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DISTRIBUTIONS OF FOURIER COEFFICIENTS OF CUSP FORMS OVER ARITHMETIC PROGRESSIONS

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Abstract

Let $\lambda_f(n)$ be the Fourier coefficients of holomorphic cusp forms $f(z)$. In this paper, we study the distributions of $\lambda_f^2(n^j)$ over arithmetic progressions for $j \geq 2$.

1. Introduction

Problems concerning the oscillation behavior of Fourier coefficients of automorphic forms have been widely researched by many mathematicians. Throughout this paper, we concentrate on the holomorphic cusp forms for the full modular group $SL_2(\mathbb{Z})$. Let H_k denote the set of normalized primitive holomorphic cusp forms of even integral weight k. The Fourier expansion of $f \in H_k$ at the cusp ∞ is given by

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi n i z},$$

where the coefficients $\lambda_f(n) \in \mathbb{R}$ are eigenvalues of Hecke operators T_n. The Ramanujan–Petersson conjecture, which was proved by Deligne [7], states that

$$|\lambda_f(n)| \leq d(n), \tag{1}$$

where $d(n)$ is the divisor function.

In 2001, Ivić [13] firstly gave an upper bound for $\sum_{n\le x}\lambda_f(n^2)$. For any holomorphic cusp form of even integral weight k satisfying $k \ll x^{\frac{1}{3}}(\log x)^{\frac{22}{3}}$, Sankaranarayanan [33] showed that

$$\sum_{n\le x}\lambda_f(n^2) \ll x^{\frac{3}{4}}(\log x)^{\frac{19}{2}}\log\log x.$$

Subsequently, Lü [23] proved that for any $\varepsilon > 0$,

$$\sum_{n\le x}\lambda_f(n^3) \ll_{f,\varepsilon} x^{\frac{3}{4}+\varepsilon} \quad\text{and}\quad \sum_{n\le x}\lambda_f(n^4) \ll_{f,\varepsilon} x^{\frac{7}{9}+\varepsilon}.$$

On the other hand, the Rankin–Selberg method (see [8, Theorem A]) yields

$$\sum_{n\le x}\lambda_f^2(n) = C_f x + O_f(x^{\frac{3}{5}}).$$

Later Lao and Sankaranarayanan [26] established

$$\sum_{n\le x}\lambda_f^2(n^j) = c_j x + O_f(x^{1-\frac{2}{(j+1)^2+1}}),$$

where $j = 2, 3, 4$.

Recently many scholars have studied the distributions of Fourier coefficients over arithmetic progressions. Ichihara [15, 16] has treated $\sum_{n\le x}\lambda_f^2(n)$ over arithmetic progressions for holomorphic cusp forms for $x \ll q^2$:

$$\begin{aligned}\sum_{\substack{n\le x\\ n\equiv l(\bmod q)}}\lambda_f^2(n) =&\frac{c}{\varphi(q)}\prod_{p\mid q}(1-\alpha_f(p)^2p^{-1})(1-p^{-1})(1-\beta_f(p)^2p^{-1})(1+p^{-1})^{-1}x\\ &+O_{f,\varepsilon}(x^{\frac{3}{5}}q^{\frac{4}{5}+\varepsilon}),\end{aligned} \tag{2}$$

where c is a constant and only depending on f. More recently, for $f \in H_k$, Jiang and Lü [19, Theorem 1.1] considered the sum of $\lambda_f^{2j}(n)$ over arithmetic progressions for $j = 2, 3, 4$, and obtained the corresponding asymptotic formulae.

In this paper, we will discuss the distributions of $\lambda_f^2(n^j)$ over arithmetic progressions. Combining classical analytic method with properties of some nice automorphic L-functions, we establish asymptotic formulae of $\sum_{n\le x}\lambda_f^2(n^j)\chi(n)$ and $\sum_{n\le x}\lambda_f^2(n^j)\chi_0(n)$. With the help of the orthogonality of Dirichlet characters, we obtain the following results.

Theorem 1. *Let $f \in H_k$ and $l \in \mathbb{Z}$. Let q be a prime with $(q, l) = 1$. For any $\varepsilon > 0$ and $j \geq 2$, if $q \leq x^{3/4\theta_j}$, then*

$$\sum_{\substack{n \leq x \\ n \equiv l (\bmod q)}} \lambda_f^2(n^j) = \frac{c_j x}{\varphi(q)} + O_{f,\varepsilon}\left(q x^{1-\frac{3}{2}\theta_j+\varepsilon}\right),$$

c_j are suitable constants, $\theta_2 = \frac{92}{597}$ and $\theta_j = \frac{92}{69(j-1)(j+3)+247}$ for $j \geq 3$.

2. Some Lemmas

In this section, we will briefly recall and establish some preliminary results which are needed in the proof of Theorem 1. According to the theory of Hecke operators and the work of Deligne [7] on the Ramanujan–Petersson conjecture, for any prime number p, the local parameters $\alpha_f(p)$ and $\beta_f(p)$ satisfy

$$\lambda_f(p) = \alpha_f(p) + \beta_f(p), \quad \alpha_f(p)\beta_f(p) = |\alpha_f(p)| = |\beta_f(p)| = 1. \tag{3}$$

The jth symmetric power L-function attached to $f \in H_k$ is defined as

$$L(s, \mathrm{sym}^j f) := \prod_p \prod_{m=0}^{j} \left(1 - \alpha_f(p)^{j-2m} p^{-s}\right)^{-1}, \quad \Re s > 1, \tag{4}$$

and it can be expressed as a Dirichlet series

$$L(s, \mathrm{sym}^j f) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s} = \prod_p \left(1 + \sum_{k \geq 1} \frac{\lambda_{\mathrm{sym}^j f}(p^k)}{p^{ks}}\right), \tag{5}$$

where $\lambda_{\mathrm{sym}^j f}(n)$ is real and multiplicative. In particular,

$$L(s, \mathrm{sym}^0 f) = \zeta(s), \quad L(s, \mathrm{sym}^1 f) = L(s, f).$$

The Rankin–Selberg L-function attached to $\mathrm{sym}^j f \times \mathrm{sym}^j f$ is defined by

$$L(s, \mathrm{sym}^j f \times \mathrm{sym}^j f) := \prod_p \prod_{m=0}^{j} \prod_{u=0}^{j} \left(1 - \alpha_f(p)^{j-2m} \alpha_f(p)^{j-2u} p^{-s}\right)^{-1}. \tag{6}$$

Similarly, for $\Re s > 1$, we have

$$L(s, \mathrm{sym}^j f \times \mathrm{sym}^j f) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f \times \mathrm{sym}^j f}(n)}{n^s} = \prod_p \left(1 + \sum_{k \geq 1} \frac{\lambda_{\mathrm{sym}^j f \times \mathrm{sym}^j f}(p^k)}{p^{ks}}\right), \tag{7}$$

where $\lambda_{\mathrm{sym}^j f\times \mathrm{sym}^j f}(n)$ is real and multiplicative.

Let χ be a Dirichlet character modulo q. Then we define the twisted jth symmetric power L-function to be the degree $j+1$ Euler product

$$L(s,\mathrm{sym}^j f\otimes\chi)=\prod_p\prod_{m=0}^{j}\left(1-\alpha_f(p)^{j-2m}\chi(p)p^{-s}\right)^{-1}, \tag{8}$$

and the Rankin–Selberg convolution of $\mathrm{sym}^j f$ and $\mathrm{sym}^j f\otimes\chi$ via the degree $2j+2$ Euler product

$$L(s,\mathrm{sym}^j f\times\mathrm{sym}^j f\otimes\chi)=\prod_p\prod_{m=0}^{j}\prod_{u=0}^{j}\left(1-\alpha_f(p)^{j-2m}\alpha_f(p)^{j-2u}\chi(p)p^{-s}\right)^{-1}. \tag{9}$$

Similarly, for $\Re s>1$, we have

$$\begin{aligned}L(s,\mathrm{sym}^j f\times\mathrm{sym}^j f\otimes\chi)&=\sum_{n=1}^{\infty}\frac{\lambda_{\mathrm{sym}^j f\times\mathrm{sym}^j f}(n)\chi(p)}{n^s}\\&=\prod_p\left(1+\sum_{k\geq 1}\frac{\lambda_{\mathrm{sym}^j f\times\mathrm{sym}^j f}(p^k)\chi^k(p)}{p^{ks}}\right).\end{aligned} \tag{10}$$

It is easy to get that

$$L(s,\mathrm{sym}^0 f\otimes\chi)=L(s,\chi),\quad L(s,\mathrm{sym}^m f\times\mathrm{sym}^0 f\otimes\chi)=L(s,\mathrm{sym}^m f\otimes\chi).$$

Lemma 1 ([3-6, 9, 20-22, 29]). *Let $f\in H_k$ be a primitive cusp form. The jth symmetric power L-function $L(s,\mathrm{sym}^j f)$ is defined by Equation (4). For $j\geq 1$, $L(s,\mathrm{sym}^j f)$ has an analytic continuation as an entire function in the whole complex plane $\mathbb{C}$ and satisfies a certain functional equation of Riemann zeta-type of degree $j+1$ (see [2, §3.2.1]).*

Proof. Based on the work of Gelbart and Jacquet [9], Kim and Shahidi [20–22], Dieulefait [6], Clozel and Thorne [3–5], we know that for $2\leq j\leq 8$, $\mathrm{sym}^j f$ is an automorphic cuspidal representation of $GL_{j+1}(\mathbb{A}_{\mathbb{Q}})$. Recently Newton and Thorne [29, Theorem A] proved the automorphy of all symmetric powers for cuspidal Hecke eigenforms of level 1 and weight $k\geq 2$. More precisely, for $j\geq 1$ and $f\in H_k$, the L-function $L(s,\mathrm{sym}^j f)$ attached to $\mathrm{sym}^j f$ is automorphic. Hence the jth symmetric power L-function can be extended as an entire function and also satisfy a nice functional equation of degree $j+1$. □

Lemma 2 ([17, 18, 27, 32, 34, 35]). *Let $f\in H_k$ be a primitive cusp form. Let $L(s,\mathrm{sym}^j f\times\mathrm{sym}^j f)$ be defined as in Equation (6). For $j\geq 1$, the complete*

$L(s, \mathrm{sym}^j f \times \mathrm{sym}^j f)$ is entire except possibly for simple poles at $s = 0, 1$ and satisfies a certain functional equation of Riemann zeta-type of degree $(j+1)^2$ (see [27, Proposition 2.1]).

Proof. Based on the work of Jacquet and Shalika [17,18], Shahidi [34,35], Rudnick and Sarnak [32], Lau and Wu [27], the L-function $L(s, \mathrm{sym}^j f \times \mathrm{sym}^j f)$ associated to $\mathrm{sym}^j f$ and $\mathrm{sym}^j f$, where $j \geq 1$ and $f \in H_k$, is an entire function except possibly for simple poles at $s = 0, 1$ and satisfies a certain Riemann zeta-type functional equation of degree $(j + 1)^2$. □

Lemma 3 ([19, 29]). *Let $f \in H_k$ be a Hecke eigencuspform and χ be a primitive character modulo a prime q. The completed L-function defined as*

$$\Lambda(s, \mathrm{sym}^j f \times \mathrm{sym}^i f \otimes \chi) = q^{(j+1)(i+1)s/2}\gamma(s)L(s, \mathrm{sym}^j f \times \mathrm{sym}^i f \otimes \chi)$$

is an entire function and satisfies a functional equation

$$\Lambda(s, \mathrm{sym}^j f \times \mathrm{sym}^i f \otimes \chi) = \epsilon(f, \chi)\Lambda(1 - s, \mathrm{sym}^j f \times \mathrm{sym}^i f \otimes \overline{\chi}),$$

where $j \geq 1$, $0 \leq i \leq j$, $|\epsilon(f, \chi)| = 1$, and $\gamma(s)$ denotes the product of some gamma functions $\Gamma((s + \kappa_n)/2)$, $n = 1, \ldots, (j + 1)(i + 1)$, with κ_n depending on the weight or spectrum of f and the parity of the character χ and $\Re\kappa_n \geq 0$.

Proof. The classical results of Lemma 3 for $1 \leq j \leq 4$ and $0 \leq i \leq j$ are due to Lemma 2.1 of Jiang and Lü [19]. From the excellent work of Newton and Thorne [29, Theorem A], we can learn that for $j \geq 1$, an automorphic cuspidal self-dual representation $\mathrm{sym}^j \pi = \bigotimes_{p\leq\infty} \mathrm{sym}^j \pi_p$ exists. On the other hand, whether χ is odd or even, we can follow the proof of Lemma 2.1 of Jiang and Lü [19] to get the desired results about $\epsilon(f, \chi)$ and $\gamma(s)$. □

From Lemma 1 and Lemma 2, for $j \geq 1$ and $f \in H_k$, we observe that $L(s, \mathrm{sym}^j f)$ and $L(s, \mathrm{sym}^j f \times \mathrm{sym}^j f)$ are general L-functions in the sense of Perelli [31]. For general L-functions, we have the following averaged or individual convexity bounds (see [24,25,31]).

Lemma 4 ([24, 25, 31]). *Suppose that $\mathfrak{L}(s)$ is a general L-function of degree m. Then for any $\varepsilon > 0$, we have*

$$\int_T^{2T} |\ \mathfrak{L}(\sigma + it)\ |^2\, dt \ll T^{\max\{m(1-\sigma),1\}+\varepsilon} \tag{11}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1$ and $T \geq 1$; and

$$|\ \mathfrak{L}(\sigma + it)\ | \ll (1+ |\ t\ |)^{\frac{m}{2}(1-\sigma)+\varepsilon} \tag{12}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1 + \varepsilon$ and $|\ t\ | \geq 1$.

In order to get a better result, we introduce the following two subconvexity bounds which are the new breakthroughs of Bourgain [1, Theorem 5] and Nunes [30, Corollary 1.2], respectively.

Lemma 5 ([1,30]). *For any $\varepsilon > 0$, $\frac{1}{2} \leq \sigma \leq 1 + \varepsilon$ and $|t| \geq 2$, we have*

$$\zeta(\sigma + it) \ll_{f,\varepsilon} (1+ \mid t \mid)^{\max\left\{\frac{13}{42}(1-\sigma),0\right\}+\varepsilon}, \tag{13}$$

$$L(\sigma + it, \mathrm{sym}^2 f) \ll_{f,\varepsilon} (1+ \mid t \mid)^{\max\left\{\frac{5}{4}(1-\sigma),0\right\}+\varepsilon}. \tag{14}$$

The next result is due to Heath-Brown [12], which will be used to prove Proposition 2.

Lemma 6 ([12]). *For any $\varepsilon > 0$ and $T \geq 1$, we have*

$$\int_1^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{12} dt \ll T^{2+\varepsilon}.$$

Lemma 7 ([10,11,18]). *Let χ be a primitive character modulo q. For any $\varepsilon > 0$ and $T \geq 1$ with $q \ll T^2$,*

$$L(\sigma + iT, \chi) \ll_\varepsilon (q(1 + |T|))^{\max\left\{\frac{1}{3}(1-\sigma),0\right\}+\varepsilon}, \tag{15}$$

$$L(\sigma + iT, \mathrm{sym}^2 f \otimes \chi) \ll_\varepsilon (q(1 + |T|))^{\max\left\{\frac{67}{46}(1-\sigma),0\right\}+\varepsilon}, \tag{16}$$

and if further q is a prime,

$$\int_0^T |L(\sigma + it, \chi)|^{12} dt \ll_\varepsilon q^{4(1-\sigma)} T^{3-2\sigma+\varepsilon}. \tag{17}$$

Proof. The above three results on the critical line $s = \frac{1}{2} + it$ were derived from Heath-Brown [11], Huang [10] and Motohashi [28], respectively. □

From Lemma 3, for $j \geq 1$ and $f \in H_k$, $L(s, \mathrm{sym}^j f \otimes \chi)$ and $L(s, \mathrm{sym}^j f \times \mathrm{sym}^j f \otimes \chi)$ are Perelli's general L-function as defined in [19, Section 2.2] and [31]. Lemma 8 follows plainly from [19, Lemma 2.4].

Lemma 8 ([19]). *Let $f \in H_k$ be a Hecke eigencuspform and χ be a primitive character modulo q. Let $\mathfrak{L}^{\mathbf{d}}_{\mathbf{m},\mathbf{n}}(s, \chi)$ be a general L-function of degree $2A$. For any $\varepsilon > 0$, we have*

$$\int_T^{2T} |\mathfrak{L}^{\mathbf{d}}_{\mathbf{m},\mathbf{n}}(\sigma + it, \chi)|^2 dt \ll (qT)^{2A(1-\sigma)+\varepsilon} \tag{18}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1$ *and* $T \geq 1$. *In addition,*

$$\mathfrak{L}^{\mathbf{d}}_{\mathbf{m},\mathbf{n}}(\sigma + it, \chi) \ll (q(|t| + 1))^{max\{A(1-\sigma),0\}+\varepsilon} \tag{19}$$

uniformly for $-\varepsilon \leq \sigma \leq 1 + \varepsilon$.

The aim of Lemma 9 is to decompose $F_j(s)$ into the product of some nice automorphic L-functions of lower ranks and the simpler Dirichlet series which converges absolutely slightly to the left of $\Re s = \frac{1}{2}$.

Lemma 9. *Let* $f \in H_k$ *be a Hecke eigencuspform. For* $j \geq 2$, *we introduce*

$$F_j(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^2(n^j)}{n^s}, \quad \Re s > 1.$$

Then we have that for $\Re s > 1$,

$$F_j(s) = L(s, \mathrm{sym}^j f \times \mathrm{sym}^j f)H_j(s) = \zeta(s) \prod_{l_1=1}^{j} L(s, \mathrm{sym}^{2l_1} f)H_j(s),$$

where $H_j(s)$ *converges uniformly and absolutely in the half-plane* $\Re s \geq \frac{1}{2} + \varepsilon$ *for any* $\varepsilon > 0$.

Proof. By Equations (3)–(5) and Hecke operator theory, we have

$$\lambda_f(p^j) = \sum_{m=0}^{j} \alpha_f(p)^{j-2m} = \lambda_{\mathrm{sym}^j f}(p), \qquad j \geq 1. \tag{20}$$

Similarly, by Equations (6), (7) and (20), we derive that for $j \geq 1$,

$$\lambda_{\mathrm{sym}^j f \times \mathrm{sym}^j f}(p) = \sum_{m=0}^{j} \sum_{u=0}^{j} \alpha_f(p)^{j-2m} \alpha_f(p)^{j-2u} = \lambda_{\mathrm{sym}^j f}(p) \lambda_{\mathrm{sym}^j f}(p). \tag{21}$$

By observing Equations (20) and (21), we find that

$$\begin{aligned} \lambda_f^2(p^j) &= \lambda_{\mathrm{sym}^j f \times \mathrm{sym}^j f}(p) = \sum_{m=0}^{j} \sum_{u=0}^{j} \alpha_f(p)^{j-2m} \alpha_f(p)^{j-2u} \\ &= \sum_{u=0}^{j} \alpha_f(p)^{2j-2u} + \sum_{u=0}^{j} \alpha_f(p)^{2j-2-2u} + ... + \sum_{u=0}^{j} \alpha_f(p)^{-2u} \\ &= \lambda_{\mathrm{sym}^{2j} f}(p) + \lambda_{\mathrm{sym}^{2j-2} f}(p) + ... + \lambda_{\mathrm{sym}^0 f}(p) \\ &= 1 + \sum_{l_1=1}^{j} \lambda_{\mathrm{sym}^{2l_1} f}(p). \end{aligned} \tag{22}$$

Thus by Equations (5), (7) and (22), $L(s, \mathrm{sym}^j f \times \mathrm{sym}^j f)$ can be writen as

$$L(s, \mathrm{sym}^j f \times \mathrm{sym}^j f) = \prod_p \left(1 + \sum_{k\geq 1} \frac{\lambda_{\mathrm{sym}^j f \times \mathrm{sym}^j f}(p^k)}{p^{ks}}\right) = \zeta(s) \prod_{l_1=1}^{j} L(s, \mathrm{sym}^{2l_1} f). \tag{23}$$

Due to the multiplicativity property of $\lambda_f^2(n^j)$, we have the Euler product identity

$$F_j(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^2(n^j)}{n^s} = \prod_p \left(1 + \sum_{k\geq 1} \frac{\lambda_f^2(p^{kj})}{p^{ks}}\right), \quad \Re s > 1. \tag{24}$$

Therefore from Equations (23)–(24), we obtain that for $\Re s > 1$,

$$\begin{aligned} F_j(s) &= L(s, \mathrm{sym}^j f \times \mathrm{sym}^j f) \prod_p \left(1 + \sum_{k=2}^{\infty} \frac{\lambda_f^2(p^{kj}) - \lambda_{\mathrm{sym}^j f \times \mathrm{sym}^j f}(p^k)}{p^{ks}}\right) \\ &= L(s, \mathrm{sym}^j f \times \mathrm{sym}^j f) H_j(s), \end{aligned}$$

where $H_j(s)$ converges uniformly and absolutely in the half-plane $\Re s \geq \frac{1}{2} + \varepsilon$ for any $\varepsilon > 0$. □

Let $f \in H_k$ and χ be a Dirichlet character modulo q. We define

$$F_j(s, \chi) = \sum_{n=1}^{\infty} \frac{\lambda_f^2(n^j)\chi(n)}{n^s}.$$

For $j = 1$, from [15],

$$F_1(s, \chi) = L(s, \chi) L(s, \mathrm{sym}^2 f \otimes \chi) L(2s, \chi^2)^{-1}.$$

Clearly, $L(2s, \chi^2)^{-1}$ is absolutely convergent and has no zeros for $\Re s \geq \frac{1}{2} + \varepsilon$ for any $\varepsilon > 0$. For the higher cases, we have the following results.

Lemma 10. *For any $\varepsilon > 0$, $\Re s > 1$ and $j \geq 2$, we have*

$$F_j(s, \chi) = L(s, \mathrm{sym}^j f \times \mathrm{sym}^j f \otimes \chi) H_j(s, \chi) = L(s, \chi) \prod_{l_2=1}^{j} L(s, \mathrm{sym}^{2l_2} f \otimes \chi) H_j(s, \chi), \tag{25}$$

where $H_j(s, \chi) := \prod_p L_{j,p}(\chi(p)p^{-s})$. Moreover, if $f \in H_k$, then $H_j(s, \chi)$ converges absolutely in $\Re s \geq \frac{1}{2} + \varepsilon$, and the convergence for all cases is uniform in q.

Proof. From Equation (22), we have

$$\lambda_f^2(p^j)\chi(p) = \lambda_{\mathrm{sym}^j f \times \mathrm{sym}^j f}(p)\chi(p) = (1 + \sum_{l_2=1}^{j} \lambda_{\mathrm{sym}^{2l_2} f}(p))\chi(p). \tag{26}$$

Thus by Equations (8), (10) and (26), we get

$$L(s, \mathrm{sym}^j f \times \mathrm{sym}^j f \otimes \chi) = L(s,\chi) \prod_{l_2=1}^{j} L(s, \mathrm{sym}^{2l_2} f \otimes \chi).$$

On noting that $\lambda_f^2(n^j)$ is a multiplicative function, $F_j(s,\chi)$ can be written as

$$F_j(s,\chi) = \sum_{n=1}^{\infty} \frac{\lambda_f^2(n^j)\chi(p)}{n^s} = \prod_p \left(1 + \sum_{k\geq 1} \frac{\lambda_f^2(p^{kj})\chi^k(p)}{p^{ks}}\right), \ \Re s > 1. \tag{27}$$

Therefore from Equations (10), (26) and (27), we have that for $\Re s > 1$,

$$\begin{aligned} F_j(s,\chi) &= L(s, \mathrm{sym}^j f \times \mathrm{sym}^j f \otimes \chi) \prod_p L_{j,p}(\chi(p)p^{-s}) \\ &= L(s, \mathrm{sym}^j f \times \mathrm{sym}^j f \otimes \chi) H_j(s,\chi), \end{aligned}$$

where

$$L_{j,p}(\chi(p)p^{-s}) = 1 + \sum_{k=2}^{\infty} \frac{(\lambda_f^2(p^{kj}) - \lambda_{\mathrm{sym}^j f \times \mathrm{sym}^j f}(p^k))\chi^k(p)}{p^{ks}}.$$

For $f \in H_k$ and any $\varepsilon > 0$, owing to the expression of $L_{j,p}(\chi(p)p^{-s})$, $H_j(s,\chi)$ converges absolutely in the region $\Re s \geq \frac{1}{2} + \varepsilon$. Furthermore, since $|\chi(p)| = 1$, the convergence in all cases is uniform in q. □

3. Proof of Theorem 1

In order to prove Theorem 1, the sum $\sum_{n\leq x} \lambda_f^2(n^j)\chi(n)$ should be firstly considered.

Proposition 1. *Let $f \in H_k$ and let χ be a primitive character modulo a prime q. For any $\varepsilon > 0$, $q \leq x^{\theta_j}$ and $j \geq 2$, we have*

$$\sum_{n\leq x} \lambda_f^2(n^j)\chi(n) = O_{f,\varepsilon}(qx^{1-\frac{3}{2}\theta_j+\varepsilon}), \tag{28}$$

where $\theta_2 = \frac{92}{597}$ and $\theta_j = \frac{92}{69(j-1)(j+3)+247}$ for $j \geq 3$.

Proof. Applying Perron's formula [14, Proposition 5.54] and Deligne's bound Equation (1) , we obtain

$$\sum_{n\leq x} \lambda_f^2(n^j)\chi(n) = \frac{1}{2\pi i} \int_{1+\varepsilon-iT}^{1+\varepsilon+iT} F_j(s,\chi) \frac{x^s}{s} ds + O_{f,\varepsilon}(x^{1+\varepsilon}/T),$$

where $s = \sigma + it$ and $1 \leq T \leq x$ is a parameter to be chosen later.

Next we move the line of integration to $\Re s = \frac{1}{2}+\varepsilon$. In the rectangle formed by the line segments joining the points $1+\varepsilon+iT$, $\frac{1}{2}+\varepsilon+iT$, $\frac{1}{2}+\varepsilon-iT$, $1+\varepsilon-iT$ and $1+\varepsilon+iT$, by Lemma 3 and Lemma 10, $F_j(s,\chi)$ has no poles and $H_j(s,\chi)$ converges absolutely and uniformly in q. By Cauchy's residue theorem, we derive

$$\begin{aligned}\sum_{n\le x}\lambda_f^2(n^j)\chi(n) &= -\frac{1}{2\pi i}\bigg(\int_{1+\varepsilon+iT}^{\frac{1}{2}+\varepsilon+iT}+\int_{\frac{1}{2}+\varepsilon-iT}^{1+\varepsilon-iT}+\int_{\frac{1}{2}+\varepsilon+iT}^{\frac{1}{2}+\varepsilon-iT}\bigg)\\ &\quad\cdot F_j(s,\chi)\frac{x^s}{s}ds + O_{f,\varepsilon}(x^{1+\varepsilon}/T)\\ &:= I_{j,1}+I_{j,2}+I_{j,3}+O_{f,\varepsilon}(x^{1+\varepsilon}/T).\end{aligned} \tag{29}$$

For the integrals over the horizontal segments, by Equations (15), (16), (19) and Lemma 10, for $q \ll T^2$ and $j \ge 2$, it follows that

$$\begin{aligned}I_{j,1}+I_{j,2} &\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} x^{\sigma}|L(s,\mathrm{sym}^j f\times\mathrm{sym}^j f\otimes\chi)|T^{-1}d\sigma\\ &\ll x^{1+\varepsilon}/T + \max_{\frac{1}{2}+\varepsilon\le\sigma\le1+\varepsilon} x^{\sigma}(qT)^{\frac{69(j-1)(j+3)+247}{138}(1-\sigma)+\varepsilon}T^{-1}\\ &\ll x^{1+\varepsilon}/T + \max_{\frac{1}{2}+\varepsilon\le\sigma\le1+\varepsilon}\left(\frac{x}{(qT)^{\frac{69(j-1)(j+3)+247}{138}}}\right)^{\sigma}(qT)^{\frac{69(j-1)(j+3)+247}{138}+\varepsilon}T^{-1}\\ &\ll x^{1+\varepsilon}/T + x^{\frac{1}{2}+\varepsilon}q^{\frac{69(j-1)(j+3)+247}{276}+\varepsilon}T^{\frac{69(j-1)(j+3)-29}{276}+\varepsilon}.\end{aligned} \tag{30}$$

For $j \ge 3$, we use Equations (15), (16) in Lemma 7 and Equation (18) in Lemma 8 to bound

$$\begin{aligned}I_{j,3} &\ll x^{\frac{1}{2}+\varepsilon}\log T\sup_{1\le T_1\le T}\left\{\frac{1}{T_1}\int_{\frac{T_1}{2}}^{T_1}\left|L\left(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^j f\times\mathrm{sym}^j f\otimes\chi\right)\right|dt\right\}\\ &\ll x^{\frac{1}{2}+\varepsilon}\log T\max_{T_1\le T}\left\{\frac{1}{T_1}L\left(\frac{1}{2}+\varepsilon+it,\chi\right)\left(\int_{\frac{T_1}{2}}^{T_1}\left|L\left(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^4 f\otimes\chi\right)\right|^2dt\right)^{\frac{1}{2}}\right.\\ &\quad\left.\times L\left(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^2 f\otimes\chi\right)\left(\int_{\frac{T_1}{2}}^{T_1}\left|\prod_{l_2=3}^{j}L\left(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^{2l_1} f\otimes\chi\right)\right|^2dt\right)^{\frac{1}{2}}\right\}\\ &\ll x^{\frac{1}{2}+\varepsilon}q^{\frac{69(j-1)(j+3)+247}{276}+\varepsilon}T^{\frac{69(j-1)(j+3)-29}{276}+\varepsilon}.\end{aligned} \tag{31}$$

Combining Equations (29), (30) and (31), we obtain Equation (28) by taking $T = x^{\frac{138}{69(j-1)(j+3)+247}}/q$. Since $q \ll T^2$, we have proved Equation (28) for $q \le x^{\frac{92}{69(j-1)(j+3)+247}}$.

To get a better error term, the proof of $j=2$ is a little different from the above cases. From Equation (30), for $j=2$, we have

$$I_{2,1}+I_{2,2} \ll x^{\frac{1}{2}+\varepsilon}q^{\frac{148}{69}+\varepsilon}T^{\frac{79}{69}+\varepsilon}+x^{1+\varepsilon}/T. \tag{32}$$

We carefully evaluate the vertical integral by Equations (16), (17) and (18),

$$\begin{aligned}
I_{2,3} &\ll x^{\frac{1}{2}+\varepsilon}\log T \sup_{1\le T_1\le T}\left\{\frac{1}{T_1}\left(\int_{\frac{T_1}{2}}^{T_1}\left|L\left(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^4 f\otimes\chi\right)\right|^2 dt\right)^{\frac{1}{2}}\right.\\
&\times\left.\left(\int_{\frac{T_1}{2}}^{T_1}\left|L\left(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^2 f\otimes\chi\right)\right|^{\frac{12}{5}} dt\right)^{\frac{5}{12}}\left(\int_{\frac{T_1}{2}}^{T_1}\left|L\left(\frac{1}{2}+\varepsilon+it,\chi\right)\right|^{12} dt\right)^{\frac{1}{12}}\right\}\\
&\ll x^{\frac{1}{2}+\varepsilon}\log T \sup_{1\le T_1\le T}\left\{\frac{1}{T_1}\left(\int_{\frac{T_1}{2}}^{T_1}\left|L\left(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^4 f\otimes\chi\right)\right|^2 dt\right)^{\frac{1}{2}}\right.\\
&\times\left(\sup_{\frac{T_1}{2}\le t\le T_1}\left|L\left(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^2 f\otimes\chi\right)\right|^{\frac{2}{5}}\int_{\frac{T_1}{2}}^{T_1}\left|L\left(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^2 f\otimes\chi\right)\right|^{2} dt\right)^{\frac{5}{12}}\\
&\times\left.\left(\int_{\frac{T_1}{2}}^{T_1}\left|L\left(\frac{1}{2}+\varepsilon+it,\chi\right)\right|^{12} dt\right)^{\frac{1}{12}}\right\}\\
&\ll x^{\frac{1}{2}+\varepsilon}\sup_{1\le T_1\le T}\left\{\frac{1}{T_1}\,(qT_1)^{\frac{5}{4}+\varepsilon}(qT_1)^{(\frac{2}{5}\times\frac{67}{92}+\frac{3}{2})\times\frac{5}{12}+\varepsilon}(qT_1)^{\frac{1}{6}+\varepsilon}\right\}\\
&\ll x^{\frac{1}{2}+\varepsilon}q^{\frac{199}{92}+\varepsilon}T^{\frac{107}{92}+\varepsilon}.
\end{aligned}\tag{33}$$

Taking into account Equations (29), (32) and (33), we get Equation (28) by choosing $T = x^{\frac{46}{199}}/q$. Since $q \ll T^2$, we also prove Equation (28) with the situation of $q \le x^{\frac{92}{597}}$. □

Proposition 2. *Let $f \in H_k$ and let χ_0 be a principal character modulo a prime q. For any $\varepsilon > 0$, $q \ll x$ and $j \ge 2$, we have*

$$\sum_{n\le x}\lambda_f^2(n^j)\chi_0(n) = c_j x + O_{f,\varepsilon}(x^{c_j+\varepsilon}),$$

where c_j are suitable constants, $c_2 = \frac{79}{103}$ and $c_j = \frac{42(j+1)^2-121}{42(j+1)^2-37}$ for $j \ge 3$.

Proof. By Lemma 10, it can easily be seen that

$$\begin{aligned}
F_j(s,\chi_0) &= \sum_{n=1}^{\infty}\frac{\lambda_f^2(n^j)\chi_0(n)}{n^s}\\
&= L_{j,p}^{-1}(q^{-s})\prod_{i=0}^{j}(1-\alpha_f(q)^{2(j-i)}q^{-s})^{i+1}\prod_{m=0}^{j-1}(1-\beta_f(q)^{2(j-m)}q^{-s})^{m+1}F_j(s),
\end{aligned}\tag{34}$$

where $F_j(s)$ is given in Lemma 9 and $H_j(s)$ is absolutely convergent in $\Re s \ge \frac{1}{2}+\varepsilon$ for $j \ge 2$. Applying the Perron's formula [14, Proposition 5.54], for $2 \le T \le x$, it follows that

$$\sum_{n\le x}\lambda_f^2(n^j)\chi_0(n) = \frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT}F_j(s,\chi_0)\frac{x^s}{s}ds + O_{f,\varepsilon}(x^{1+\varepsilon}/T).$$

Then we move the line of integration to the parallel segment with $\Re s = \frac{1}{2}+\varepsilon$. From Equation (34) and the expression of $L(s, \mathrm{sym}^j f \times \mathrm{sym}^j f)$, we observe that the point $s = 1$ is the only pole in the region $\frac{1}{2}+\varepsilon \le \sigma \le 1+\varepsilon$. By Cauchy's residue theorem, we obtain

$$\begin{aligned}\sum_{n\le x} \lambda_f^2(n^j)\chi_0(n) = &-\frac{1}{2\pi i}\bigg(\int_{1+\varepsilon+iT}^{\frac{1}{2}+\varepsilon+iT} + \int_{\frac{1}{2}+\varepsilon-iT}^{1+\varepsilon-iT} + \int_{\frac{1}{2}+\varepsilon+iT}^{\frac{1}{2}+\varepsilon-iT}\bigg) F_j(s,\chi_0)\frac{x^s}{s}ds \\ &+ \underset{s=1}{Res}\, F_j(s,\chi_0)\frac{x^s}{s} + O_{f,\varepsilon}(x^{1+\varepsilon}/T),\end{aligned} \tag{35}$$

where $\underset{s=1}{Res}\, F_j(s,\chi_0)\frac{x^s}{s} = c_j(x)$ and c_j are suitable constants. We also have

$$\prod_{i=0}^{j}(1-\alpha_f(q)^{2(j-i)}q^{-s})^{i+1} \ll (1+q^{-\sigma})^{j(i+1)} \ll 1, \quad \prod_{m=0}^{j-1}(1-\beta_f(q)^{2(j-m)}q^{-s})^{m+1} \ll 1.$$

In addition, $H_j(s,\chi)$ converges absolutely in $\Re s \ge \frac{1}{2}+\varepsilon$ and uniformly in q. So Equation (35) turns into

$$\sum_{n\le x} \lambda_f^2(n^2)\chi_0(n) := J_{j,1} + J_{j,2} + J_{j,3} + c_j x + O_{f,\varepsilon}(x^{1+\varepsilon}/T). \tag{36}$$

Next we begin to handle the three terms $J_{j,1}$, $J_{j,2}$ and $J_{j,3}$. The estimates for the integrals follow from the similar procedure as in the proof of Proposition 1. By Equation (12), Lemma 5 and Lemma 9, we obtain

$$\begin{aligned}J_{j,1}+J_{j,2} &\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} x^\sigma |L(s,\mathrm{sym}^j f\times \mathrm{sym}^j f)| T^{-1} d\sigma \\ &\ll x^{1+\varepsilon}/T + \max_{\frac{1}{2}+\varepsilon\le\sigma\le 1+\varepsilon} x^\sigma T^{\frac{42(j+1)^2-37}{84}(1-\sigma)+\varepsilon} T^{-1} \\ &\ll x^{1+\varepsilon}/T + x^{\frac{1}{2}+\varepsilon} T^{\frac{42(j+1)^2-205}{168}+\varepsilon}.\end{aligned} \tag{37}$$

For $j \ge 3$, we use Lemmas 4–5 and Cauchy's inequality to get

$$\begin{aligned}J_{j,3} &\ll x^{\frac{1}{2}+\varepsilon}\log T \sup_{1\le T_1\le T}\left\{\frac{1}{T_1}\int_{\frac{T_1}{2}}^{T_1}\left|L\left(\frac{1}{2}+\varepsilon+it, \mathrm{sym}^j f\times \mathrm{sym}^j f\right)\right| dt\right\} \\ &\ll x^{\frac{1}{2}+\varepsilon}\log T \max_{T_1\le T}\left\{\frac{1}{T_1}\zeta\left(\frac{1}{2}+\varepsilon+it\right)\left(\int_{\frac{T_1}{2}}^{T_1}\left|L\left(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^4 f\right)\right|^2 dt\right)^{\frac{1}{2}}\right. \\ &\left.\times L\left(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^2 f\right)\left(\int_{\frac{T_1}{2}}^{T_1}\left|\prod_{l_1=3}^{j} L\left(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^{2l_1} f\right)\right|^2 dt\right)^{\frac{1}{2}}\right\} \\ &\ll x^{\frac{1}{2}+\varepsilon} T^{\frac{42(j+1)^2-205}{168}+\varepsilon}.\end{aligned} \tag{38}$$

Inserting the bounds in Equations (37) and (38) into Equation (36),

$$\sum_{n\leq x}\lambda_f^2(n^j)\chi_0(n) = c_jx + O(x^{\frac{1}{2}+\varepsilon}T^{\frac{42(j+1)^2-205}{168}+\varepsilon}) + O_{f,\varepsilon}(x^{1+\varepsilon}/T). \tag{39}$$

Selecting $T = x^{\frac{84}{42(j+1)^2-37}}$, Equation (39) turns into

$$\sum_{n\leq x}\lambda_f^2(n^j)\chi_0(n) = c_jx + O_{f,\varepsilon}(x^{\frac{42(j+1)^2-121}{42(j+1)^2-37}+\varepsilon}).$$

Finally, we consider the case $j = 2$. From Equation (37), we have

$$J_{2,1} + J_{2,2} \ll x^{\frac{1}{2}+\varepsilon}T^{\frac{173}{168}+\varepsilon} + x^{1+\varepsilon}/T. \tag{40}$$

After applying Lemmas 4–6 and Hölder's inequality, we get

$$\begin{aligned}
J_{2,3} \ll{}& x^{\frac{1}{2}+\varepsilon}\log T \sup_{1\leq T_1\leq T}\left\{\frac{1}{T_1}\left(\int_{\frac{T_1}{2}}^{T_1}\left|L\left(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^4 f\right)\right|^2 dt\right)^{\frac{1}{2}}\right.\\
&\times\left(\int_{\frac{T_1}{2}}^{T_1}\left|L\left(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^2 f\right)\right|^{\frac{12}{5}} dt\right)^{\frac{5}{12}}\left.\left(\int_{\frac{T_1}{2}}^{T_1}\left|\zeta\left(\frac{1}{2}+\varepsilon+it\right)\right|^{12} dt\right)^{\frac{1}{12}}\right\}\\
\ll{}& x^{\frac{1}{2}+\varepsilon}\log T \sup_{1\leq T_1\leq T}\left\{\frac{1}{T_1}\left(\int_{\frac{T_1}{2}}^{T_1}\left|L\left(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^4 f\right)\right|^2 dt\right)^{\frac{1}{2}}\right.\\
&\times\left(\sup_{\frac{T_1}{2}\leq t\leq T_1}\left|L\left(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^2 f\right)\right|^{\frac{2}{5}}\int_{\frac{T_1}{2}}^{T_1}\left|L\left(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^2 f\right)\right|^2 dt\right)^{\frac{5}{12}}\\
&\times\left.\left(\int_{\frac{T_1}{2}}^{T_1}\left|\zeta\left(\frac{1}{2}+\varepsilon+it\right)\right|^{12} dt\right)^{\frac{1}{12}}\right\}\\
\ll{}& x^{\frac{1}{2}+\varepsilon}\sup_{1\leq T_1\leq T}\left\{\frac{1}{T_1}\;T_1^{\frac{5}{4}+\varepsilon}T_1^{(\frac{2}{5}\times\frac{5}{8}+\frac{3}{2})\times\frac{5}{12}+\varepsilon}T_1^{\frac{1}{6}+\varepsilon}\right\}\\
\ll{}& x^{\frac{1}{2}+\varepsilon}T^{\frac{55}{48}+\varepsilon}.
\end{aligned} \tag{41}$$

From Equations (36), (40) and (41), we have

$$\sum_{n\leq x}\lambda_f^2(n^2)\chi_0(n) = c_2x + O(x^{\frac{1}{2}+\varepsilon}T^{\frac{55}{48}+\varepsilon}) + O_{f,\varepsilon}(x^{1+\varepsilon}/T). \tag{42}$$

On taking $T = x^{\frac{24}{103}+\varepsilon}$ in Equation (42), we conclude that

$$\sum_{n\leq x}\lambda_f^2(n^2)\chi_0(n) = c_2x + O_{f,\varepsilon}(x^{\frac{79}{103}+\varepsilon}).$$

□

Proof of Theorem 1. Let χ be a Dirichlet character modulo a prime q. By orthogonality, we get

$$\sum_{\substack{n\le x\\ n\equiv l(\bmod q)}}\lambda_f^2(n^j)=\frac{1}{\varphi(q)}\sum_{\chi(\bmod q)}\overline{\chi}(l)\sum_{n\le x}\lambda_f^2(n^j)\chi(n)$$
$$=\frac{1}{\varphi(q)}\sum_{n\le x}\lambda_f^2(n^j)\chi_0(n)+O\Big(\sum_{n\le x}\lambda_f^2(n^j)\chi(n)\Big),$$

where $\varphi(q)$ is the Euler function and $\varphi(q)=q-1$.

According to proposition 1 and 2, noting that $1-\frac{3}{2}\theta_j>c_j$ for $j\ge 2$, we have

$$\sum_{\substack{n\le x\\ n\equiv l(\bmod q)}}\lambda_f^2(n^j)=\frac{c_jx}{\varphi(q)}+O_{f,\varepsilon}(qx^{1-\frac{3}{2}\theta_j+\varepsilon}).$$

This completes the proof of Theorem 1.

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