MORE ABOUT EXACT SLOW k-NIM

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Received: 2/6/21, Accepted: 8/30/21, Published: 9/27/21

Abstract

Given n piles of tokens and a positive integer $k \leq n$, the game $\text{NIM}^1_{n,=k}$ of exact slow k-NIM is played as follows. Two players move alternately. In each move, a player chooses exactly k non-empty piles and removes one token from each of them. A player whose turn it is to move but has no move loses (if the normal version of the game is played, and wins if it is the misère version). In Integers 20 (2020), #G3, Gurvich et al. gave an explicit formula for the Sprague-Grundy function of $\text{NIM}^1_{4,=2}$, for both its normal and misère versions. Here we extend this result and obtain an explicit formula for the P-positions of the normal version of $\text{NIM}^1_{5,=2}$ and $\text{NIM}^1_{6,=2}$.

1. Introduction and Main Results

Games $\text{NIM}^1_{n,=k}$ and $\text{NIM}^1_{n,\leq k}$ of Exact and Moore's Slow k-NIM were introduced in 2015 [4]. The present paper extends some results obtained in [3] for $\text{NIM}^1_{4,=2}$ to $\text{NIM}^1_{5,=2}$ and $\text{NIM}^1_{6,=2}$. All basic definitions (impartial games in normal and misère versions, positions, moves, P- and N-positions, Sprague-Grundy function) can be found in the introduction of [3]; so we will not repeat them. Here we need only P-positions of the normal version of $\textsc{Nim}^1_{n,=k}$.

Positions of $\textsc{Nim}^1_{n,=k}$ are represented by nonnegative integer n-vectors. An n-vector $x = (x_1, \dots, x_n)$ is called *nondecreasing* if $x_1 \le \dots \le x_n$. We will always assume that positions of $\textsc{Nim}^1_{n,=k}$ are represented by nondecreasing vectors, yet this assumption may hold for x but fail for x' after a move $x \to x'$. In this case we reorder coordinates of x' to maintain the assumption; see more details in [3].

Note that $\textsc{Nim}^1_{n',=k}$ is a subgame of $\textsc{Nim}^1_{n,=k}$ whenever $n' \le n$. Indeed, the set of n-vectors $x = (x_1, \dots, x_n)$ (which are positions of $\textsc{Nim}^1_{n,=k}$) whose first $n - n'$ coordinates are zeros is in an obvious one-to-one correspondence with the set of n'-vectors (which are the positions of $\textsc{Nim}^1_{n',=k}$). Thus, a formula for the P-positions or the Sprague-Grundy function of $\textsc{Nim}^1_{n,=k}$ works for $\textsc{Nim}^1_{n',=k}$ as well.

By definition, any move $x \to x'$ in $\textsc{Nim}^1_{n,=k}$ reduces exactly k coordinates of x by exactly one each; hence, $\sum_{i=1}^n x_i = \sum_{i=1}^n x'_i \mod k$. Thus $\textsc{Nim}^1_{n,=k}$ is split into k disjoint subgames $\textsc{Nim}^1_{n,=k}[j]$ for $j = 0, \dots, k-1$ such that $\sum_{i=1}^n x_i = j \mod k$.

Explicit formulas were obtained in [3] for the Sprague-Grundy function of both the normal and misère versions of $\textsc{Nim}^1_{4,=2}$. Here we extend this result and give explicit formulas for the P-positions of the normal versions of $\textsc{Nim}^1_{5,=2}$ and $\textsc{Nim}^1_{6,=2}$. We will assume that $\textsc{Nim}^1_{n,=k}$ and $\textsc{Nim}^1_{n,=k}[j]$ refer to the normal version of the game unless it is explicitly said otherwise.

Given $x = (x_1, \dots, x_n)$, its *parity vector* $p(x) = (p(x_1, \dots, p(x_n))$ is defined as follows: $p(x)$ has n coordinates taking values $p(x_i) = e$ if x_i is even and $p(x_i) = o$ if x_i is odd, for $i = 1, \dots, n$.

Theorem 1. *The P-positions of* $\textsc{Nim}^1_{6,=2}[0]$ *are characterized by the parity vectors:*

$$(e, e, o, o, o, o), (o, o, e, e, o, o), (o, o, o, o, e, e), (e, e, e, e, e, e). \tag{1}$$

Remark 1. Somewhat surprisingly, the same characterization holds for the P-positions of Moore's Slow $\textsc{Nim}^1_{6,\le 3}$; see part (5) of Theorem 2 in [3].

The set of P-positions of $\textsc{Nim}^1_{6,=2}[0]$ can be defined by a system of linear equations modulo 2 and the nondecreasing condition. More specifically, the system of equations is

$$\begin{aligned} x_2 - x_1 &\equiv 0 \pmod 2, \\ x_4 - x_3 &\equiv 0 \pmod 2, \\ x_5 - x_4 - x_1 &\equiv 0 \pmod 2. \end{aligned} \tag{2}$$

Note that Equations (2) imply $x_6 - x_5 \equiv 0 \pmod 2$ since

$$x_6 - x_5 + (x_4 - x_3) + (x_2 - x_1) \equiv x_6 + x_5 + x_4 + x_3 + x_2 + x_1 \equiv 0 \pmod 2$$

(the total number of tokens is even). It is easy to see that Equation (1) provides all four solutions of Equations (2) such that $x_6 \equiv x_5 \pmod 2$.

The following concept will play an important role. Given a game $\text{NIM}^1_{n,=k}$ or $\text{NIM}^1_{n,=k}[j]$, a nonnegative nondecreasing n-vector y is called a *P-shift* if for any nondecreasing n-vector x we have: either both x and $x+y$ are P-positions of the considered game, or both are not. Obviously, if y' and y'' are P-shifts then $y = y' + y''$ is a P-shift too.

By Theorem 1, the nonnegative nondecreasing 6-vectors with even coordinates and also vectors $(0,0,1,1,1,1)$ and $(1,1,1,1,2,2)$ are P-shifts in $\text{NIM}^1_{6,=2}[0]$. (Equations (2) help to check this claim.)

As for $\text{NIM}^1_{6,=2}[1]$, the set of P-positions has a more complicated structure. To describe it, we introduce several conditions on a nondecreasing 6-vector

$$(x_1, x_2, x_3, x_4, x_5, x_6).$$

Define $E(x)$ to be true if Equations (2) are satisfied by x; $F(x)$ to be true if $x_4 - x_3$ is even; and $K(x)$ to be true if the total number of tokens $N(x)$ has residue 1 modulo 4, i.e.

$$N(x) = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \equiv 1 \pmod 4.$$

We define the functions

$$\begin{aligned} r(x) &= x_1 + x_4 - x_5, \\ s(x) &= x_1 - x_2 + x_3 - x_4 - x_5 + x_6, \\ u(x) &= s(x) - 2r(x) + 1. \end{aligned}$$

Note that since $N(x)$ is odd, $s(x)$ is also odd, and $u(x)$ is even. Using these functions we define $T(x) = \min(s(x), u(x))$ and identify three regions as follows:

$$A = \{x : T(x) > 0\}, \quad B = \{x : T(x) = 0\}, \quad C = \{x : T(x) < 0\}.$$

Theorem 2. *The P-positions of* $\text{NIM}^1_{6,=2}[1]$ *coincide with the set*

$$\mathcal{P} = \{x : \big(E(x) \& (x \in A)\big) \vee \big(K(x) \& F(x) \& (x \in B)\big) \vee \big(K(x) \& (x \in C)\big)\}.$$

To get a description of P-positions for $\text{NIM}^1_{5,=2}$ one should set $x_1 = 0$ in Theorems 1 and 2 and shift indices by 1. This gives two corollaries.

Corollary 1. *The P-positions of* $\text{NIM}^1_{5,=2}[0]$ *are characterized by the parity vectors:*

$$(e, o, o, o, o), (e, e, e, e, e). \tag{3}$$

Proof. The two other parity vectors correspond to odd values of x_1 in Equation (1). $\square$

If $y = (y_1, y_2, y_3, y_4, y_5)$ and $x = (0, y_1, y_2, y_3, y_4, y_5)$, then define

$$\begin{aligned}
r'(y) &= r(x) = y_3 - y_4,\\
s'(y) &= s(x) = -y_1 + y_2 - y_3 - y_4 + y_5,\\
u'(y) &= u(x) = s'(y) - 2r'(y) + 1,\\
E'(y) &= E(x) = \big(y_1 \equiv y_3 - y_2 \equiv y_4 - y_3 \equiv 0 \pmod 2\big),\\
K'(y) &= K(x) = \big(y_1 + y_2 + y_3 + y_4 + y_5 \equiv 1 \pmod 4\big).
\end{aligned}$$

Since the components of x and y are in sorted order, $r'(y) \le 0$ and $u'(y) > s'(y)$, and so $T'(y) = \min(s'(y), u'(y)) = s'(y)$.

Corollary 2. *The P-positions of* $\textsc{Nim}^1_{5,=2}[1]$ *are characterized by the formula*

$$(T' < 0) \& K' \vee (T' > 0) \& E'.$$

Proof. The conditions of this corollary are obtained from the conditions of Theorem 2 by setting $x = (0, y_1, y_2, y_3, y_4, y_5)$ as in the definitions above. Since $T'(y) = s'(y) \equiv y_1 + y_2 + y_3 + y_4 + y_5 \equiv 1 \pmod 2$, the case $T'(y) = 0$, corresponding to $x \in B$, is impossible. The correspondence $x \in A$ if $T'(y) > 0$ and $x \in C$ if $T'(y) < 0$ gives the P-positions for $\textsc{Nim}^1_{5,=2}[1]$. $\square$

There are fewer invariant shifts in $\textsc{Nim}^1_{6,=2}[1]$ than in $\textsc{Nim}^1_{6,=2}[0]$.

Proposition 1. *Shifts by vectors*

$$\begin{aligned}
y^{(1)} &= (0,0,1,1,1,1), \quad y^{(2)} = (1,1,1,1,2,2),\\
y^{(3)} &= (0,0,0,2,2,4), \quad y^{(4)} = (0,2,2,2,2,4)
\end{aligned}$$

preserve the sets described in Theorems 1 *and* 2.

Thus, Theorems 1 and 2 imply that these vectors are invariant shifts in $\textsc{Nim}^1_{6,=2}$.

Proof. As mentioned above, shifts by these vectors preserve the set described in Theorem 1. It remains to prove that they preserve the set described in Theorem 2.

A shift by a vector $y^{(i)}$, $1 \le i \le 4$, preserves the parities of $x_2 - x_1$, $x_4 - x_3$, $x_5 - x_4 - x_1$. So the conditions E and F are invariant under the shifts.

A shift by $y^{(i)}$, $1 \le i \le 4$, changes the total number of tokens by a multiple of 4. So the condition K is also invariant under the shifts.

Note that

$$\begin{aligned}
y_1^{(i)} + y_4^{(i)} - y_5^{(i)} &= 0,\\
y_1^{(i)} - y_2^{(i)} + y_3^{(i)} - y_4^{(i)} - y_5^{(i)} + y_6^{(i)} &= 0.
\end{aligned}$$

for $1 \le i \le 4$. Thus the shifts preserve the values of each of the functions r, s, u and T. $\square$

The rest of the paper is organized as follows. In Section 2 we study P-shifts (as well as some "stronger" concepts). In Section 3.1 we prove Theorem 1, and in Section 3.2. we prove Theorem 2.

2. Invariant Shifts

Note that vector $(0,0,0,2,2)$ is a P-shift in $\textsc{Nim}^1_{5,=2}$ assuming Corollaries 1 and 2. The shift by this vector preserves the set described by Equation (3). It changes the total number of tokens by a multiple of 4 and it preserves the condition E' and the function T'. Thus it preserves the set described in Corollaries 1 and 2.

There are more examples: $(1,1,1,1)$ and $(1,1,1,1,1,1)$ are P-shifts for $\textsc{Nim}^1_{4,=2}$ and $\textsc{Nim}^1_{6,=3}$, respectively; $(0,1,1,1,1)$ and $(0,1,1,1,1,1,1)$ are P-shifts for $\textsc{Nim}^1_{5,=2}$ and $\textsc{Nim}^1_{7,=3}$, respectively; $(0,2,2,2)$ and $(0,2,2,2,2)$ are P-shifts for $\textsc{Nim}^1_{4,=3}$ and $\textsc{Nim}^1_{5,=4}$, respectively.

Our next statement, generalizing all six examples, will require the following definitions and notation.

Given a game $\textsc{Nim}^1_{n,=k}$ or $\textsc{Nim}^1_{n,=k}[j]$, a nonnegative nondecreasing n-vector y will be called a *g-shift, or g^--shift, or $g^\pm$-shift* if adding y to any position x of the considered game preserves the Sprague-Grundy (SG) value, misère SG value, or both values, respectively; in other words, if for any nonnegative nondecreasing n-vector x we have $g(x) = g(x+y)$, or $g^-(x) = g^-(x+y)$, or both equalities hold, respectively.

The reader may recall the definitions of the normal and misère SG functions from the introduction of [3]. As usual, by a^ℓ we denote a symbol (in particular, a number) a repeated ℓ times.

Theorem 3. *Vectors (1^{2k}), $(0,1^{2k})$, and $(0,2^k)$) are $g^\pm$-shifts for the exact k-*$\textsc{Nim}$ *games:* $\textsc{Nim}^1_{2k,=k}$, $\textsc{Nim}^1_{2k+1,=k}$, *and* $\textsc{Nim}^1_{k+1,=k}$, *respectively.*

Also, vector (2^k) is a $g^\pm$-shift in $\textsc{Nim}^1_{k,=k}$. However, this game is trivial: $g(x) = 1 - g^-(x) = x_1 \bmod 2$.

The *support*, $\mathrm{supp}(z)$, of a nonnegative n-vector z is defined as the set of its positive coordinates.

Note also that in each of the three games considered in Theorem 3, for the corresponding shift y and any position x we have:

(i) the set difference $\mathrm{supp}(x) \setminus \mathrm{supp}(y)$ contains at most one element;

(ii) the sum of the coordinates of y equals $2k$.

Both observations will be essential in the proof.

Proof of Theorem 3. We have to show that $g(x) = g(x+y)$ and $g^-(x) = g^-(x+y)$ for every position x in each of the three games considered. We will prove all three claims simultaneously by induction on the height $h(x)$ of a position x. Recall that $h(x)$ is defined as the maximum number of successive moves that can be made from x.

To a move $x \to x'$ we assign the move from $x + y \to x''$ that reduces the same coordinates. Then, $g(x') = g(x'')$ holds by the induction hypothesis, implying that $g(x + y) \geq g(x)$. Assume for contradiction that $g(x + y) > g(x)$. Then there is a move $x+y \to z'$ such that $g(z') = g(x)$. If there exists a move $x \to z''$ reducing the same coordinates then the previous arguments work and by induction we obtain that $g(z') = g(z'')$. Hence, $g(z') \neq g(x)$, because otherwise $g(x) = g(z'')$, while $x \to z''$ is a move, which is a contradiction.

However, the required move $x \to z''$ may fail to exist. Obviously, in this case $x_1 = 0$ must hold in all three cases. For $\text{NIM}^1_{2k+1,=k}$ and $\text{NIM}^1_{k+1,=k}$ we also have $y_1 = 0$ and hence, move $x + y \to z'$ cannot reduce $x_1 + y_1 = 0$. But then, in all three cases there exists a move $z' \to x$. Indeed, in $\text{NIM}^1_{k+1,=k}$ we just repeat the previous move reducing the same piles. In contrast, in $\text{NIM}^1_{2k,=k}$ and in $\text{NIM}^1_{2k+1,=k}$ we make the "complementary" move, reducing the k piles that were not touched by the previous move. Note that in the second case we also do not touch the first pile, because it is empty. (Note also that both of the above conditions (i,ii) are essential in all three cases.)

Finally, the existence of the move $z' \to x$ implies that $g(x) \neq g(z')$, Thus, no move from $x+y$ can reach the SG value $g(x)$, implying that equality $g(x) = g(x+y)$ still holds.

The same arguments work in the misère case as well, since the misère SG function g^- is defined by the same recursion as g, differing from it only by the initialization.

It remains to verify the base of induction for both g and g^- and for all three games considered. Note that all terminal positions (from which there are no moves), not only $(0, \ldots, 0)$, must be considered. It is easily seen that the terminal positions in $\text{NIM}^1_{2k,=k}$, $\text{NIM}^1_{2k+1,=k}$, and $\text{NIM}^1_{k+1,=k}$ are exactly the positions with at least $k + 1, k + 2$, and 2 zero coordinates, respectively. For each game we have to verify that $g(x + y) = g(x) = 0$ and $g^-(x + y) = g^-(x) = 1$ for the corresponding shift y and any terminal position x of the game. It is enough to check that

- there is no move $x + y \to x$;
- for any move $x + y \to z'$ there exists a move $z' \to x$ to a terminal position x;
- there exists a move $x + y \to z''$ such that each move from z'' results in a terminal position.

We leave this tedious but simple case analysis to the careful reader. □

Note that Theorem 3 may simplify the analysis of the three games considered. Without loss of generality, we can restrict ourselves to the positions x such that

- $x_1 = 0$ in $\text{NIM}^1_{2k,=k}$; in other words, we reduce $\text{NIM}^1_{2k,=k}$ to $\text{NIM}^1_{2k-1,=k}$; (for $k = 2$ see Theorem 4 and Corollary 1 in [3]);

- $x_1 = x_2$ in $\textsc{Nim}^1_{2k+1,=k}$;
- $x_2 \le x_1 + 1$ for $\textsc{Nim}^1_{k+1,=k}$.

Let us mention a few more observations related to P-shifts.

Vector $(0, 0, 0, 2, 2)$ is not a P-shift in the misère version of $\textsc{Nim}^1_{5,=2}$. For example, direct computations show that $(3, 3, 3, 4, 8)$ is a P-position, while $(3, 3, 3, 6, 10)$ is not. We have no explicit formula for the P-positions in this case.

Vector $(0, 0, 0, 0, 4)$ is not a P-shift in $\textsc{Nim}^1_{5,=2}$. For example, $x' = (2, 2, 3, 4, 6)$ is a P-position, while $x'' = (2, 2, 3, 4, 10)$ is not, by Corollary 2. Indeed, $T'(x') < 0$ $(-2+2-3-4+6 < 0)$ and $K'(x')$ is true $(2+2+3+4+6 = 17 \equiv 1 \pmod 4)$. On the other hand, $T'(x'') > 0$ $(-2+2-3-4+10 > 0)$, and $E'(x'') = (2 \equiv 3-2 \equiv 4-3 \equiv 0 \pmod 2)$ is false. Thus, in accordance with Corollary 2, x' is a P-position, while x'' is not.

Finally, let us consider shift $(0, 0, 0, 2, 2, 2)$ in $\textsc{Nim}^1_{6,=3}$. Our computations suggest the conjecture that it is a P-shift in the subgames $\textsc{Nim}^1_{6,=3}[0]$ and $\textsc{Nim}^1_{6,=3}[1]$. Yet, it is not a g-shift in these games, for example,

$$3 = g(1, 2, 2, 2, 4, 4) \ne g(1, 2, 2, 4, 6, 6) = 5;$$
$$1 = g(1, 2, 3, 3, 3, 4) \ne g(1, 2, 3, 5, 5, 6) = 3.$$

Also, it is not a P-shift in $\textsc{Nim}^1_{6,=3}[2]$; for example,

$$0 = g(0, 7, 7, 7, 7, 10) \ne g(0, 7, 7, 9, 9, 12) = 3.$$

Finally, it is not a g^--shift, and not even a P-shift in the misère version of $\textsc{Nim}^1_{6,=3}[j]$ for all $j = 0, 1, 2$. For example,

$$0 = g^-(1, 2, 3, 3, 3, 3) \ne g^-(1, 2, 3, 5, 5, 5) = 1;$$
$$0 = g^-(1, 2, 3, 3, 3, 4) \ne g^-(1, 2, 3, 5, 5, 6) = 3;$$
$$0 = g^-(0, 1, 2, 2, 2, 4) \ne g^-(0, 1, 2, 4, 4, 6) = 3.$$

3. Proofs of Main Theorems

3.1. Proof of Theorem 1

We have to prove that the set of P-positions of $\textsc{Nim}^1_{6,=2}[0]$ coincides with the set $\mathcal{P}$ satisfying (1). It is enough to show that (I) there is no move $x \to x'$ such that $x, x' \in \mathcal{P}$ and (II) for any $x \notin \mathcal{P}$ there is a move $x \to x'$ such that $x' \in \mathcal{P}$.

`Case (I).` By (1), the Hamming distance between any two parity vectors of distinct positions of $\mathcal{P}$ is exactly 4, while for any move $x \to x'$ the Hamming distance between $p(x)$ and $p(x')$ is exactly 2, in accordance with the rules of the game.

`Case (II).` Fix a position $x = (x_1, \ldots, x_6) \notin \mathcal{P}$ in $\textsc{Nim}^1_{6,=2}[0]$. Since $\sum_{i=1}^6 x_i$ is even, the number j of odd coordinates of x is even too and we have to consider the following three cases.

If $j = 6$, that is, all six piles are odd, $p(x) = (o, o, o, o, o, o)$. In this case we can reduce x_1 and x_2, getting a move $x \to x'$ such that $p(x') = (e, e, o, o, o, o)$.

If $j = 2$, that is, there are exactly two odd piles. Reducing them we get a move $x \to x'$ such that $p(x') = (e, e, e, e, e, e)$.

If $j = 4$, that is, there are four odd piles and two even. Let us match coordinates 1 and 2, 3 and 4, 5 and 6. Make the (unique) move $x \to x'$ reducing in x the larger even pile and the odd one that is matched with the smaller even pile. If the two even piles are of the same size, $x_i = x_{i+1}$, then we agree that pile i is smaller than pile $i + 1$.

It is easy to verify that in all the above cases the chosen move $x \to x'$ is possible, because it reduces (by one token) two non-empty piles; furthermore, in all cases $x' \in \mathcal{P}$, by (1). $\square$

3.2. Proof of Theorem 2

It is convenient to change coordinates for positions. We adopt 'differential coordinates':

$$\begin{aligned} q_1 &= x_1, & q_2 &= x_2 - x_1, & q_3 &= x_3 - x_2, \\ q_4 &= x_4 - x_3, & q_5 &= x_5 - x_4, & q_6 &= x_6 - x_5. \end{aligned}$$

Recall that x is assumed to be nondecreasing. Thus $q_i \geq 0$ for all i.

We define a move, reducing two piles by one token each, to be *legal* if the resulting piles are still in nondecreasing order of size. For a move i, j, where $i < j$ this condition is that $q_i > 0$, and if $i < j - 1$ then also $q_j > 0$.

In these coordinates the conditions used in Theorem 2 are expressed as follows: $r(q) = q_1 - q_5$; $s(q) = -q_2 - q_4 + q_6$; $u(q) = -2q_1 - q_2 - q_4 + 2q_5 + q_6 + 1$; and $T(q) = \min(s(q), u(q))$. The three polyhedral regions are as before:

$$\begin{aligned} A &= \{q : T(q) > 0\}, \\ B &= \{q : T(q) = 0\}, \\ C &= \{q : T(q) < 0\}. \end{aligned} \tag{4}$$

The set described in the theorem is expressed in the differential coordinates as

$$\mathcal{P} = \{q : \big(E(q)\&(q \in A)\big) \vee \big(K(q)\&F(q)\&(q \in B)\big) \vee \big(K(q)\&(q \in C)\big)\}, \tag{5}$$

where

$$\begin{aligned} E(q) &:= (q_2 \equiv q_4 \equiv q_1 - q_5 \equiv 0 \pmod 2), \\ F(q) &:= (q_4 \equiv 0 \pmod 2), \\ K(q) &:= (N(q) \equiv -2q_1 + q_2 - q_4 + 2q_5 + q_6 \equiv 1 \pmod 4). \end{aligned}$$

We have to prove that the set of P-positions of $\textsc{Nim}^1_{6,=2}[1]$ coincides with the set $\mathcal{P}$. The parity condition

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \equiv q_2 + q_4 + q_6 \equiv 1 \pmod 2 \tag{6}$$

implies that $s(q)$ is odd and $u(q)$ is even.

We will use several simple observations on the conditions (5) before and after a move.

Proposition 2. *Let $q \to q'$ be a move in $\textsc{Nim}^1_{6,=2}[1]$. Then $K(q) \oplus K(q') = 1$.*

Proof. The total number of tokens N decreases by 2 after each move. So, the (odd) residue of N modulo 4 alternates between 1 and 3. □

Proposition 3. *Let $q \to q'$ be a move in $\textsc{Nim}^1_{6,=2}[1]$. Then $E(q)$ is true implies that $E(q')$ is false.*

Proof. We repeat the argument from the proof of Theorem 1, case (I). If the total number of tokens is odd and $E(q)$ is true, then the parities of vector x have the form $(a, a, b, b, c, \bar{c})$ where $c = a \oplus b$. Therefore, for distinct positions q, q' satisfying the condition $E(\cdot)$, the Hamming distance between q, q' is exactly 4, while for any move $x \to x'$ the Hamming distance between $p(x)$ and $p(x')$ is exactly 2. □

In the case analysis below, Table 1 is helpful. It shows, for each move, the changes of q-coordinates, $r(q)$, $s(q)$, and $u(q)$.

As in the proof of Theorem 1, it is enough to show that (I) there is no move $q \to q'$ such that $q, q' \in \mathcal{P}$ and (II) for any $q \notin \mathcal{P}$ there is a move $q \to q'$ such that $q' \in \mathcal{P}$.

We prove (I) by contradiction. Suppose that $q, q' \in \mathcal{P}$ for a move $q \to q'$. Propositions 2 and 3 imply that either (i) $K(q)\&\neg E(q)\&\neg K(q')\&E(q')$ or (ii) $\neg K(q)\&E(q)\&K(q')\&\neg E(q')$.

`Case` (i): $K(q)\&\neg E(q)\&\neg K(q')\&E(q')$. In this case, $T(q') > 0$, $E(q')$, and either $(T(q) = 0)\&F(q)$ or $T(q) < 0$.

If $(T(q) = 0)\&F(q)$ then from $K(q)$ and $T(q) = u(q) = 0$, we have

$$\begin{aligned} &-2q_1 + q_2 - q_4 + 2q_5 + q_6 && \equiv 1 \pmod 4, \text{ and} \\ &-2q_1 - q_2 - q_4 + 2q_5 + q_6 + 1 && = 0, \end{aligned}$$

which implies that q_2 is odd. Since $F(q)$ (i.e., q_4 is even) and $E(q')$, any move taking q to q' must have Δq_4 even and Δq_2 odd. From Table 1 we see that there are just four moves with this property but all of these have $\Delta u = 0$, which contradicts the required increase in T.

Otherwise, $T(q) < 0$, so $u(q) \le -2$ or $s(q) \le -1$ but $u(q') \ge 2$ and $s(q') \ge 1$. Since $\Delta u \le 4$ and $\Delta s \le 2$ for all the moves, either $u(q') = 2$ or $s(q') = 1$. From

Moves	Δq_1	Δq_2	Δq_3	Δq_4	Δq_5	Δq_6	Δr	Δs	Δu
$\{1,2\}$	-1	0	$+1$	0	0	0	-1	0	$+2$
$\{1,3\}$	-1	$+1$	-1	$+1$	0	0	-1	-2	0
$\{1,4\}$	-1	$+1$	0	-1	$+1$	0	-2	0	$+4$
$\{1,5\}$	-1	$+1$	0	0	-1	$+1$	0	0	0
$\{1,6\}$	-1	$+1$	0	0	0	-1	-1	-2	0
$\{2,3\}$	0	-1	0	$+1$	0	0	0	0	0
$\{2,4\}$	0	-1	$+1$	-1	$+1$	0	-1	$+2$	$+4$
$\{2,5\}$	0	-1	$+1$	0	-1	$+1$	$+1$	$+2$	0
$\{2,6\}$	0	-1	$+1$	0	0	-1	0	0	0
$\{3,4\}$	0	0	-1	0	$+1$	0	-1	0	$+2$
$\{3,5\}$	0	0	-1	$+1$	-1	$+1$	$+1$	0	-2
$\{3,6\}$	0	0	-1	$+1$	0	-1	0	-2	-2
$\{4,5\}$	0	0	0	-1	0	$+1$	0	$+2$	$+2$
$\{4,6\}$	0	0	0	-1	$+1$	-1	-1	0	$+2$
$\{5,6\}$	0	0	0	0	-1	0	$+1$	0	-2

Table 1: Results of moves $q \to q'$ in 'differential coordinates', $\Delta q_i = q'_i - q_i$, $\Delta r = r(q') - r(q)$, $\Delta s = s(q') - s(q)$, $\Delta u = u(q') - u(q)$.

$\neg K(q')\&E(q')$, we have

$$N(q') \equiv -2q'_1 + q'_2 - q'_4 + 2q'_5 + q'_6 \equiv 3 \pmod 4 \text{ and}$$
$$q'_2 \equiv q'_4 \equiv q'_5 - q'_1 \equiv 0 \pmod 2.$$

Note that

$$3 \equiv N(q') \equiv u(q') + 2q'_2 - 1 \pmod 4 \text{ and}$$
$$3 \equiv N(q') \equiv s(q') + 2q'_2 - 2r(q') - 2 \pmod 4,$$

so $u(q') = 2$ implies q'_2 odd, and $s(q') = 1$ implies $q'_2 - r(q')$ odd. Either of these contradict $E(q')$ which has q'_2 and $r(q')$ both even.

`Case` $\neg K(q)\&E(q)\&K(q')\&\neg E(q')$. In this case, $T(q) > 0$, $E(q)$, and either $(T(q') = 0)\&F(q')$ or $T(q') < 0$. The analysis is almost the same as above, exchanging q' for q, replacing "increase" by "decrease", and similar.

Now we prove (II). Thus, $q \notin \mathcal{P}$ and we are going to indicate a move $q \to q'$ such that $q' \in \mathcal{P}$.

`Case` $q \in C$. This implies $\neg K(q)$. By Proposition 2, $q' \in \mathcal{P}$ for any move $q \to q'$ preserving region C.

Table 1 shows many moves that preserve region C, including $\{2,3\}$, $\{5,6\}$, and $\{1,6\}$. If $q_2 > 0$ or $q_5 > 0$ then $\{2,3\}$ or $\{5,6\}$ is legal and we are done. Now we assume $q_2 = q_5 = 0$.

If $q_6 > 0$ then $\{1,6\}$ is legal provided that $q_1 > 0$. If $q_1 = 0$ then $u(q) = -q_4 + q_6 + 1 > s(q) = -q_4 + q_6 = T(q) < 0$. Move $\{4,6\}$ is legal since $q_4 > q_6 > 0$ and gives $\Delta s = 0$ from Table 1. Therefore $s(q') = s(q) < 0$ and $q' \in \mathcal{P}$.

Otherwise, if $q_6 = 0$ then $s(q) = -q_4 < 0$ and $u(q) = -2q_1 - q_4 + 1 < 0$. If $q_1 > 0$ then $\{1,4\}$ is legal and gives $\Delta s = 0$, thus preserving region C. However, if $q_1 = 0$ then $u(q) = -q_4 + 1 < 0$. Since u is always even, this implies $q_4 \geq 3$. Move $\{4,5\}$ is legal and gives $\Delta s = 2$, so $s(q') = s(q) + 2 = -q_4 + 2 < 0$ and region C is preserved.

`Case` $q \in B$. In this case $T(q) = 0$ and, since this is even, $s(q) > u(q) = T(q) = 0$. Therefore $r(q) > 0$, i.e., $q_1 > q_5 \geq 0$, and $q_6 \geq s(q) > 0$.

`Subcase` $q \in B$ `and` $\neg K(q)$. Proposition 2 proves $K(q')$. Since $2q_2 \equiv 1 + N(q) \equiv 0 \pmod 4$, q_2 is even. We will show that at least one of the moves $\{1,6\}$, $\{4,5\}$ or $\{4,6\}$ gives $q' \in \mathcal{P}$.

If q_4 is even, we use $\{1,6\}$, which is legal since $q_1 > 0$ and $q_6 > 0$. Table 1 shows that $\Delta q_4 = 0$ and $\Delta T \leq 0$, therefore $K(q')\&F(q')\&(q' \in B \cup C)$ and so $q' \in \mathcal{P}$.

If q_4 is odd, then $q_4 > 0$ and both $\{4,5\}$ and $\{4,6\}$ are legal. If $r(q)$ is even, then we see from Table 1 that $\{4,5\}$ gives $\Delta q_2 = 0$, $\Delta q_4 = -1$, $\Delta r = 0$, $\Delta T = 2$, so $E(q')\&(T(q') = 2)$ and therefore $q' \in \mathcal{P}$. Otherwise, if $r(q)$ is odd then $\{4,6\}$ ensures in a similar way that $E(q')\&(T(q') > 0)$ and therefore $q' \in \mathcal{P}$.

`Subcase` $q \in B$ `and` $K(q)$. Since $2q_2 \equiv 1 + N(q) \equiv 2 \pmod 4$, q_2 is odd, and since $q \notin \mathcal{P}$, F is false, i.e., q_4 is odd. Also $s(q) = -q_2 - q_4 + q_6$ is always odd, so q_6 is odd too. We have $q_1 > 0$ and both of q_2 and q_4 odd, so moves $\{1,4\}$ and $\{2,4\}$ are legal. Since $q \in B$, $s(q) > u(q) = 0$ and so Table 1 shows that each of these moves gives $T(q') = \min(s(q'), u(q')) > 0$, i.e., $q' \in A$. To prove $q' \in \mathcal{P}$ we have only to show $E(q')$.

If $r(q)$ is even then $\{1,4\}$ gives $\Delta q_2 = +1$, $\Delta q_4 = -1$, $\Delta r = -2$, so $E(q')$.

If $r(q)$ is odd then $\{2,4\}$ gives $\Delta q_2 = -1$, $\Delta q_4 = -1$, $\Delta r = -1$, so $E(q')$.

`Case` $q \in A$. In this case $E(q)$ is false. For the positions in this case, we are going to indicate a move $q \to q'$ such that $q' \in A$ and $E(q')$ is true, so $q' \in \mathcal{P}$. Depending on the parities of q_2, q_4, $r(q)$, suitable moves are listed in Table 2. It is easy to check using Table 1 that the listed moves make q_2, q_4 and $r(q)$ even, ensuring $E(q')$, and furthermore they also satisfy $\Delta T \geq 0$, preserving A. We now need to check that these moves are legal.

In the subcase A_2 the move $\{4,5\}$ is always legal, since $q_4 > 0$. Similarly, for the move $\{2,3\}$ in the subcase A_6 and for the move $\{2,4\}$ in the subcase A_7. Other subcases require a more detailed analysis.

`Subcase` A_1. If $q_1 > 0$ then $\{1,2\}$ is legal, otherwise $q_1 = 0$. This implies that $q_5 > 0$ since $r(q) = q_1 - q_5$ is odd in this subcase. Therefore move $\{5,6\}$ is legal and changes the parities to $q_2' = q_2 \equiv q_4' = q_4 \equiv (q_5' - q_1') = q_5 - 1 - q_1 \equiv 0 \pmod 2$. Thus $E(q')$ is true, but we still need to prove that $q' \in A$. In general, this move

Subcase	q_2 mod 2	q_4 mod 2	$r(q)$ mod 2	moves to $\mathcal{P}$
A_1	0	0	1	$\{1,2\}$
A_2	0	1	0	$\{4,5\}$
A_3	0	1	1	$\{4,6\}$
A_4	1	0	0	$\{2,6\}$
A_5	1	0	1	$\{2,5\}$
A_6	1	1	0	$\{2,3\}$
A_7	1	1	1	$\{2,4\}$

Table 2: Winning moves in the case $q \in A$

does not preserve the region A: $\Delta s = 0$ but $\Delta u = -2$. Since $r(q) = q_1 - q_5 \leq -1$ and $T(q) = s(q) \geq 1$, we have $u(q) = s(q) - 2r(q) + 1 \geq 4$, and so $T(q') > 0$ and $q' \in A$.

`Subcase` A_3. Since $s(q) \geq 0$ and q_4 is odd, $q_6 > 0$ in this case. Thus the move $\{4, 6\}$ is legal.

`Subcase` A_4. Again, since $s(q) \geq 0$ and q_2 is odd, $q_6 > 0$ in this case, and the move $\{2, 6\}$ is legal.

`Subcase` A_5. If $\{2, 5\}$ is legal then we are done. Otherwise, $q_5 = 0$, since q_2 is odd in this case. This implies that $q_1 > 0$ since $r(q) = q_1 - q_5$ is odd. Again, $q_6 > 0$ because q_2 is odd and $s(q) \geq 0$. Thus, the move $\{1, 6\}$ is legal and it changes the parities to $q_2' = q_2 + 1 \equiv q_4' = q_4 \equiv r(q') = r(q) - 1 \equiv 0 \pmod 2$. Thus $E(q')$ is true. For $\{1, 6\}$, $\Delta s = -2$ and $\Delta u = 0$. From $u(q) = s(q) - 2r(q) + 1 \geq 2$ and $r(q) > 0$ we conclude that $s(q) \geq 3$ and so $s(q') \geq 1$. Therefore $T(q') > 0$ and $q' \in A$. $\square$

4. Open Questions and Conjectures

P-positions of $\textsc{Nim}^1_{6,=2}$ are characterized by the explicit formulas of Theorems 1 and 2. Although they look complicated, they immediately provide a polynomial time algorithm to solve the following problem: given a nonnegative integer vector $(x_1, x_2, x_3, x_4, x_5, x_6)$, whose coordinates x_i are represented in binary, check whether $(x_1, x_2, x_3, x_4, x_5, x_6)$ is a P-position in $\textsc{Nim}^1_{6,=2}$.

Note that the set of P-positions of $\textsc{Nim}^1_{6,=2}$ is semilinear. Recall that a *semilinear set* is a set of vectors from $\mathbb{N}^d$ that can be expressed in Presburger arithmetic; see, e.g., [6]. Presburger arithmetic admits quantifier elimination. So, a semilinear set can be expressed as a finite union of solutions of systems of linear inequalities and equations modulo some integer (they are fixed for the set).

This observation leads to a natural conjecture.

Conjecture 1. For any n, k P-positions of $\textsc{Nim}^1_{n,=k}$ form a semilinear set.

Conjecture 1 implies that there exists a polynomial time algorithm deciding P-positions of $\textsc{Nim}^1_{n,=k}$ for all n, k.

It is well-known that evaluating formulas in Presburger arithmetic is a very hard problem. The first doubly exponential time bounds for this problem were established by Fischer and Rabin [2]. Berman proved that the problem requires doubly exponential space [1].

But these hardness results are irrelevant for our needs. In solving $\textsc{Nim}^1_{n,=k}$ we deal with sets of dimension $O(1)$ and may hardwire the description of the set of P-positions in an algorithm solving the game. Verifying a linear inequality takes polynomial time as well as verifying an equation modulo an integer. Thus, any semilinear set of dimension $O(1)$ can be recognized by a polynomial time algorithm.

Note that for a wider class of games, so-called multidimensional subtraction games with a fixed difference set, P-positions can be not semilinear. Moreover, it was recently proven that there is no polynomial time algorithm solving a specific game of this sort [5].

In the game from [5] it is allowed to add tokens to some piles by a move, provided that the total number of tokens is strictly decreasing. We consider this feature crucial for hardness results.

To put it more formally, we need to make formal definitions. A game from the class FDG is specified by a finite set $\mathcal{D} \subset \mathbb{Z}^d$ which is called the *difference set*. We require that

$$\sum_{i=1}^{d} a_i > 0$$

for each $(a_1, \dots, a_d) \in \mathcal{D}$.

Positions of the game are vectors $x = (x_1, \dots, x_d)$ with nonnegative coordinates. A move from x to y is possible if $x - y \in \mathcal{D}$.

It is proven in [5] that for some constant d there exists a set $\mathcal{D}$ such that there is no algorithm solving the game with the difference set $\mathcal{D}$ and running in time $O(2^{n/16})$, where n is the input size (the length of the binary representation of the position $(x_1, \dots, x_d)$).

Also, it was proved by Larsson and Wästlund [7] that the equivalence problem for FDG is undecidable. The existence of difference vectors with negative coordinates is essential for both results.

$\textsc{Nim}^1_{n,=k}$ belongs to the class FDG: difference vectors are (0,1)-vectors with exactly k coordinates equal to 1. The solution of $\textsc{Nim}^1_{6,=2}$ encourages us to suggest a stronger conjecture.

Conjecture 2. For any FDG game such that $a_i \geq 0$ for each $(a_1, \dots, a_d) \in \mathcal{D}$, the set of P-positions is semilinear.

Conjecture 2 also implies that the equivalence problem for FDG with nonnegative difference vectors is decidable, since Presburger arithmetic is decidable.

Why do we believe in Conjecture 2 in spite of results from [7] and [5]? The first idea is to apply inductive arguments in the case of nonnegative difference vectors. Note that Conjecture 2 holds for a 1-dimensional FDG. If a coordinate becomes zero it remains zero during the rest of the game (if all difference vectors are nonnegative). So to prove Conjecture 2 one needs to provide a reduction in dimension of a game.

This idea does not work in a straightforward manner. It can be shown that arbitrary semilinear boundary conditions do not imply semilinearity of the solution for the recurrence determining the set of P-positions [8]. For the boundary conditions determined by FDG games the question is open and we still have a hope to develop a dimension reducing technique.

In any case, characterization of FDG games with semilinear P-positions would be an important problem for future research.

Acknowledgements. The paper was prepared within the framework of the HSE University Basic Research Program; the fifth author was funded partially by the RFBR grant 20-01-00645 and State Assignment, theme no. 0063-2019-0003.

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