in \mathbb{Z}$ becomes a dummy variable.

Definition 6. Let $x \in \mathbb{Z}$ and $S_x, T_x \subseteq \mathbb{Z}$, and let $d : S_x \cup T_x \to \mathbb{Z}$. Then we define $\mathcal{C}_x = (S_x, T_x, d)$ as a *cut set* if and only if these four requirements hold:

(a) For all $y \in S_x$, we have $y < x + \frac{1}{2}$ and $y + d(y) > x + \frac{1}{2}$.

(b) For all $y \in T_x$, we have $y > x + \frac{1}{2}$ and $y + d(y) < x + \frac{1}{2}$.

(c) $\#S_x = \#T_x < \infty$.

(d) For all $y, y' \in S_x \cup T_x$, we have $y + d(y) = y' + d(y')$ if and only if $y = y'$.

With each cut set, we associate the sets

$$S_x^* = \{y + d(y) | y \in T_x\}, \quad T_x^* = \{y + d(y) | y \in S_x\}.$$

Lemma 11. *Let $x, \tilde{x} \in \mathbb{Z}$. Define a relation between two cut sets $\mathcal{C}_x$ and $\tilde{\mathcal{C}}_{\tilde{x}} = \left(\tilde{S}_{\tilde{x}}, \tilde{T}_{\tilde{x}}, \tilde{d}\right)$ by $\mathcal{C}_x \sim \tilde{\mathcal{C}}_{\tilde{x}}$ if and only if $T_x \sim \tilde{T}_{\tilde{x}}$, $T_x^* \sim \tilde{T}_{\tilde{x}}^*$, and $d(y) = \tilde{d}(y - x + \tilde{x})$ for all $y \in S_x \cup T_x$. Then $\sim$ is an equivalence relation.*

Proof. For all $x \in \mathbb{Z}$ we see that $\mathcal{C}_x \sim \mathcal{C}_x$, as $\sim$ defined on subsets is an equivalence relation. With $\tilde{x} \in \mathbb{Z}$ and $\mathcal{C}_x \sim \tilde{\mathcal{C}}_{\tilde{x}}$, assume $y \in \tilde{S}_{\tilde{x}} \cup \tilde{T}_{\tilde{x}}$. Then, we have $y - \tilde{x} + x \in S_x \cup T_x$ with

$$\tilde{d}(y) = \tilde{d}((y - \tilde{x} + x) - x + \tilde{x}) = d(y - \tilde{x} + x),$$

so $\tilde{\mathcal{C}}_{\tilde{x}} \sim \mathcal{C}_x$.

When $\overline{C}_{\overline{x}} = \left(\overline{S}_{\overline{x}}, \overline{T}_{\overline{x}}, \overline{d}\right)$, assuming $\mathcal{C}_x \sim \tilde{\mathcal{C}}_{\tilde{x}} \sim \overline{\mathcal{C}}_{\overline{x}}$, transitivity is shown by

$$d(y) = \tilde{d}(y - x + \tilde{x}) = \overline{d}((y - x + \tilde{x}) - \tilde{x} + \overline{x}) = \overline{d}(y - x + \overline{x}).$$

□

Each element $\mathcal{C}_x$ in an equivalence class $[\mathcal{C}_0]$ will correspond to an $x \in \mathbb{Z}$.

We now need to replace the sets $(\mathcal{Y}_x)$. For this, we will use exclusions.

Definition 7. Let $\mathcal{C}_x = (S_x, T_x, d)$ be a cut set.

(i) Let $x' \in S_x$. For all $x'' \in S_x$ with $x' < x''$ and $x' + d(x') > x'' + d(x'')$, the difference value $x'' + d(x'') - x'$ is *excluded* at the index x', and for all $y \in \{x + 1, \ldots, x' + d(x') - 1\} \setminus T_x^*$, the value $y - x'$ is excluded at the index x'. These are the *positive exclusions*.

(ii) Let $y'' \in T_x$. For all $y' \in \{x + 1, \ldots, y'' - 1\}$ with either $y' \notin T_x$, or $y' \in T_x$ with $y' + d(y') > y'' + d(y'')$, the difference value $y'' + d(y'') - y'$ is excluded at the index y'. These are the *negative exclusions*.

(iii) We say there is a *possible zero* (in difference value) after the cut if and only if $x + 1 \notin T_x \cup T_x^*$. We say there is a possible zero before the cut if and only if $x \notin S_x \cup S_x^*$.

We cannot find exlusions of the value 0 by looking at one cut set only. For this, we need to have two cut sets in succession.

Definition 8. Let $\mathcal{C}_x, \tilde{\mathcal{C}}_{\tilde{x}}$ be two cut sets. We call $\tilde{\mathcal{C}}_{\tilde{x}}$ a *direct successor* of $\mathcal{C}_x$ if and only if there exists a cut set $\mathcal{C}_{x+1} = \left(S_{x+1}, T_{x+1}, \tilde{d}\right)$, which fulfills $\mathcal{C}_{x+1} \sim \tilde{\mathcal{C}}_{\tilde{x}}$ and the following condition, where we initially set $S_{x+1} = S_x \setminus \{x' \in S_x \mid x' + d(x') = x+1\}$ and $T_{x+1} = T_x \setminus \{x+1\}$. Then we adjoin one or zero indices to S_{x+1} and/or T_{x+1}, based on these criteria:

(i) With $x+1 \in T_x \cap T_x^*$, we cannot adjoin any indices.

(ii) With $x+1 \in T_x^* \setminus T_x$, we adjoin $x+1$ to S_{x+1}, and $\tilde{d}(x+1) > 0$ can be set to any value where $x+1+\tilde{d}(x+1) \notin T_x^*$.

(iii) With $x+1 \in T_x \setminus T_x^*$, we adjoin $x' \in \{y \in \mathbb{Z} \,|\, y > x+1\} \setminus T_x$ to T_{x+1}, and set $\tilde{d}(x') = x' - (x+1)$.

(iv) With $x+1 \notin T_x \cup T_x^*$, meaning that $\mathcal{C}_x$ has a possible zero after the cut, we have two options:

(ivA) We adjoin $x+1$ to S_{x+1} and a new index x' to T_{x+1} such that $\tilde{d}(x+1)$ fulfills the conditions from (iii), and x' fulfills the condtions from (iv). Then, we say that the value 0 is *excluded* at the index $x+1$.

(ivB) We add nothing, meaning that $\mathcal{C}_{x+1}$ has a possible zero before the cut. Then, we say that $(\mathcal{C}_x, \mathcal{C}_{x+1})$ has a *matching zero.*

We recognize this process of removing and adjoining values from the proofs of Lemmas 6 and 8. Note that these criteria will guarantee that $\#S_{x+1} = \#T_{x+1}$.

To handle exclusions, we need to look at collections of cut sets.

Definition 9. Let $x \in \mathbb{Z}$ and $R \in \mathbb{N}_1$. Then we define $\mathcal{C}_x = (\mathcal{C}_{x,r})_{r=0}^{R-1}$ as a *cut set with R rows*, if for all $r, r' \in \{0, \ldots, R-1\}$, where $\mathcal{C}_{x,r} = (S_{x,r}, T_{x,r}, d_r)$ is a cut set, we have:

(a) $T_{x,r} \sim T_{x,r'}$ and $T_{x,r}^* \sim T_{x,r'}^*$ if and only if $r = r'$.

(b) If the difference value $k \in \mathbb{Z}$ is excluded at the index x' in $\mathcal{C}_{x,r}$, then $x' \in S_{x,r'} \cup T_{x,r'}$ implies that $d_{r'}(x') \neq k$.

We say that two cut sets $\mathcal{C}_x$ and $\tilde{\mathcal{C}}_{\tilde{x}} = \left(\tilde{\mathcal{C}}_{\tilde{x},r}\right)_{r=0}^{R-1}$ with the same number of rows are *equivalent*, written $\mathcal{C}_x \sim \tilde{\mathcal{C}}_{\tilde{x}}$ if and only if $\mathcal{C}_{x,r} \sim \tilde{\mathcal{C}}_{\tilde{x},r}$ for all $r \in \{0, \ldots R-1\}$.

Definition 10. Let $\mathcal{C}_x, \tilde{\mathcal{C}}_{\tilde{x}}$ be two cut sets with R rows. We call $\tilde{\mathcal{C}}_{\tilde{x}}$ a *direct successor* of $\mathcal{C}_x$ if and only if there exists a cut set $\mathcal{C}_{x+1} = (\mathcal{C}_{x+1,r})_{r=0}^{R-1}$ with $C_{x+1,r} = \left(S_{x+1,r}, T_{x+1,r}, \tilde{d}_r\right)$, which fulfills $\mathcal{C}_{x+1} \sim \tilde{\mathcal{C}}_{\tilde{x}}$ and the following conditions for all $r, r' \in \{0, \ldots, R-1\}$:

(a) $C_{x+1,r}$ is a direct successor of $C_{x,r}$.

(b) If $(C_{x,r}, C_{x+1,r})$ excludes the difference value 0 at the index $x+1$, then $(C_{x,r'}, C_{x+1,r'})$ cannot have a matching zero.

Definition 11. Let $R \in \mathbb{N}_1$, $\underline{M} \in -\mathbb{N}_1$ and $\overline{M} \in \mathbb{N}_1$. Then $\Gamma(R, \underline{M}, \overline{M}) = \big(V(R, \underline{M}, \overline{M}), E(R, \underline{M}, \overline{M})\big)$ is the digraph, where the vertices $V(R, \underline{M}, \overline{M})$ are all cut sets $\mathcal{C}_x = (S_{x,r}, T_{x,r}, d_r)_{r=0}^{R-1}$ with R rows which fulfill

$$\bigcup_{r=0}^{R-1} d_r\left(S_{x,r} \cup T_{x,r}\right) \subseteq \left\{\underline{M}, \ldots, \overline{M}\right\}.$$

The edges $E(R, \underline{M}, \overline{M})$ are defined by the direct successors from Definition 10. With $J \in \mathbb{N}_0$, we call $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_J$ for a *path* in $\Gamma(R, \underline{M}, \overline{M})$ if and only if $(\mathcal{C}_j, \mathcal{C}_{j+1}) \in E(R, \underline{M}, \overline{M})$ for all $j \in \{0, \ldots, J-1\}$. We also use the terms *ancestor* and *successor*, defined in the standard way for a digraph.

We will usually abbreviate the notation as $\Gamma = (V, E)$.

We think of a cut set as an $R \times M$ matrix of difference values. The entries with undefined difference functions are left blank. A path with J elements can be considered as an $R \times (M+J-1)$ matrix that defines R common difference functions $(d_r)_{r=0}^{R-1}$, one function for each row. For all indices j in the path where there is a matching zero in the rth row after the cut, we define $d_r(j) = 0$.

Example 4. Here is an example with a cut set $\mathcal{C}_0$ and a possible direct successor $\mathcal{C}_1$ in $\Gamma(3, -2, 3)$:

$$\mathcal{C}_0 : \begin{array}{ccc|cc} +3 & \_\_ & \_\_ & -1 & \_\_ \\ \_\_ & \_\_ & +2 & -2 & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ & \_\_ \end{array} \quad \hookrightarrow \quad \mathcal{C}_1 : \begin{array}{ccc|cc} \_\_ & \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & +2 & \_\_ & -1 & \_\_ \\ \_\_ & \_\_ & +2 & -1 & \_\_ \end{array}$$

(The vertical lines mark the cut between positive and negative difference values.)

◊

Example 5. Let $R \in \mathbb{N}_1$, and let us look at the digraph $\Gamma(R, -2, 2)$. It follows from Theorem 3 and Definition 9 (a) that if $R > \binom{|-2|+2}{2} = 6$, then the digraph is empty. Here, we set $R = 6$ and construct an infinite path in Γ with a cycle of four cut sets $\mathcal{C}_0, \ldots, \mathcal{C}_3, \mathcal{C}_4 \sim \mathcal{C}_0$:

$$\begin{array}{cc|cccc|cc} +2 & +2 & -2 & -2 & +2 & +2 & -2 & -2 \\ \_\_ & +2 & +2 & -2 & -2 & +2 & \_\_ & -2 \\ \_\_ & \_\_ & +2 & +2 & -2 & -2 & \_\_ & \_\_ \\ +2 & \_\_ & -2 & +2 & +2 & -2 & -2 & \_\_ \\ \_\_ & +2 & -2 & +2 & -2 & +2 & -2 & \_\_ \\ +2 & \_\_ & +2 & -2 & +2 & -2 & \_\_ & -2 \end{array}$$

(The first vertical line marks the end of $S_{0,r}$, and the second line marks the beginning of $T_{4,r}$.)

It can be shown that these four cut sets form a *connected component*, and that all connected components in $\Gamma(6,-2,2)$ consist of exactly four cut sets, which may be created by applying the same permutation of the six rows to our four cut sets. Note that the path has two separate row cycles, one 4-cycle which cycles the top four rows, and one 2-cycle which cycles the bottom two rows. This implies that a Nim sequence with $\underline{M} = -2$, $\overline{M} = 2$ and a difference period of length $6p$ cannot exist, as we will show. ♢

Definition 12. Let $\mathcal{C}_x, \tilde{\mathcal{C}}_{\tilde{x}}$ be two cut sets with R rows. We call $\tilde{\mathcal{C}}_{\tilde{x}}$ the *cycled cut set of* $\mathcal{C}_x$ if and only if for all $r \in \{0,\ldots,R-1\}$

$$\mathcal{C}_{x,r} \sim \tilde{\mathcal{C}}_{\tilde{x},\pi(r)},$$

where π is the row permutation that cycles all rows of $\mathcal{C}_x$ upwards, defined by $\pi(r) = (r-1) \bmod R$.

We now establish the connection between this abstract setup and the Nim sequence G. We need the additive period length of G to be a multiple of p, which explains the clumsy definition of R in the following lemma.

Lemma 12. *Let G be a Nim sequence over $(\mathcal{Y}_x)$ that has additive period length p. Let G have difference function d, preperiod length $\tilde{P}$ and additive period length $\overline{p}$, and let (S_x, T_x) be the cut after $x \in \mathbb{N}_0$ defined by G. Set $R = \frac{\operatorname{lcm}(p,\overline{p})}{p}$. Then there exists a path of cut sets $\mathcal{C}_0,\ldots,\mathcal{C}_p$ with R rows, where $\mathcal{C}_j = (S_{j,r}, T_{j,r}, d_r)_{r=0}^{R-1}$, which fulfills:*

(a) $\mathcal{C}_p$ is the cycled cut set of $\mathcal{C}_0$;

(b) For all $x \geq \tilde{P}$, where $j \in \{0,\ldots,p-1\}$, $r \in \{0,\ldots,R-1\}$, $C \in \mathbb{N}_0$ is uniquely defined by $x = j + rp + CRp$, we have:

$$S_{j,r} \sim S_x \ , \ T_{j,r} \sim T_x \ ;$$

(c) With $k \in \mathbb{Z}$, where $j + k \in S_{j,r} \cup T_{j,r}$, we have $d_r(j+k) = d(x+k)$.

Proof. Set $x' = \left(\left\lceil \frac{\tilde{P}}{p} \right\rceil + C'\right) p$ for some sufficiently large $C' \in \mathbb{N}_0$. Write the difference period of G, as it begins at the index x', in a $R \times p$ matrix. Extend the matrix to the left with the elements of $d(S_{x'+rp})$ and to the right with the elements of $d(T_{x'+p-1+rp})$ in each row with index r, as in Example 5, so the path $\mathcal{C}_0,\ldots,\mathcal{C}_{p-1}$ will be represented by the matrix. Conditions (a) and (b) now follow directly. □

Example 6. We demonstrate this transformation from Nim sequence to path. In Example 2, G_3 is a Nim sequence over a sequence $(\mathcal{Y}_x)$. We see that G_3 has period length 6, and $(\mathcal{Y}_x)$ has period length $\operatorname{lcm}(1,3,3) = 3$. Thus, $R = 6/3 = 2$. We write the difference period of G_3 in a 2×3 matrix, extended to both sides, as follows:

$$\begin{array}{ccc|ccc|cccc} \_\_ & +3 & \_\_ & +2 & +3 & -2 & -4 & \_\_ & \_\_ & \_\_ \\ \_\_ & +3 & \_\_ & -4 & +3 & -2 & \_\_ & \_\_ & \_\_ & \_\_ \end{array}$$

This matrix represents a path $\mathcal{C}_0, \ldots, \mathcal{C}_3$ in $\Gamma(2, -4, 3)$. For instance, we have

$$\mathcal{C}_1 : \begin{array}{ccc|cccc} +3 & \_\_ & +2 & \_\_ & -2 & -4 & \_\_ \\ +3 & \_\_ & \_\_ & \_\_ & -2 & \_\_ & \_\_ \end{array} .$$

◊

Lemma 13. *Let $p \in \mathbb{N}_1$, and $\mathcal{C}_0, \ldots, \mathcal{C}_p$ be a path of cut sets with R rows. For all $r \in \{0, \ldots, R-1\}$, there exists an injective function*

$$G_r : \bigcup_{i=0}^{p} (S_{i,r} \cup T_{i,r}) \cup \{1, \ldots, p\} \to \bigcup_{i=0}^{p} \left(S^*_{i,r} \cup T^*_{i,r}\right) \cup \{1, \ldots, p\},$$

defined by

$$G_r(x) = \begin{cases} x + d_r(x) & \text{if } j \in S_{x,r} \cup T_{x-1,r}, \\ x & \text{if } j \notin S_{x,r} \cup T_{x-1,r}. \end{cases}$$

For all $y \in \{1, \ldots, p\}$, there exists $x \in \{1, \ldots, p\}$ with $G_r(x) = y$.

Proof. First, G_r is injective: if $x \notin S_{x,r} \cup T_{x-1,r}$, we have $x \in \{1, \ldots, p\}$. Now $x \in T^*_{x,r}$ leads to a contradiction with (iii) or (ivA) in Definition 8, which gives $x \in T_{x,r}$ or $x \in S_{x-1,r}$. Similarly, $x \in S^*_{x,r}$ leads to a contradiction with (i) or (ii), so we have $d_r(x) = 0$. If $x \in S_{x,r} \cup T_{x-1,r}$, injectivity is guaranteed by (d) in Definition 6 and rules (i)-(ivB) in Definition 8.

Second, given $y \in \{1, \ldots, p\}$, assume $G_r(x) \neq y$ for all $x \in \bigcup_{i=0}^{p} (S_{i,r} \cup T_{i,r}) \cup \{1, \ldots, p\}$. This implies $y \notin \bigcup_{i=0}^{p} \left(S^*_{i,r} \cup T^*_{i,r}\right)$, and $G_r(y) \neq y$. If $G_r(y) > y$, then $\#S_{y-1,r} + 1 = \#S_{y,r}$ and $\#T_{y-1,r} = \#T_{y,r}$, which contradicts (c) in Definition 8. If $G_r(y) < y$, then $\#T_{y-1,r} + 1 = \#T_{y,r}$ and $\#S_{y,r} = \#S_{y-1,r}$, which is impossible for the same reason. □

The following lemma is the converse of Lemma 12.

Lemma 14. *Let $\mathcal{C}_0, \ldots, \mathcal{C}_p$ be a path of cut sets with R rows, where $\mathcal{C}_p$ is the cycled cut set of $\mathcal{C}_0$, and none of $\mathcal{C}_1, \ldots, \mathcal{C}_{p-1}$ is the cycled cut set of $\mathcal{C}_0$. Then there exist an additive periodic sequence $(\mathcal{Y}_x)_{x=0}^{\infty}$ of finite subsets of $\mathbb{Z}$ with period length p, and a Nim sequence G over $(\mathcal{Y}_x)$ with additive period length $\overline{p}$ that divides Rp, and preperiod length $\tilde{P}$. For all $x \geq \tilde{P}$, where $i, j \in \{0, \ldots, p-1\}$, $r, r' \in \{0, \ldots, R-1\}$, $C, C' \in \mathbb{N}_0$ is uniquely defined by $x = j + rp + CRp$, $x - 1 = i + r'p + C'Rp$, we have $G(x)$ defined by*

$$G(x) = \begin{cases} x + d_r(j) & \text{if } j \in S_{j,r}, \\ x + d_r(i+1) & \text{if } j \in T_{i,r'}, \\ x & \text{if } j \notin S_{j,r} \cup T_{i,r'}. \end{cases}$$

Proof. We see that G is uniquely defined: If $j \in T_{i,r'}$, either (i) or (iii) from Definition 8 is true, and in both cases $j \notin S_{j,r}$. Injectivity of G is shown as in the proof of Lemma 13.

Set $\tilde{P} = |\min_{r,j} d_r(j)|$. For $x \geq \tilde{P}$, we define the difference function $d(x) = G(x) - x$, and the cuts (S_x, T_x) as in Definition 5. Then $S_x \sim S_{j,r}$, $T_x \sim T_{j,r}$, so $\#S_x = \#T_x$ for $x \geq 2\tilde{P}$, when we define the seed $\left[g_0, \ldots, g_{\tilde{P}-1}\right]$ as

$$\begin{aligned} & \left\{0, \ldots, 2\tilde{P}-1\right\} \setminus G\left(\left\{\tilde{P}, \ldots, 3\tilde{P}-1\right\}\right) \\ = \; & \left\{0, \ldots, 2\tilde{P}-1\right\} \\ & \setminus \left(S^*_{\tilde{P}-1} \cup S^*_{2\tilde{P}-1} \cup \left(G\left(\left\{\tilde{P}, \ldots, 2\tilde{P}-1\right\}\right) \setminus S^*_{\tilde{P}-1} \setminus T^*_{2\tilde{P}-1}\right)\right), \end{aligned}$$

where $S^*_{\tilde{P}-1}$ and $S^*_{2\tilde{P}-1}$ are disjoint, as $\tilde{P}$ is the maximum of $d_r(j)$.

Now the additive period begins directly after the seed, as

$$\begin{aligned} \tilde{P} &= 2\tilde{P} - \left(\#S_{\tilde{P}-1} + \#S_{2\tilde{P}-1} + \left(\tilde{P} - \#S_{\tilde{P}-1} - \#S_{2\tilde{P}-1}\right)\right) \\ &= \#\left(\left\{0, \ldots, 2\tilde{P}-1\right\} \setminus G\left(\left\{\tilde{P}, \ldots, 3\tilde{P}-1\right\}\right)\right). \end{aligned}$$

The seed is then equal to $\mathbb{N}_0 \setminus \left\{G(x) | x \geq \tilde{P}\right\}$. When we set $G(x) = g_x$ for $x < \tilde{P}$, then G will be injective.

Finally, G is surjective. If we assume $x \in \mathbb{N}_0 \setminus G(\mathbb{N}_0)$ with $G(x) > x$, we have $\#S_{j,r} = \#S_{i,r'} + 1$ and $\#T_{j,r} = \#T_{i,r'}$, which contradicts c) in Definition 6. We reach a similar contradiction with $G(x) < x$ and $G(x) = x$.

Now we define $\mathcal{Y}_x$ for $x \geq \tilde{P}$ as

$$\begin{aligned} \mathcal{Y}_x = \; & \Big(\{z \in \mathbb{N}_1 \mid z \text{ is positively excluded at index } j \text{ in } \mathcal{C}_{j,r}\} \\ & \cup \{z \in -\mathbb{N}_1 \mid z \text{ is negatively excluded at index } j \text{ in } \mathcal{C}_{i,r'}\} \\ & \cup \left\{z = 0 \mid j \in S_{j,r} \cap S^*_{j,r}\right\}\Big) + x, \end{aligned}$$

and show that $G(x) = \operatorname{mex}\left(\{G(x') | x' < x\} \cup \mathcal{Y}_x\right)$. Assume that (x, y) is an inversion of G. If $G(y) - x > 0$, then $G(y) - x$ is positively excluded in $\mathcal{C}_{j,r}$, so $G(y) = (G(y) - x) + x \in \mathcal{Y}_x$. We find negative and zero exclusions in the same way.

It follows from (b) in Definition 9 and (b) in Definition 10 that $d(x)$ cannot be excluded at the index x, so $G(x) = x + d(x) \notin \mathcal{Y}_x$. $\square$

A direct conclusion from these lemmas is that given a Nim sequence G over $(\mathcal{Y}_x)$, we can construct a seed of length no larger than $|\underline{M}|$ as in the previous proof. This seed will define, together with $(\mathcal{Y}_x)$, another Nim sequence G' that will have the same difference period as G. Of course, we could also have proven this fact directly.

4. Optimization

Whenever we use the notation $\mathcal{C}_0$ in this section, it should be understood as a representation of the equivalence class $[\mathcal{C}_0]$.

While our new framework has removed the sets $(\mathcal{Y}_x)$, we are now forced to deal with exclusions. However, all excluded values are strictly bounded between $\underline{M}$ and $\overline{M}$, so if all difference values in a cut set are either $\underline{M}$ or $\overline{M}$, they could never be excluded. This motivates our next definition.

Definition 13. Let

$$\hat{\mathcal{C}}_0 = \left(\hat{S}_{0,r}, \hat{T}_{0,r}, \hat{d}_r\right)_{r=0}^{R-1} \in V\left(R, \underline{M}, \overline{M}\right).$$

We call a difference value $\hat{d}_r(x)$ *optimized*, if $\hat{d}_r(x) \in \left\{\underline{M}, \overline{M}\right\}$. We call $\hat{\mathcal{C}}_0$ an *optimized cut set* if and only if all of its difference values are optimized, so

$$\bigcup_{r=0}^{R-1} \hat{d}_r\left(\hat{S}_r \cup \hat{T}_r\right) \subseteq \left\{\underline{M}, \overline{M}\right\}.$$

Let $\mathcal{C}_0 = (S_{0,r}, T_{0,r}, d_r)_{r=0}^{R-1} \in V$. We call $\hat{\mathcal{C}}_0$ the *optimized cut set of* $\mathcal{C}_0$ if and only if $T_{0,r} \sim \hat{T}_{0,r}$ and $T^*_{0,r} \sim \hat{T}^*_{0,r}$ for all $r \in \{0, \ldots, R-1\}$. Given $\mathcal{C}_0$, its optimized cut set $\hat{\mathcal{C}}_0$ is constructed by setting

$$\hat{d}_r(x) = \begin{cases} \underline{M} & \text{if } x \in T_{0,r}, \\ \overline{M} & \text{if } x + \overline{M} \in T^*_{0,r}. \end{cases}$$

We call a row $\mathcal{C}_{0,r}$ of a cut set *optimized*, if all difference values in the row are optimized.

Definition 14. Let $\left(\hat{\mathcal{C}}_0, \hat{\mathcal{C}}_1\right) \in E$, where $\hat{\mathcal{C}}_0, \hat{\mathcal{C}}_1$ both are optimized cut sets. We call $\hat{\mathcal{C}}_1$ the *optimized successor of* $\hat{\mathcal{C}}_0$ if and only if $\left(\hat{\mathcal{C}}_0, \hat{\mathcal{C}}_1\right)$ has no matching zeros. Given $\hat{\mathcal{C}}_0$, its optimized successor is constructed by only adjoining optimized difference values following the criteria of Definition 8.

The constructions of the optimized cut set of $\mathcal{C}_0$ and the optimized successor of $\hat{\mathcal{C}}_0$ result in uniquely defined cut sets.

Lemma 15 (Optimization Lemma)**.** *Let* $\mathcal{C}_0 \in V\left(R, \underline{M}, \overline{M}\right)$*. Then its optimized cut set* $\hat{\mathcal{C}}_0$ *is a successor of* $\mathcal{C}_0$*.*

Proof. We create a path $\mathcal{C}_0, \ldots, \mathcal{C}_{M-1}, \mathcal{C}_M \sim \hat{\mathcal{C}}_0$, where the difference values not defined by $\mathcal{C}_0$ are set inductively for $x = 1, \ldots, M$ by

$$d_r(x) = \begin{cases} \underline{M} & \text{if } x + \underline{M} > 0 \text{ and } x + \underline{M} \notin T^*_{x+\underline{M}-1}, \\ \overline{M} & \text{otherwise.} \end{cases}$$

We show that $\mathcal{C}_M \sim \hat{\mathcal{C}}_0$. If $x \in T^*_{0,r}$, then $d_r(x + |\underline{M}|) = \overline{M}$, so $M + x = |\underline{M}| + \overline{M} + x \in T^*_{M,r}$. If instead $x \notin T^*_{0,r}$, either $d_r(x + |\underline{M}|) = \underline{M} \neq \overline{M}$, which implies $x + |\underline{M}| + \overline{M} = M + x \notin T^*_{M,r}$. Or $d_r(x - \overline{M}) = \overline{M}$, which can only happen when $\overline{M} < x$, which implies $M + x \notin T^*_{M,r}$.

If $x \in T_{0,r}$, then $x \leq |\underline{M}|$ and $d_r(x) \neq \overline{M}$. Then $d_r(M + x) = \underline{M}$, so $M + x + \underline{M} \leq M$, implying $M + x \in T_{M,r}$. If instead $x \notin T_{0,r}$, either $d_r(x) = \overline{M}$, so $d_r(M + x) \neq \underline{M}$, or $d_r(x) = \underline{M}$ with $|\underline{M}| < x$. In both cases, we have $M + x \notin T_{M,r}$.

Now $\mathcal{C}_M$ will be optimized, as no difference value in $\mathcal{C}_0$ can be a part of $\mathcal{C}_M$. □

The process of creating the path $\mathcal{C}_0, \ldots, \hat{\mathcal{C}}_0$ is called the *optimization of* $\mathcal{C}_0$. We can optimize an already optimized cut set this way, which implies that if there exists a connected component in Γ containing only one element, it must be a non-optimized cut set.

With $C \in \mathbb{N}_1$, when the process is repeated C times, we create a path $\hat{\mathcal{C}}_0, \ldots, \hat{\mathcal{C}}_{CM}$, where $\hat{\mathcal{C}}_0 \sim \hat{\mathcal{C}}_M \sim \ldots \sim \hat{\mathcal{C}}_{CM}$. For all $C' \leq C$ and $x \leq CM$, we have $d_r(x) = d_r(x + C'M)$, when both difference values are defined.

Example 7. Here we find the optimization $\mathcal{C}_0, \ldots, \hat{\mathcal{C}}_0$ in $\Gamma(3, -2, 3)$ of the cut set from Example 4.

$$\begin{array}{ccc|cc} +3 & \_\_ & \_\_ & -1 & \_\_ \\ \_\_ & \_\_ & +2 & -2 & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ & \_\_ \end{array} \quad : \quad \begin{array}{ccc|ccccc|cc} +3 & \_\_ & \_\_ & -1 & +3 & +3 & -2 & -2 & -2 & \_\_ \\ \_\_ & \_\_ & +2 & -2 & +3 & -2 & +3 & -2 & -2 & \_\_ \\ \_\_ & \_\_ & \_\_ & +3 & +3 & -2 & -2 & -2 & \_\_ & \_\_ \end{array}$$

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Lemma 16. *Suppose* $(\mathcal{C}_0, \mathcal{C}_1) \in E(R, \underline{M}, \overline{M})$*, and let* $\hat{\mathcal{C}}_0 = \left(\hat{S}_{0,r}, \hat{T}_{0,r}, \tilde{d}_r\right)_{r=0}^{R-1}$ *be the optimized cut set of* $\mathcal{C}_0$*. Then* $\hat{\mathcal{C}}_0$ *has a direct successor* $\tilde{\mathcal{C}}_1 = (\tilde{S}_{1,r}, \tilde{T}_{1,r}, \tilde{d}_r)_{r=0}^{R-1}$ *with* $T_{1,r} \sim \tilde{T}_{1,r}$ *and* $T^*_{1,r} \sim \tilde{T}^*_{1,r}$ *for all* $r \in \{0, \ldots, R-1\}$ *, defined by*

$$\begin{aligned} \tilde{S}_{1,r} &= \left(\hat{S}_{0,r} \setminus \{-\overline{M} + 1\}\right) \cup \{1 \mid 1 \in S_{1,r}\}, \\ \tilde{T}_{1,r} &= \left(\hat{T}_{0,r} \setminus \{1\}\right) \cup \{k + 1 \mid k \in \mathbb{N}_0, d_r(k+1) = -k\}, \end{aligned}$$

and the possible adjoined difference values will be $\tilde{d}_r(1) = d_r(1)$*, and* $\tilde{d}_r(k+1) = -k$*.*

Proof. As $\hat{T}_{0,r} \sim T_{0,r}$, $\hat{T}^*_{0,r} \sim T^*_{0,r}$, and the two possible non-optimized difference values in $\tilde{\mathcal{C}}_1$ are the ones adjoined to $\mathcal{C}_1$ when it is defined as a successor of $\mathcal{C}_0$, it is clear that $T_{1,r} \sim \tilde{T}_{1,r}$ and $T^*_{1,r} \sim \tilde{T}^*_{1,r}$.

The adjoined values stem from $\mathcal{C}_1$ and cannot exclude themselves. An adjoined positive value must occur at index 1 and cannot be positively excluded by the values of $\hat{S}_0$, and an adjoined negative value at index $k+1$ has $k+1+d_r(k+1) = 1$ and cannot be negatively excluded by the values of $\hat{T}_0$. Thus, $\tilde{\mathcal{C}}_1$ is a valid cutset. □

Lemma 17. *Suppose $\mathcal{C}_0 \in V(R, \underline{M}, \overline{M})$ lies in a connected component with more than one element. Then, its optimized cut set $\hat{\mathcal{C}}_0$ is an ancestor of $\mathcal{C}_0$.*

Proof. We find a path $\mathcal{C}_0, \ldots, C_{j-1}, C_j \sim \mathcal{C}_0$ for some $j > |\underline{M}|$. Then, we create a path $\hat{\mathcal{C}}_0, \tilde{\mathcal{C}}_1, \ldots, \tilde{\mathcal{C}}_{j-1}, \tilde{\mathcal{C}}_j \sim \mathcal{C}_j$ by adjoining difference values from $\mathcal{C}_1, \ldots, \mathcal{C}_j$ to $\hat{\mathcal{C}}_0$ as in Lemma 16. □

It follows from Lemma 15 and Lemma 17 that each connected component in Γ containing more that one element is fully characterized by its optimized cut sets, and the cut sets that connect them.

Lemma 18. *Let $\mathcal{C}_{-M}, \ldots, \mathcal{C}_{M+1}$ be a path in $\Gamma\left(R, \underline{M}, \overline{M}\right)$ with difference functions (d_r). Suppose that in the rth row, the difference value $d_r(1)$ is non-optimized, and all other difference values defined in this row are optimized. Then,*

$$d_r(-M+1) = \overline{M} \quad ; \quad d_r(M+1) = \underline{M}.$$

Also,

$$d_r(-\overline{M}+1+d_r(1)) = \underline{M} \quad ; \quad d_r(|\underline{M}|+1+d_r(1)) = \overline{M}.$$

The difference value $d_r(1)$ will be excluded at the index $-M+1$. For all $l \in \left\{-\underline{M}+1, \ldots, \overline{M}-1\right\} \setminus \{d_r(1)\}$, we have $d_r(-\overline{M}+l+1) = d_r(|\underline{M}|+l+1)$.

Proof. By Lemma 13, for all $y \in \{-M+1, \ldots, M+1\}$ there must exist $x \in \mathbb{Z}$ so $x + d_r(x) = y$. As $1 + d_r(1) \neq \underline{M} + 1$, and all other difference values in the row are optimized, we must have $d_r(-M+1) = \overline{M}$, so $-M+1+d_r(-M+1) = \underline{M}+1$. Similarly, as $1+d_r(1) \neq \overline{M}+1$, we have $d_r(M+1) = \underline{M}$, so $M+1+d_r(M+1) = \overline{M}+1$.

By Lemma 13, for all $x, x' \in \mathbb{Z}$ with $x \neq x'$, we have $x + d_r(x) \neq x' + d_r(x')$. As $-\underline{M}+1+d_r(1)+d_r(-M+1+d_r(1)) \neq d_r(1)+1$, we must have $d_r(-M+1+d_r(1)) \neq \overline{M}$, so $d_r(-M+1+d_r(1)) = \underline{M}$. Similarly, $d_r(|\underline{M}|+1+d_r(1)) = \overline{M}$.

The value

$$-M+1+d_r(-M+1+d_r(1)) - (-\overline{M}+1+d_r(1)) = d_r(1)$$

is by definition excluded at the index $-M+1$.

Assume $l \in \left\{-\underline{M}+1, \ldots, \overline{M}-1\right\} \setminus \{d_r(1)\}$. If $d_r(-\overline{M}+l+1) = \overline{M}$, we have $-\overline{M}+l+1+\overline{M} = l+1 \neq d_r(1)+1$. By Lemma 13, we have $d_r(|\underline{M}|+l+1) \neq \underline{M}$, and as this difference value is optimized, we have $d_r(|\underline{M}|+l+1)=\overline{M}$. If instead $d_r(-\overline{M}+l+1)=\underline{M}$, then $-\overline{M}+l+1+\overline{M}\neq d_r(1)+1$, so $d_r(|\underline{M}|+l+1)=\underline{M}$. □

When $(\mathcal{C}_0,\mathcal{C}_1)\in E$, it follow from Lemma 16 and Lemma 15 that $\hat{\mathcal{C}}_1$ is a successor of $\hat{\mathcal{C}}_0$. The following lemma shows how to construct a path between these two optimized cut sets, where the non-optimized difference values are adjoined in gradual steps.

Lemma 19. *Let $(\mathcal{C}_0,\mathcal{C}_1)\in E\left(R,\underline{M},\overline{M}\right)$ with difference functions (d_r). Then there exists a path $\mathcal{D}_0,\ldots,\mathcal{D}_{(M-1)M+1}$ in Γ with difference functions $\left(\tilde{d}_r\right)$, where $\mathcal{D}_0=\hat{\mathcal{C}}_0$, and $\mathcal{D}_{(M-1)M+1}\sim\hat{\mathcal{C}}_1$. All difference values in the $\mathcal{D}$–path are optimized, and there are no matching zeroes, with the following exceptions:*

(i) There are matching zeros in the rows of $(\mathcal{D}_0,\mathcal{D}_1)$, where $(\mathcal{C}_0,\mathcal{C}_1)$ also has matching zeros. If this is true for at least one row, then in all rows of $\mathcal{C}_0$, where there is a possible zero after the cut, there is a matching zero in $(\mathcal{D}_0,\mathcal{D}_1)$.

(ii) For all $k\in\left\{1,\ldots,\overline{M}-1\right\}$, we have $\tilde{d}_r((\overline{M}-k)M+1)=k$ for all row indices r, where $d_r(1)=k$. If this is true for at least one row, then for all row indices r', where the difference value k is excluded at the index 1 in the $\mathcal{D}$–path, we have $d_{r'}(1)=k$ and $\tilde{d}_{r'}((\overline{M}-k)M+1)=k$.

(iii) For all $k\in\{1,\ldots,|\underline{M}+1|\}$, we have $\tilde{d}_r((\overline{M}+k-1)M+k+1)=-k$ for all row indices r, where $d_r(k+1)=-k$. If this is true for at least one row, then for all row indices r', where the difference value $-k$ is excluded at the index $k+1$ in the $\mathcal{D}$–path, we have $d_{r'}(k+1)=-k$ and $\tilde{d}_{r'}((\overline{M}+k-1)M+k+1)=-k$.

Proof. We use the notation $\mathcal{D}_x=\left(\tilde{S}_{x,r},\tilde{T}_{x,r},\tilde{d}_r\right)_{r=0}^{R-1}$.

(i) Suppose $(\mathcal{C}_0,\mathcal{C}_1)$ has a matching zero in the rth row. Similar to Lemma 16, we can construct a successor $\mathcal{D}_1$ of $\mathcal{D}_0$, where $(\mathcal{D}_{0,r},\mathcal{D}_{1,r})$ has a matching zero. For all row indices r', where there is a possible zero in $\mathcal{C}_{0,r'}$, it follows from (b) in Definition 10 that $(\mathcal{D}_{0,r'},\mathcal{D}_{1,r'})$ also must have a matching zero. For other row indices r'', only optimized difference values will be adjoined to $\mathcal{D}_{1,r''}$, and they cannot exclude the matching zero.

Since no more non-optimized difference values can be added to $\mathcal{C}_1$ in the rth row, it follows that $\tilde{d}_r(x)$ will be optimized for all $x>1$. It follows from Lemma 18 that for all $C\in\mathbb{N}_0$, $\tilde{d}_r(CM+1)=\underline{M}$ and $\tilde{d}_r\left(CM+|\underline{M}|+1\right)=\overline{M}$. Also, $\tilde{T}_{CM+1,r}\sim\hat{T}_{1,r}$, and $\tilde{T}^*_{CM+1,r}\sim\hat{T}^*_{1,r}$.

(ii) Suppose that $d_r(1)=k$. We know that the $\mathcal{D}$–path in this row before the index $\left(\overline{M}-k\right)M$ is formed via repeated optimization, so we have $\tilde{T}_{\left(\overline{M}-k\right)M,r}\sim\hat{T}_{0,r}\sim T_{0,r}$, and $\tilde{T}^*_{\left(\overline{M}-k\right)M,r}\sim T^*_{0,r}$. Then, we can set $\tilde{d}_r((\overline{M}-k)M+1)=k$. For all row indices r', where a difference value $k'\in\left\{k+1,\ldots,\overline{M}-1\right\}$ is adjoined to

$\mathcal{C}_{1,r'}$, or there is a matching zero, Lemma 18 shows that $\tilde{d}_{r'}((\overline{M}-k')M+1) = \underline{M}$, so k cannot be excluded at the index $(\overline{M}-k)M+1$, or at the index 1, in this row.

Also by Lemma 18, when $d_{r'}(1) = k$, then k is excluded at the index 1 in the $\mathcal{D}$–path.

For all row indices r' where $d_{r'}(1) < 0$, or $d_{r'}(1) > k$, then k cannot be excluded at the index 1 in the $\mathcal{D}$–path. This would imply that k is excluded at the index 1 in $\mathcal{C}_1$, which is impossible.

It is possible that a negative non-optimized difference value is adjoined to $\mathcal{C}_{1,r}$, so $d_r(k'+1) = -k'$ for some $k \in \{1, \ldots, |\underline{M}+1|\}$. Then $1 \notin T^*_{0,r}$. After we adjoin k to the $\mathcal{D}$–path, we would have $\tilde{d}_r((\overline{M}-k)M+1+|\underline{M}|) = \underline{M}$, so we have $CM+1 \notin \tilde{T}^*_{CM,r}$ for all $C \geq \overline{M}-k$. So it would be possible to set $\tilde{d}_r((\overline{M}+k'+1)M+k'+1) = -k'$ in step (iii).

If instead $\mathcal{C}_{1,r}$ has no negative non-optimized difference values, we would have $\tilde{T}_{(\overline{M}-k)M+1,r} \sim \hat{T}_{1,r}$, and $\tilde{T}^*_{(\overline{M}-k)M+1,r} \sim \hat{T}^*_{1,r}$. As the remainder of the $\mathcal{D}$–path is formed via optimization in the rth row, we have $\tilde{T}_{(M-1)M+1} \sim \hat{T}_{1,r}$ and $\tilde{T}^*_{(M-1)M+1} \sim \hat{T}^*_{1,r}$.

(iii) Suppose that $d_r(k+1) = -k$. Using the same argument as above, we can set $\tilde{d}_r((\overline{M}+k+1)M+k+1) = -k$. For all row indices r', where a difference value $-k'$ is adjoined with $k' \in \{1, \ldots, k-1\}$, Lemma 18 (after adjusting the indices of the lemma with $+k'$) shows that

$$\tilde{d}_{r'}((\overline{M}+k)M+|\underline{M}|+k'+1-k') = \tilde{d}_{r'}((\overline{M}+k)M+|\underline{M}|+1) = \overline{M},$$

so $-k$ cannot be excluded at the index $(\overline{M}+k+1)M+k+1$ in this row. The same argument shows that $-k$ cannot be excluded in rows with a matching zero.

Since we have now adjoined all non-optimized difference values in the rth row, we argue as above that at the end of the $\mathcal{D}$–path, we have $\tilde{T}_{(M-1)M+1} \sim \hat{T}_{1,r}$ and $\tilde{T}^*_{(M-1)M+1} \sim \hat{T}^*_{1,r}$.

As we have now shown that $\tilde{T}_{(M-1)M+1} \sim \hat{T}_{1,r}$ and $\tilde{T}^*_{(M-1)M+1} \sim \hat{T}^*_{1,r}$ for all rows, and $\mathcal{D}_{(M-1)M+1}$ is an optimized cut set, we conclude that $\mathcal{D}_{(M-1)M+1} \sim \hat{\mathcal{C}}_1$. □

Example 8. Here we show a $\mathcal{D}$–path between $\hat{\mathcal{C}}_0$ from Example 7 and the optimized cut set of $\mathcal{C}_1$ from Example 4. Note that parts of any $\mathcal{D}$–path may be simple optimization; for instance, when there are no row indices r for which $d_r(1) = k$, where k is non-optimized, the path $\mathcal{D}_{(\overline{M}-k)M+1}, \ldots, \mathcal{D}_{(\overline{M}-k+1)M+1}$ is optimized. For brevity, we omit these parts here.

$$\begin{array}{ccc|ccccccccccc|cc}
+3 & \underline{\quad} & \underline{\quad} & -2 & +3 & +3 & -2 & -2 & -2 & +3 & +3 & -2 & -2 & -2 & \underline{\quad} & \underline{\quad} \\
\underline{\quad} & +3 & \underline{\quad} & -2 & +3 & -2 & +3 & -2 & -2 & -1 & +3 & +3 & -2 & -2 & -2 & \underline{\quad} \\
\underline{\quad} & \underline{\quad} & \underline{\quad} & +2 & +3 & -2 & -2 & +3 & -2 & -1 & +3 & -2 & +3 & -2 & -2 & \underline{\quad}
\end{array}$$

◊

5. Binary representations

Definition 15. Let $\mathcal{C}_0 \in V\left(R, \underline{M}, \overline{M}\right)$. The *binary representation of* $\mathcal{C}_0$ is an $R \times M$ matrix $\mathcal{B}_0 = [b_{0,r,m}]_{r=0,\dots,R-1, m=-\overline{M}+1,\dots,|\overline{M}|}$ with entries in $\{+,-\}$ defined by

$$b_{0,r,m} = \begin{cases} + & \text{if } m+\overline{M} \in T^*_{0,r} \text{ for } m \le 0, \text{ or } m \notin T_{0,r} \text{ for } m > 0, \\ - & \text{if } m+\overline{M} \notin T^*_{0,r} \text{ for } m \le 0, \text{ or } m \in T_{0,r} \text{ for } m > 0. \end{cases}$$

Let c_m be the symbol for the mth column of $\mathcal{B}_0$, so

$$c_m = \begin{bmatrix} b_{0,0,m} \\ \vdots \\ b_{0,R-1,m} \end{bmatrix}.$$

We can use the binary representation $\mathcal{B}_0$ of $\mathcal{C}_0$ to define the optimized cut set $\hat{\mathcal{C}}_0$. Its difference functions $\left(\hat{d}_r\right)$ will be defined as

$$\hat{d}_r(m) = \begin{cases} \overline{M} & \text{if } x \le 0 \text{ and } b_{0,r,m} = +, \\ \underline{M} & \text{if } x > 0 \text{ and } b_{0,r,m} = -, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

It follows that there is a one-to-one correspondence between the binary representations and the optimized cut sets of Γ. In an optimized path $\hat{\mathcal{C}}_0, \dots, \hat{\mathcal{C}}_M \sim \hat{\mathcal{C}}_0$, the entry $b_{x,r,m}$ is equal to the sign of $\hat{d}_r(x+m)$.

Let τ be the column permutation that cycles all columns of $\mathcal{B}_0$, defined by

$$\tau(m) = \begin{cases} |\underline{M}| & \text{for } m = -\overline{M}+1, \\ m-1 & \text{for } m > -\overline{M}+1. \end{cases}$$

The permutation τ represents the rotation of the columns from $\mathcal{B}_0$ to $\mathcal{B}_1$, where $\mathcal{B}_1$ is the binary represantion of a direct successor $\mathcal{C}_1$ of $\mathcal{C}_0$. For instance, if $\hat{\mathcal{C}}_1$ is the optimized successor of $\hat{\mathcal{C}}_0$, then $\mathcal{B}_{0,r,m} = \mathcal{B}_{1,r,\tau(m)}$ for all r, m.

Note that $\mathcal{B}_0$ cannot have two identical rows, as it would violate condition (a) in Definition 9.

Example 9. Continuing Example 4, here are the binary representations of $\mathcal{C}_0$ and $\mathcal{C}_1$.

$$\begin{array}{ccc|cc} + & - & - & - & + \\ - & + & - & - & + \\ - & - & - & + & + \end{array} \quad \hookrightarrow \quad \begin{array}{ccc|cc} - & - & - & + & + \\ + & - & - & - & + \\ - & + & - & - & + \end{array}$$

When we ignore the τ–rotation of the columns, we see that (c_{-2}, c_2) and (c_0, c_1) in the ancestor have been swapped. $\Diamond$

Lemma 20. *Let* $\left(\hat{\mathcal{C}}_0, \mathcal{C}_1\right) \in E(R, \underline{M}, \overline{M})$*, where* $\hat{\mathcal{C}}_0$ *is optimized. Let* $r \in \{0, \ldots, R-1\}$*, and* $m \in \left\{-\overline{M}+1, \ldots, |\underline{M}|\right\}$*. Then* $b_{0,r,m} = b_{1,r,\tau(m)}$*, with these exceptions:*

(i) With $k \in \left\{1, \ldots, \overline{M}-1\right\}$*, where* $d_r(1) = k$*, then*

$$b_{0,r,1} = +, b_{1,r,0} = - \,;\; b_{0,r,-\overline{M}+k+1} = -, b_{1,r,-\overline{M}+k} = +.$$

(ii) With $k \in \{1, \ldots, |\underline{M}+1|\}$*, where* $d_r(k+1) = -k$*, then*

$$b_{0,r,k+1} = +, b_{1,r,k} = - \,;\; b_{0,r,-\overline{M}+1} = -, b_{1,r,|\underline{M}|} = +.$$

(iii) When $\left(\hat{\mathcal{C}}_0, \mathcal{C}_1\right)$ *has a matching zero in the* r*th row, then*

$$b_{0,r,1} = +, b_{1,r,0} = - \,;\; b_{0,r,-\overline{M}+1} = -, b_{1,r,|\underline{M}|} = +.$$

Proof. Construct a path $\tilde{\mathcal{C}}_{-M}, \ldots, \tilde{\mathcal{C}}_M$ with difference functions $\left(\tilde{d}_r\right)$ as in Lemma 18, where $\hat{\mathcal{C}}_0 \sim \tilde{\mathcal{C}}_0$, $\mathcal{C}_1 \sim \tilde{\mathcal{C}}_1$. Then the changes in value from $b_{0,r,m}$ to $b_{1,r,\tau(m)}$ correspond to sign changes from $\tilde{d}_r(-M+1)$ to $\tilde{d}_r(M+1)$, or from $\tilde{d}_r(-\overline{M}+1+d_r(1))$ to $\tilde{d}_r(|\underline{M}|+1+d_r(1))$.

For instance, if $k \in \{1, \ldots, M-1\}$ with $d_r(1) = k$, we have $\tilde{d}_r(-M+1) = \overline{M}$, and $\tilde{d}_r(M+1) = \underline{M}$. As $\tilde{\mathcal{C}}_{-M} \sim \tilde{\mathcal{C}}_0$, and $\tilde{\mathcal{C}}_1 \sim \tilde{\mathcal{C}}_{M+1}$, and the binary representations are equal to the sign of the optimized values, we have $b_{0,r,1} = +$, and $b_{1,r,0} = -$.

The other claims about changes are shown in the same way. When $d_r(k+1) = -k$, we must adjust the indices of Lemma 18 with $+k$ in the proof.

It also follows by this lemma, or by examining (T_0, T_0^*), (T_1, T_1^*), that $b_{0,r,m} = b_{1,r,\tau(m)}$ otherwise. □

Remark 3. To avoid confusion of indices, we will mostly refer to $\mathcal{B}_0/\tau$ instead of $\mathcal{B}_0$, and index the columns of $\mathcal{B}_0/\tau$ with $m \in \{0, \ldots, M-1\}$. Given $\left(\hat{\mathcal{C}}_0, \mathcal{C}_1\right)$ from Lemma 20, we summarize the changes of the binary entries in this table:

Value	In $\mathcal{B}_0/\tau$ at column:				In $\mathcal{B}_1/\tau$ at column:			
adjoined:	0	k	$\lvert\underline{M}\rvert$	$k+\lvert\underline{M}\rvert$	0	k	$\lvert\underline{M}\rvert$	$k+\lvert\underline{M}\rvert$
$+k$	$+$			$-$	$-$			$+$
$-k$		$+$	$-$			$-$	$+$	

We have normalized the column indices, so $+k$ is adjoined at the index 0 in $\mathcal{B}_1/\tau$, and $-k$ is adjoined at the index k. For instance, the change from $b_{0,r,1} = +$ to $b_{1,r,0} = -$ corresponds to the change at column index 0. We can interpret a matching zero as either $+0$ or -0.

The next lemma uses $\mathcal{B}_0/\tau$ to give a condition for connectivity of cut sets.

Lemma 21. *Let* $\left(\hat{\mathcal{C}}_0, \mathcal{C}_1\right) \in E\left(R, \underline{M}, \overline{M}\right)$*, where* $\hat{\mathcal{C}}_0$ *is optimized. Then* $\hat{\mathcal{C}}_0$ *and* $\mathcal{C}_1$ *lie in the same connected component if and only if we can construct* $\mathcal{B}_1$ *by some column permutation of* $\mathcal{B}_0$*.*

Proof. If $\mathcal{C}_1$ is the optimized successor of $\hat{\mathcal{C}}_0$, then $\mathcal{B}_1 = \tau(\mathcal{B}_0)$. To create a path from $\mathcal{C}_1$ to $\hat{\mathcal{C}}_0$, we optimize $\mathcal{C}_1$.

Assume instead that $\mathcal{C}_1$ has non-optimized difference values, or a matching zero. Examining the table in Remark 3, we find that for all $m_1, m_2 \in \{0, \ldots, M-1\}$ and $r \in \{0, \ldots, R_1\}$, where $(b_{0,r,m_1}, b_{0,r,m_2}) = (+, -)$, we can find a successor $\mathcal{C}_j$ of $\mathcal{C}_0$, where these entries are swapped in $\mathcal{B}_j/\tau$. . We let $\hat{\mathcal{C}}_1, \ldots, \hat{\mathcal{C}}_{j-1}$ be optimized successors until we reach a point, where we can adjoin a non-optimized difference value to set $(b_{j,r,m_1}, b_{j,r,m_2}) = (-, +)$.

Given a row index r, we can now create a successor $\hat{\mathcal{C}}_j$ of $\mathcal{C}_1$, where $\hat{\mathcal{C}}_{j,r} \sim \hat{\mathcal{C}}_{0,r}$. First, we locate the binary entries we need to swap. There are at most one non-optimized positive and one non-optimized negative difference value adjoined in $\mathcal{C}_{1,r}$, so there are at most two different pair of binary entries swapped from $\mathcal{B}_{0,r}/\tau$ to $\mathcal{B}_{1,r}/\tau$. As above, we optimize $\mathcal{C}_1$ until we reach a point where we can adjoin a non-optimized value to swap one pair of binary entries. If we need to swap another pair, we optimize again until we can adjoin another non-optimized value. Following another optimization, we reach a successor $\hat{\mathcal{C}}_{j'}$, where $b_{j',r,m} = b_{0,r,m}$ for all $m \in \{0, \ldots, M-1\}$. Now, we examine what will happen in the other rows.

Let $r' \in \{0, \ldots, R-1\} \setminus \{r\}$, and $x \in \mathbb{N}_0$. Assume we adjoin a non-optimized difference value in the rth row of $\mathcal{C}_{x+1}$, which results in the pair $(b_{x,r,m_1}, b_{x,r,m_2}) = (+, -)$ being swapped in $\mathcal{B}_{x+1}/\tau$. If $(b_{x,r',m_1}, b_{x,r',m_2}) = (+, -)$, then this difference value would be excluded in the row with index r', unless it is adjoined in $\mathcal{C}_{x+1,r'}$. Then we also have a swap of binary entries in this row, so $(b_{x+1,r',m_1}, b_{x+1,r',m_2}) = (-, +)$.

If $(b_{x,r',m_1}, b_{x,r',m_2}) \neq (+, -)$, the adjoined difference value is not excluded in $C_{x+1,r'}$, and by Lemma 18, it cannot be adjoined to this row. It follows that $(b_{x+1,r',m_1}, b_{x+1,r',m_2}) = (b_{x,r',m_1}, b_{x,r',m_2})$.

The same lemma shows that for all $m \notin \{m_1, m_2\}$, we have $b_{x+1,r',m} = b_{x,r',m}$. It follows that if $(b_{x,r',m_1}, b_{x,r',m_2}) \neq (-, +)$ for all row indices r', we are swapping exactly two columns c_{m_1}, c_{m_2} when we go from $\mathcal{B}_x/\tau$ to $\mathcal{B}_{x+1}/\tau$.

However, if $(b_{x,r',m_1}, b_{x,r',m_2}) = (-, +)$ for some row index r', the swap will result in c_{m_1} and c_{m_2} being changed, as each column now will contain a different amount of pluses and minuses after the swap. We now have $(b_{x+1,r,m_1}, b_{x+1,r,m_2}) = (b_{x+1,r',m_1}, b_{x+1,r',m_2}) = (-, +)$. For any further swaps in the path at $\mathcal{B}_{x'}/\tau$ with $x' > x$, we see that $b_{x+1,r,m_1}, b_{x+1,r',m_1}$ now shift together. The same is true for $b_{x+1,r,m_2}, b_{x+1,r',m_2}$.

In any path we take $\mathcal{C}_{x+1}, \mathcal{C}_{x+2}, \ldots$, it follows from Lemma 19 that each step of the path $C_{x'}, C_{x'+1}$ can be replaced with $\mathcal{C}_{x'}, \ldots, \hat{\mathcal{C}}_{x'}, \ldots, \mathcal{C}_{x'+M(M-1)+1} \sim \hat{\mathcal{C}}_{x'+1}$, where in each step of the new path, at most two columns in the binary represention will be exchanged. If $\mathcal{C}_{x'}, \mathcal{C}_{x'+1}$ lie in the same connected component, it follows from Lemma 17 that there is a path $\hat{\mathcal{C}}_{x'+1}, \ldots, \mathcal{C}_{x'+1}$. If they do not lie in the same component, there can be no path from $\mathcal{C}_{x'+1}$ to $\hat{\mathcal{C}}_0$.

Now, if $\mathcal{B}_1$ is not equal to some column permutation of $\mathcal{B}_0$, for some values of r, r', m_1, m_2 we must have $(b_{0,r,m_1}, b_{0,r,m_2}) = (+,-)$ with $(b_{0,r',m_1}, b_{0,r',m_2}) = (-,+)$, where the entries b_{0,r,m_1}, b_{0,r,m_2} are swapped in $\mathcal{B}_1/\tau$. Then, there can no path from $\mathcal{C}_1$ to $\hat{\mathcal{C}}_0$. The entries $b_{x+1,r,m_1}, b_{x+1,r',m_1}$ and $b_{x+1,r,m_1}, b_{x+1,r',m_1}$ will shift together in any path from $\mathcal{C}_1$, so c_{m_1}, c_{m_2} can never be equal to their corresponding columns in $\mathcal{B}_0/\tau$.

If instead $\mathcal{B}_1$ is equal to a column permutation of $\mathcal{B}_0$, we construct a path from $\mathcal{C}_1$ to $\hat{\mathcal{C}}_0$ by swapping back two columns at a time, similar to the proof of Lemma 19. □

Example 10. We construct a path in $\Gamma(3,-2,2)$ based on Lemma 21. The path begins with $\mathcal{C}_1$ from Example 4 and ends with $\hat{\mathcal{C}}_0$, which we found in Example 7.

$$\begin{array}{ccc|cccccccccccccc|cc}
\underline{\quad} & \underline{\quad} & \underline{\quad} & +3 & +3 & -2 & -2 & -2 & +3 & +3 & -2 & -2 & -2 & +3 & +3 & -2 & -2 & -2 & \underline{\quad} \\
\underline{\quad} & +2 & \underline{\quad} & -1 & +3 & +3 & -2 & -2 & -2 & +2 & +3 & -2 & -2 & +3 & -2 & +3 & -2 & -2 & \underline{\quad} \\
\underline{\quad} & \underline{\quad} & +2 & -1 & +3 & -2 & -1 & +3 & -2 & +2 & -2 & -2 & +3 & +3 & -2 & -2 & -2 & \underline{\quad} & \underline{\quad}
\end{array}$$

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Corollary 3. *Let $(\mathcal{C}_0, \mathcal{C}_1) \in V(R, \underline{M}, \overline{M})$. If $\mathcal{C}_0$ and $\mathcal{C}_1$ lie in the same connected component, then $\mathcal{B}_1$ is equal to some column permutation of $\mathcal{B}_0$. In this case, we say there is a* reversible shift *from $\mathcal{B}_0$ to $\mathcal{B}_1$.*

Proof. The path $\mathcal{C}_0, \ldots, \mathcal{C}_1, \ldots, \mathcal{C}_0$ can be extended with Lemma 19, so in each step of the path, at most two columns are swapped. Then we apply Lemma 21. □

Now, we can prove our main theorem.

Theorem 4. *Any Nim sequence G is additively periodic, and the length $\overline{p}$ of its period is bounded by $\overline{p} \leq K_{\underline{M},\overline{M}}\, p$. Here, $K_{\underline{M},\overline{M}}$ is the maximal value of $\operatorname{lcm}(p_1, \ldots, p_n)$, where $p_1, \ldots, p_n \in \mathbb{N}_1$ for some $n \in \mathbb{N}_1$ with the constraints that $\sum_{i=1}^{n} p_i = M - 1$, and $n \leq \min\left(|\underline{M}|, \overline{M}\right)$. Also, there exists a Nim sequence G for which $\overline{p} = K_{\underline{M},\overline{M}}\, p$.*

Proof. Given a Nim sequence G with additive period length $\overline{p}$, we set $R = \frac{\operatorname{lcm}(\overline{p},p)}{p}$ and create the path $\mathcal{C}_0, \ldots, \mathcal{C}_p$ as in Lemma 12, where $\mathcal{C}_p$ is the cycled cut set of $\mathcal{C}_0$. As we can continue the path to reach $\mathcal{C}_{Rp} \sim \mathcal{C}_0$, it follows that $\mathcal{B}_p$ is created by a column permutation of $\mathcal{B}_0$. Let $c_0, \ldots, c_{M-1}$ be the columns of $\mathcal{B}_0$, and let $\pi(c_m) = \left[b_{0,\pi(r),m}\right]_{r=0}^{R-1}$. It follows from Definition 12 that $\pi(c_0), \ldots, \pi(c_{M-1})$ is a permutation of $c_0, \ldots, c_{M-1}$.

We say that $\left\{c_m, \pi\left(c_m\right), \ldots, \pi^{p_i-1}\left(c_m\right)\right\}$ constitute a p_i–cycle, if $p_i \in \mathbb{N}_1$ is the smallest number such that $\pi^{p_i}\left(c_m\right) = c_m$. The first elements $\left[b_{0,r,m}\right]_{r=0}^{p_i-1}$ will be repeated in c_m, so $b_{0,r,m} = b_{0,r+ap_i,m}$ for $a \in \{0, \ldots, R/p_i - 1\}$. Now the elements of c_m are all identical if and only if $\{c_m\}$ is a 1–cycle. If there are n column cycles of length $p_1, \ldots, p_n$ in $\mathcal{B}_0$, we must have $R = \operatorname{lcm}\left(p_1, \ldots, p_n\right)$, because every p_i must divide R, and two rows in $\mathcal{B}_0$ cannot be identical.

We need to examine how the columns shift from $\mathcal{B}_0$ to $\mathcal{B}_p$. If p_i happens to divide M, we can space out the columns of the p_i–cycle at the indices $0, R/p_i, \ldots, R - R/p_i$, and use the τ–rotation to shift the columns; otherwise, we need to swap at least one column of the p_i–cycle. If $p_i \geq 2$ and c_{m_1}, c_{m_2} belong to the same p_i–cycle, then there must exist row indices r', r'' such that $\left(b_{0,r',m_1}, b_{0,r',m_2}\right) = \left(b_{0,r'',m_2}, b_{0,r'',m_1}\right) = (+,-)$, so we cannot swap these columns by Corollary 3. If c_{m_1} belongs to a p_i–cycle and c_{m_2} belongs to a $p_{i'}$–cycle with p_i, $p_{i'}$ coprime, we cannot swap c_{m_1} and c_{m_2} for the same reason. However, if $\{c_{m_1}\}$ is a 1–cycle, we can swap c_{m_1} with any another column in a reversible shift. To ensure a reversible shift is possible from $\mathcal{B}_0$ to $\mathcal{B}_p$, we just need to include a 1–cycle in $\mathcal{B}_0$.

The problem of maximizing R is equal to finding $\max\left(\operatorname{lcm}\left(p_1, \ldots, p_n\right)\right)$ under the constraint that $\sum_{i=1}^{n} p_i = M - 1$. Finally, there is a bound for $\underline{M}$ and $\overline{M}$. As each of the n cycles must contain both one $+$ and one $-$, we must have $\min\left(|\underline{M}|, \overline{M}\right) \geq n$, before we can achieve the maximum value for R.

A Nim sequence G with maximal period length given $\underline{M}$ and $\overline{M}$ can be found by first creating a binary representation which permits the desired number of p_i–cycles with lengths $1, p_1, \ldots, p_n$, which are found under the constraints given above. For each column swap, we find the binary representation, convert them to optimized cut sets, and connect them as in Lemma 19. Finally, we use Lemma 14 to find $\left(\mathcal{Y}_x\right)$ and the seed. □

Proof of Theorem 1. To maximize R, we need to have as many prime factors as possible in $p_1, \ldots, p_n$. So if $M - 1$ is equal to the sum of the n smallest prime numbers, the solution is $\prod_{i=1}^{n} p_i$, where $p_1 = 2$, $p_2 = 3$, $\ldots$, and p_n is the nth smallest prime.

By an extension of the Prime Number Theorem (see [8], [10]), the asymptotic value of the sum of all primes smaller or equal to $x \in \mathbb{N}_1$ is

$$\sum_{p_i \leq x} p_i \sim \operatorname{Li}\left(x^2\right) \quad \text{as } x \to \infty \ ,$$

and the product of all primes smaller or equal to x, the *primorial* (see [9]), has the asymptotic value

$$\prod_{p_i \leq x} p_i \sim \exp(x) \quad \text{as } x \to \infty \ .$$

Combining these formulas, we find

$$\max\left(\operatorname{lcm}\left(p_1,\ldots,p_n\right)\right) \sim \exp\left(\sqrt{\operatorname{Li}^{-1}\left(M-1\right)}\right) \quad \text{as } M \to \infty$$

which gives the asymptotic value of $K_{\underline{M},\overline{M}}$. □

Remark 4. The proof assumes that we can choose the elements of each $\mathcal{Y}_x$ freely between the difference bounds. If $(\mathcal{Y}_x)$ is given, the maximal bound $K_{\underline{M},\overline{M}}p$ may shrink, if the excluded elements do not permit the optimal number of column cycles. The bounds of Theorem 3 may also shrink.

Example 11. Let us find the maximum value of R in $\Gamma(R,-3,3)$. As $M = 6 = 1+2+3$, we construct a binary representation with a 1–cycle, a 2–cycle and a 3–cycle (example, left below). It gives rise to a path defining a difference period of length 30, which we write in a 6×5 matrix, extended to both sides (example, right below):

$$\begin{array}{ccc|ccc} + & - & - & + & - & + \\ - & + & - & - & + & + \\ - & - & + & + & - & + \\ + & - & - & - & + & + \\ - & + & - & + & - & + \\ - & - & + & - & + & + \end{array} \; ; \quad \begin{array}{ccc|ccccc|ccc} +3 & \_\_ & \_\_ & +3 & -3 & 0 & +3 & -3 & -1 & \_\_ & \_\_ \\ \_\_ & +3 & \_\_ & -1 & +3 & 0 & -3 & +3 & \_\_ & -3 & \_\_ \\ \_\_ & \_\_ & +3 & +3 & -3 & +3 & -3 & -3 & -1 & \_\_ & \_\_ \\ +3 & \_\_ & \_\_ & -1 & +3 & 0 & +3 & -3 & \_\_ & -3 & \_\_ \\ \_\_ & +3 & \_\_ & +3 & -3 & 0 & -3 & +3 & -1 & \_\_ & \_\_ \\ \_\_ & \_\_ & +3 & -1 & +3 & +3 & -3 & -3 & \_\_ & -3 & \_\_ \end{array}$$

There are two column swaps in the path: first c_2 and c_5, followed by c_2 and c_4. The shift (c_2, c_5) corresponds to the columns where the zeros are adjoined. The shift (c_2, c_4) corresponds to the columns with the difference values -1.

As it happens that $K_{-3,3} = 2 * 3 = 6$, we could alternatively define a binary representation with one 6–cycle, which cycles its columns by the τ–rotation (example, right below).

$$\begin{array}{ccc|ccc} + & + & + & - & - & - \\ - & + & + & + & - & - \\ - & - & + & + & + & - \\ - & - & - & + & + & + \\ + & - & - & - & + & + \\ + & + & - & - & - & - \end{array}$$

In this case, $\{-2,\ldots,2\}$ are excluded at every index, so we must have $p = 1$, and the difference period will be $[+3,+3,+3,-3,-3,-3]$, as in Example 1.

If we look at $\Gamma(R,-4,2)$, we can modify the first binary representation by changing the 1–cycle from all pluses to all minuses, as follows:

$$\begin{array}{ccc|ccc} + & - & - & + & - & - \\ - & + & - & - & + & - \\ - & - & + & + & - & - \\ + & - & - & - & + & - \\ - & + & - & + & - & - \\ - & - & + & - & + & - \end{array}$$

We could also modify the 6–cycle, so the difference period now will be equal to $[+2, +2, +2, +2, -4, -4]$.

If we look at $\Gamma(R, -5, 1)$, we can no longer modify the first example, but we can construct a 6–cycle so the difference period becomes $[+1, +1, +1, +1, +1, -5]$. $\Diamond$

We now expand our framework, so we can give a bound for the length of the preperiod, or rather $\tilde{P} - L$, where L is the length of the seed. This bound must also depend on the seed, as we can make the preperiod length arbitrarily long by including large elements in the seed.

Given $\overline{R}, R \in \mathbb{N}_1$, we now consider a cut set $\mathcal{C}_x = (S_{x,r}, T_{x,r}, d_r)_{r=0}^{\overline{R}+R-1}$ to have $\overline{R}+R$ rows, where the top $\overline{R}$ rows define the preperiod. For $r \in \{\overline{R}, \ldots, R+\overline{R}-1\}$, we redefine the row rotation of the period as $\pi(r) = ((r - \overline{R} - 1) \bmod R) + \overline{R}$, so it cycles the bottom R rows. For $r \in \{0, \ldots, \overline{R} - 1\}$, we define the row rotation of the preperiod as

$$\overline{\pi}(r) = (r - 1) \bmod \overline{R}.$$

This rotation cycles all rows with indices $0, \ldots, \overline{R} - 2$, as the bottom row of the preperiod will continue into the period.

A Nim sequence with preperiod length $\left(\overline{R} - 1\right) p + x'$ and period length Rp, where $x' \in \{1, \ldots, p\}$, corresponds to a path $\mathcal{C}_0, \ldots, \mathcal{C}_{p-1}$ in $\Gamma\left(\overline{R} + R, \underline{M}, \overline{M}\right)$. For $j < p - x'$, the cut sets $\mathcal{C}_j = (S_{j,r}, T_{j,r}, d_r)_{r=1}^{\overline{R}+R-1}$ lack one top row compared to the remaining cut sets in the path. This 0th row will be added in $\mathcal{C}_{p-x'}$. One cut set $\mathcal{C}_p$, a direct successor of $\mathcal{C}_{p-1}$, is adjoined at the end of the path, where

$$\mathcal{C}_{0,r} \sim \begin{cases} \mathcal{C}_{p,\overline{\pi}(r)} & \text{for } r \in \{1, \ldots, \overline{R} - 1\}, \\ \mathcal{C}_{p,\pi(r)} & \text{for } r \in \{\overline{R}, \ldots, \overline{R} + R - 1\}, \\ \mathcal{C}_{p,\overline{R}-1} & \text{for } r = \overline{R}. \end{cases}$$

We have $\mathcal{C}_{p,\overline{R}-1} \sim \mathcal{C}_{p,\overline{R}+R-1}$, where the preperiod shifts into the period, so $\mathcal{C}_p$ will violate condition a) in Definition 9. We can define a legal cut set $\tilde{\mathcal{C}}_p$ by removing the illegal row at the index $\overline{R} - 1$ before we move all other rows in the preperiod one index down. When now $x' < p$, $\tilde{\mathcal{C}}_p \sim \mathcal{C}_0$; or with $x' = p$, the cut sets will be equivalent when we ignore the top row of $\mathcal{C}_0$.

Example 12. The values of the Nim sequence G_3 from Wythoff's Game in Example 2 corresponds to a path $\mathcal{C}_0, \ldots, \mathcal{C}_6$ in $\Gamma(5, -5, 3)$.

				+3	+3	+3	+3	−2	−5	−5	___	___	___
___	+3	+3	+3	+3	−2	−5	−5	+2	___	___	−4	___	___
+3	+3	___	−5	−5	+2	+2	+3	−2	−4	___	___	___	___
___	+3	___	+2	+3	−2	−4	+3	−2	___	___	___	−4	___
___	+3	___	−4	+3	−2	+2	+3	−2	−4	___	___	___	___

(The horizontal line separates the preperiod from the period.) $\Diamond$

Theorem 5. *Let G be a Nim sequence over $(\mathcal{Y}_x)_{x=0}^{\infty}$ that has period length p. Let G have seed $[g_0, \ldots, g_{L-1}]$, period length $\overline{p}$ and preperiod length $\tilde{P}$. Set $\hat{K} = \max_{x<L} g_x - L$. Then the preperiod length is bounded by*

$$\tilde{P} - L \leq \left(\frac{M}{2}\right)^2 p + \hat{K},$$

and for $M \geq 11$, we have

$$\tilde{P} - L + \overline{p} \leq K_{\underline{M},\overline{M}}\, p + \hat{K},$$

where $K_{\underline{M},\overline{M}}$ is defined as in Theorem 4.

Proof. Let $c_0, \ldots, c_{M-1}$ be the columns of $\mathcal{B}_0/\tau$. We say that $\overline{\pi}(c_{m_1}) = c_{m_2}$ in $\mathcal{B}_j/\tau$ if and only if $b_{j,r,m_1} = b_{j,\overline{\pi}(r),m_2}$ for all $r \in \{0, \ldots, \overline{R}-1\}$, ignoring the values of the period. Then $\left\{c_m, \overline{\pi}\left(c_m\right), \ldots, \overline{\pi}^{p_i-1}\left(c_m\right)\right\}$ constitutes a p_i–cycle in the preperiod, if $p_i \in \mathbb{N}_1$ is the smallest number such that $\overline{\pi}^{p_i}\left(c_m\right) = c_m$.

Any shift of the columns in $\mathcal{B}_p$ affects both the period and the preperiod. It follows that if we have a p_i–cycle in the preperiod, the columns of the cycle must correspond to $p_{i'}$–cycle in the period, where either $p_{i'}$ divides p_i, or p_i divides $p_{i'}$, or $p_i = p_{i'}$.

As written above, the difference values of G define a path $\mathcal{C}_0, \ldots, \mathcal{C}_p$, where $\mathcal{C}_p$ has two equivalent rows. As $\mathcal{C}_{p-1}$ is a legal cut set, and the shift from $\mathcal{B}_{p-1}$ to $\mathcal{B}_p$ creates two identical rows in $\mathcal{B}_p$, it follows that this shift is irreversible. We find two columns c_{m_1}, c_{m_2} where this shift swaps $b_{p-1,\overline{R}-1,m_1} = +$ with $b_{p-1,\overline{R}-1,m_2} = -$, so these entries become equal to $\left(b_{p,\overline{R}+R-1,m_1}, b_{p,\overline{R}+R-1,m_2}\right) = (-,+)$. This shift cannot affect any other rows, as the shift will also be irreversible in these rows, which should all cycle back to the entries of $\mathcal{B}_0$. So $(b_{p-1,r,m_1}, b_{p-1,r,m_2}) \neq (+,-)$ for all $r \neq \overline{R}-1$. (There might be other irreversible shifts affecting the preperiod in the path, but they too cannot affect any other rows than the row $\overline{R}-1$.)

Assume there are two column cycles in the preperiod of length p_1 and p_2 with p_1, p_2 coprime, and that all other cycles have length p_i which either divides p_1 or p_2. In B_{p-1}/τ, we define the first column c_{m_1} of the p_1–cycle and the first column c_{m_2} of the p_2–cycle by

$$b_{p-1,r,m_1} = \begin{cases} + & \text{if } (r \bmod p_1) = p_1 - 1 \ , \\ - & \text{if } (r \bmod p_1) \neq p_1 - 1 \ , \end{cases}$$

$$b_{p-1,r,m_2} = \begin{cases} + & \text{if } (r \bmod p_2) = p_2 - 1 \ , \\ - & \text{if } (r \bmod p_2) \neq p_2 - 1 \ , \end{cases}$$

where $\overline{R} = p_1 p_2$, and $r \in \left\{0, \ldots, \overline{R}-1\right\}$. There is exactly one row index $r = p_1 p_2 - 1$, where $(b_{p-1,r,m_1}, b_{p,r,m_2}) = (+,-)$. Now set $R = 1$. Set $\left(b_{p-1,\overline{R},m_1}, b_{p,\overline{R},m_2}\right) = (-,+)$ and $b_{p-1,\overline{R},m} = b_{p-1,\overline{R}-1,m}$ for $m \in \{0, \ldots, M-1\} \setminus \{m_1, m_2\}$. Now the swapping of $(b_{p,p_1p_2-1,m_1}, b_{p,p_1p_2-1,m_2})$ is irreversible, and makes the rows $\overline{R}-1$ and $\overline{R}$ identical.

As we must have a common 1–cycle in the preperiod and period to swap the columns from $\mathcal{B}_0/\tau$ to $\mathcal{B}_p/\tau$, we find the maximal value of $\overline{R}$ as $\overline{R} = p_1p_2$ under the constraint $p_1+p_2 = M-1$. So if say M is even, we have $\max(p_1p_2) = \frac{M}{2}\left(\frac{M}{2}-1\right) \to \left(\frac{M}{2}^2\right)$ as $\underline{M} \to -\infty, \overline{M} \to \infty$.

Now assume there are three column cycles in the preperiod of length p_1, p_2, p_3, all coprime. Then the columns of the p_1–cycle must have a pattern with p_1 entries that repeats p_2p_3 times, and similarly, the columns of the p_2–cycle must have a pattern with p_2 entries that repeats p_1p_3 times. Let c_{m_1} be part of the p_1–cycle, and let c_{m_2} be part of the p_2–cycle. Let $r \in \left\{0, \ldots, \overline{R}-1\right\}$ such that $(b_{p-1,r,m_1}, b_{p-1,r,m_2}) = (+,-)$. Then there are at least p_3-1 other row indices $r' \in \left\{0, \ldots, \overline{R}-1\right\} \setminus \{r\}$ such that $(b_{p-1,r',m_1}, b_{p-1,r',m_2}) = (+,-)$. Thus, it is impossible to make an irreversible shift that only affects one row in the preperiod.

The same argument can be used when there are four or more column cycles of coprime lengths, which shows that there cannot be more than two column cycles with coprime period lengths.

To show that $\tilde{P} - L + \overline{p} \leq K_{\underline{M},\overline{M}}\, p$, it is enough to examine the extreme case where $\overline{p} = \max\left(\operatorname{lcm}\left(p_1, \ldots, p_n\right)\right) p$ with $\sum_{i=1}^{n} p_i = M-1$. If we have two columns c_{m_1}, c_{m_2} that either belong to a p_i– and a $p_{i'}$–cycle, or the same p_i–cycle, it is impossible to find an irreversible shift between c_{m_1} and c_{m_2} in the preperiod that would not affect rows in the period. As the 1–cycle that is used to swap columns must have identical elements in the period and the preperiod, we cannot find an irreversible shift that affects this 1–cycle. This assumes that we can fit three cycles of coprime period lengths in the period, and $11 = 1+2+3+5$ is the smallest value where this is possible. So with $M \geq 11$ and $\overline{p}$ maxed out, we cannot have any values in the preperiod at all, unless we have large elements in the seed.

We can adjoin some additional $\hat{K}$ values to the start of the preperiod, but only if they are not excluded or cause any exclusions. So we must adjoin negative difference values equal or smaller to $\underline{M}$. The maximal number of negative values we can adjoin this way is $\max_{x<L} g_x - L$, as shown in Lemma 3. □

Example 13. Let us modify Example 11 to find the longest possible preperiod in $\Gamma\left(\overline{R}+R, -3, 3\right)$. We find a preperiod of length $6 * 5 + 1 = 31$, and a difference period of length 5. The seed contains one value. We have $\overline{R} + R = 7 > 6 = K_{-3,3}$, which is possible, as $M = 6 < 11$.

$$
\begin{array}{ccc|ccccc|ccc}
\_\_ & \_\_ & [+3] & -1 & +3 & +3 & -3 & -3 & \_\_ & -3 & \_\_ \\
+3 & \_\_ & \_\_ & +3 & -3 & 0 & +3 & -3 & -1 & \_\_ & \_\_ \\
\_\_ & +3 & \_\_ & -1 & +3 & 0 & -3 & +3 & \_\_ & -3 & \_\_ \\
\_\_ & \_\_ & +3 & +3 & -3 & +3 & -3 & -3 & -1 & \_\_ & \_\_ \\
+3 & \_\_ & \_\_ & -1 & +3 & 0 & +3 & -3 & \_\_ & -3 & \_\_ \\
\_\_ & +3 & \_\_ & +3 & -3 & 0 & -3 & 0 & \_\_ & \_\_ & \_\_ \\
\hline
\_\_ & \_\_ & \_\_ & +3 & +3 & 0 & -3 & -3 & \_\_ & \_\_ & \_\_
\end{array}
$$

We show the binary representations $\mathcal{B}_4$ and $\mathcal{B}_5$ below. Note how the swaps in row 1 and 3 correspond to the rows where the difference value -1 is adjoined, while the irreversible shift in row 5 corresponds to the zero at $d_5(5)$.

$$
\begin{array}{ccc|ccc}
+ & + & - & - & + & - \\
- & - & + & - & + & + \\
+ & - & - & + & + & - \\
- & + & - & - & + & + \\
+ & - & + & - & + & - \\
- & - & - & + & + & + \\
\hline
+ & - & - & - & + & +
\end{array}
\quad \hookrightarrow \quad
\begin{array}{ccc|ccc}
+ & - & - & + & - & + \\
- & + & - & - & + & + \\
- & - & + & + & - & + \\
+ & - & - & - & + & + \\
- & + & - & + & - & + \\
- & - & - & + & + & + \\
\hline
- & - & - & + & + & +
\end{array}
$$

$\Diamond$

6. Conclusion

In this paper, we have established bounds for the period length and the preperiod length of Nim sequences. To do this, we defined several concepts, such as cut sets and binary representations. These concepts could possibly be used in the study of specific combinatorial games, like Wythoff's game and Chomp.

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References

[1] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning Ways for Your Mathematical Plays, Second Edition*, **1**, A. K. Peters, Wellesley, Massachusetts, 2001.

[2] A. Dress, A. Flammenkamp, and N. Pink, Additive periodicity of the Sprague-Grundy function of certain Nim games, *Adv. Appl. Math.* **22** (1999), 249–270.

[3] H. A. Landman, A simple FSM-based proof of the additive periodicity of the Sprague-Grundy function of Wythoff's Game, in *More Games of No Chance*, Cambridge University Press, Cambridge, 2002, 383–386.

[4] S. Huddleston and J. Shurman, Transfinite Chomp, in *More Games of No Chance*, Cambridge University Press, Cambridge, 2002, 183–212.

[5] P. Hegarty and U. Larsson, Permutations of the natural numbers with prescribed difference multisets, *Integers* **6** (2006), Article A3.

[6] L. Abrams and D. Cowen-Morton, Algebraic structure in a family of Nim-like arrays, *Pure Appl. Algebra* **214** (2010), 165–176.

[7] W. A. Wythoff, A modification of the Game of Nim, *Nieuw Arch. Wiskd.* **7** (1907), 199–202.

[8] N. J. A. Sloane (Editor), Sum of primes <= n, *The On-Line Encyclopedia of Integer Sequences*, retrieved February 4th 2019, Sequence A034387.

[9] N. J. A. Sloane (Editor), Primorial numbers (second definition), *The On-Line Encyclopedia of Integer Sequences*, retrieved February 4th 2019, Sequence A034386.

[10] T. Salát and S. Znám, On the sums of the prime powers, *Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math.* **21** (1968), 21–24.