#A100 INTEGERS 22 (2022)

# ON THE DIOPHANTINE EQUATIONS $F_N = P_M \pm P_\ell$ AND $P_N = F_M \pm F_\ell$ INVOLVING FIBONACCI AND PELL NUMBERS

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*Received: 3/12/22, Accepted: 10/16/22, Published: 11/4/22*

**Abstract**

In this paper, we use Baker's method to find all the Fibonacci numbers which are sums or differences of two nonzero Pell numbers, and all Pell numbers which are sums or differences of two nonzero Fibonacci numbers, i.e., we solve the Diophantine equations appearing in the title of this paper.

## 1. Introduction

The Fibonacci sequence $(F_n)_{n\geq 0}$ and Pell sequence $(P_n)_{n\geq 0}$ are given by $F_0 = P_0 = 0$, $F_1 = P_1 = 1$ and

$$F_{n+2} = F_{n+1} + F_n \quad \text{and} \quad P_{n+2} = 2P_{n+1} + P_n \quad \text{for all } n \geq 0,$$

respectively. In [1], Alekseyev showed that the commun terms between the sequences $(F_n)$ and $(P_n)$ are $0, 1, 2, 5$. In [3], Ddamulira et al. find all the Fibonacci numbers which are the product of two Pell numbers and all Pell numbers which are the product of two Fibonacci numbers. This result was extended to the product of three such numbers in [6].

The goal of this paper is to find all the Fibonacci numbers that can be expressed as the sum or difference of two nonzero Pell numbers, and all the Pell numbers can be expressed as the sum or difference of two nonzero Fibonacci numbers. So, we will solve the following two Diophantine equations,

$$F_n = P_m \pm P_\ell \tag{1}$$

and

$$P_n = F_m \pm F_\ell, \tag{2}$$

where $m, n$ and $\ell$ are positive integers.

The paper is organized as follows. In the next section, we introduce some necessary properties of Fibonacci and Pell numbers, a result due to Matveev concerning a lower bound for a linear in logarithms of algebraic numbers, as well a variant of a reduction result due to de Weger. These results are useful for the proofs of our theorems. The last section is devoted to the proofs of our main results.

## 2. Preliminary Resuts

### 2.1. Some properties of Fibonacci and Pell numbers

Here, we recall a few important properties of the Fibonacci sequence $(F_n)_{n\geq 0}$ and Pell sequence $(P_n)_{n\geq 0}$. Let $(\alpha, \beta) = ((1+\sqrt{5})/2, (1-\sqrt{5})/2)$ be the roots of the characteristic equation $x^2 - x - 1 = 0$ of the Fibonacci sequence $(F_n)_{n\geq 0}$. Binet's formula for $F_n$ is

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad \text{for all } n \geq 0. \tag{3}$$

One can prove by induction that

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \tag{4}$$

holds for all $n \geq 1$.

On the other hand, let $(\gamma, \delta) = (1+\sqrt{2}, 1-\sqrt{2})$ be the roots of the characteristic equation $x^2 - 2x - 1 = 0$ of the Pell sequence $(P_n)_{n\geq 0}$. Binet's formula for $P_n$ is

$$P_n = \frac{\gamma^n - \delta^n}{\sqrt{2}} \quad \text{for all } n \geq 0. \tag{5}$$

One can prove by induction that

$$\gamma^{n-2} \leq P_n \leq \gamma^{n-1} \tag{6}$$

holds for all $n \geq 1$.

### 2.2. Linear Forms in Logarithms

The proof of our main theorem uses lower bounds for linear forms in logarithms of algebraic numbers and a version of the Baker-Davenport reduction method. So let us recall some results.

For any nonzero algebraic number $\eta$ of degree $d$ over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a\prod_{i=1}^{d}\left(X - \eta^{(i)}\right)$ (with $a > 0$), we denote by

$$h(\eta) = \frac{1}{d}\left(\log a + \sum_{i=1}^{d} \log\max\left(1, \left|\eta^{(i)}\right|\right)\right)$$

the usual absolute logarithmic height of $\eta$. We recall the result of Bugeaud et al. ([2], Theorem 9.4), which is a modified version of the result of Matveev [5], which is one of our main tools in this paper.

**Theorem 1.** *Let $\eta_1, \ldots, \eta_s$ be real algebraic numbers and let $b_1, \ldots, b_s$ be integers. Let $D$ be the degree of the number field $\mathbb{Q}(\eta_1, \ldots, \eta_s)$ over $\mathbb{Q}$ and let $A_j$ be a positive real number satisfying*

$$A_j = \max\{Dh(\eta_j), |\log \eta_j|, 0.16\} \quad \textit{for} \quad j = 1, \ldots, s.$$

*Assume that*

$$B \geq \max\{|b_1|, \ldots, |b_s|\}.$$

*If $\eta_1^{b_1} \cdots \eta_s^{b_s} - 1 \neq 0$, then*

$$|\eta_1^{b_1} \cdots \eta_s^{b_s} - 1| \geq \exp(-1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot D^2(1 + \log D)(1 + \log B)A_1 \cdots A_s).$$

### 2.3. Reduction Method

In this section we discuss a computational method for reducing upper bounds for solutions of Diophantine equations. A different method is given in [4].

Let $\nu_1, \nu_2, \beta \in \mathbb{R}$ be given, and let $x_1, x_2 \in \mathbb{Z}$ be unknowns. Let

$$\Gamma = \beta + x_1\nu_1 + x_2\nu_2. \tag{7}$$

Let $c$ and $\rho$ be positive constants. Set $X = \max\{|x_1|, |x_2|\}$. Let $X_0$ be a (large) positive constant. Assume that

$$|\Gamma| \leq c \cdot \exp(-\rho \cdot Y), \tag{8}$$

$$Y \leq X \leq X_0. \tag{9}$$

When $\beta \neq 0$ in (7), put $\nu = -\nu_1/\nu_2$ and $\psi = \beta/\nu_2$. Then we have

$$\frac{\Gamma}{\nu_2} = \psi - x_1\nu + x_2.$$

Let $p/q$ be a convergent of $\nu$ with $q > X_0$. For a real number $x$, we let $||x|| = \min\{|x - n|, n \in \mathbb{Z}\}$ be the distance from $x$ to the nearest integer. We have the following result.

**Lemma 1** (See Lemma 3.3 in [8])**.** *Suppose that*

$$||q\psi|| > \frac{2X_0}{q}.$$

*Then, the solutions of* (8) *and* (9) *satisfy*

$$Y < \frac{1}{\rho} \log\left(\frac{q^2 c}{|\nu_2| X_0}\right).$$

We conclude this section by recalling the following lemma that we need in the sequel.

**Lemma 2.** [8, Lemma 2.2] *Let $a, x \in \mathbb{R}$ and $0 < a < 1$. If $|x| < a$, then*

$$|\log(1+x)| < \frac{-\log(1-a)}{a}|x|$$

*and*

$$|x| < \frac{a}{1-e^{-a}}\left|e^x - 1\right|.$$

## 3. Main Theorems and Their Proofs

**Theorem 2.** *The solutions of the Diophantine equation* (1) *in positive integers $n, m$ and $\ell$ with $m \geq \ell$ are given by*

$$(n, m, \ell) \in \{(1,2,1),(2,2,1),(4,3,2),(4,2,1),(7,4,1),(9,5,3),(16,9,2)\}.$$

*Namely,*

$$F_1 = F_2 = P_2 - P_1, \quad F_4 = P_3 - P_2,$$

*and*

$$F_4 = P_2 + P_1, \quad F_7 = P_4 + P_1, \quad F_9 = P_5 + P_3, \quad F_{16} = P_9 + P_2.$$

*Proof.* Assume throughout that Equation (1) holds. Observe that if $\ell = m$, then Equation (1) becomes $F_n = 0$ or $F_n = 2P_m = P_{m+1} - P_{m-1}$, so we can assume that $m > \ell$.

For $1 \leq \ell < m \leq 100$, a search with *Maple* in this range gave only the solutions displayed in the statement of Theorem 2. Thus, throughout the remainder of the proof, we shall suppose that $m > 100$.

By using the estimates (4) and (6), we have

$$\alpha^{n-2} \leq F_n = P_m \pm P_\ell \leq 2P_m \leq \gamma^m,$$

from which it follows that $n < 2m$ because $m > 100$. Therefore, it suffices to bound $m$. To get this bound, we examine Equation (1) in two different ways.

Binet's formulas Equation (3) and Equation (5) enable us to express Equation (1) as

$$\frac{\alpha^n}{\sqrt{5}} - \frac{\gamma^m}{2\sqrt{2}} = \frac{\beta^n}{\sqrt{5}} - \frac{\delta^m}{2\sqrt{2}} \pm P_\ell,$$

which yields

$$\left|\frac{\alpha^n}{\sqrt{5}} - \frac{\gamma^m}{2\sqrt{2}}\right| \leq \gamma^{\ell-1} + 0.81.$$

Now, we multiply through by $2\sqrt{2}\gamma^{-m}$ to arrive at

$$\left|\sqrt{\frac{8}{5}}\alpha^n\gamma^{-m} - 1\right| < 2\sqrt{2}\gamma^{\ell-m}(\gamma^{-1} + 0.81\gamma^{-\ell}) < \frac{2.2}{\gamma^{m-\ell}}. \tag{10}$$

To get an upper bound for the left-hand side of inequality (10), we set in Theorem 1:

$$s := 3, \quad (\eta_1, b_1) := \left(\sqrt{\frac{8}{5}}, 1\right), \quad (\eta_2, b_2) := (\alpha, n), \quad \text{and} \quad (\eta_3, b_3) := (\gamma, -m).$$

Thus, we have

$$\Gamma_1 := \sqrt{\frac{8}{5}}\alpha^n\gamma^{-m} - 1. \tag{11}$$

Clearly $\eta_1, \eta_2, \eta_3 \in \mathbb{L} := \mathbb{Q}(\sqrt{2}, \sqrt{5})$, and $d_{\mathbb{L}} = 4$. Further, one can easily see that

$$h(\eta_1) = \frac{\log 8}{2}, \quad h(\eta_2) = \frac{\log \alpha}{2} \quad \text{and} \quad h(\eta_3) = \frac{\log \gamma}{2},$$

so, we choose

$$A_1 := 2\log 8 = \max\{4h(\eta_1), |\log \eta_1|, 0.16\},$$
$$A_2 := 2\log \alpha = \max\{4h(\eta_2), |\log \eta_2|, 0.16\},$$

and

$$A_3 := 2\log \gamma = \max\{4h(\eta_3), |\log \eta_3|, 0.16\}.$$

Finally, we can take $D := 2m > \max\{1, n, m\}$. We next show that $\Gamma_1$ is nonzero. Suppose that $\Gamma_1 = 0$. Then we get $\alpha^n\gamma^{-m} = \sqrt{\frac{8}{5}}$. However, the left-hand side of the above relation is a unit in $\mathbb{L}$, whereas the right-hand side is not as its norm over $\mathbb{L}$ is $\frac{64}{25}$. So, $\Gamma_1 \neq 0$. Applying Theorem 1 gives

$$\log|\Gamma_1| > -1.4 \times 30^6 \times 3^{4.5} \times 4^2(1 + \log 4)(1 + \log(2m))(2\log\alpha)(2\log\gamma)(2\log 8).$$

The above inequality together with inequality (10) tell us

$$(m - \ell)\log\gamma < 5.3 \times 10^{10}(1 + \log(2m)). \tag{12}$$

Returning to Equation (1), we reorganize it as

$$\frac{\alpha^n}{\sqrt{5}} - \frac{\gamma^m \mp \gamma^\ell}{2\sqrt{2}} = \frac{\beta^n}{\sqrt{5}} - \frac{\delta^m \mp \delta^\ell}{2\sqrt{2}}.$$

Consequently, we get

$$\left|\frac{\alpha^n}{\sqrt{5}} - \frac{\gamma^m\left(1 \mp \gamma^{\ell-m}\right)}{2\sqrt{2}}\right| < 1.2.$$

Multiplying through by $2\sqrt{2}\gamma^{-m}(1 \mp \gamma^{\ell-m})^{-1}$, we obtain

$$\left|\sqrt{\frac{8}{5}}\left(1 \mp \gamma^{\ell-m}\right)^{-1}\alpha^n\gamma^{-m} - 1\right| < \frac{1.2 \times 2\sqrt{2}\left(1 \mp \gamma^{\ell-m}\right)^{-1}}{\gamma^m} < \frac{6}{\gamma^m}, \tag{13}$$

where we have used the fact that $\left(1 \mp \gamma^{\ell-m}\right)^{-1} < 1.71$. Now, we will again apply Theorem 1, but this time we take

$$\Gamma_2 := \sqrt{\frac{8}{5}} \left(1 \mp \gamma^{\ell-m}\right)^{-1} \alpha^n \gamma^{-m} - 1, \tag{14}$$

with $s := 3$,

$$(\eta_1, b_1) := \left( \sqrt{\frac{8}{5}} \left(1 \mp \gamma^{\ell-m}\right)^{-1}, 1 \right), \; (\eta_2, b_2) := (\alpha, n), \text{ and } (\eta_3, b_3) := (\gamma, -m).$$

We again take $\mathbb{L} = \mathbb{Q}(\sqrt{2}, \sqrt{5})$, for which $d_{\mathbb{L}} = 4$. As before we can choose

$$A_2 = 2 \log \alpha \quad \text{and} \quad A_3 = 2 \log \gamma.$$

Let us estimate $A_1$. To get this estimate, we need to find the maximum of the quantities $h(\eta_1)$ and $|\log \eta_1|$. By the properties of the absolute logarithmic height, we have

$$h(\eta_1) \leq h(\sqrt{8/5}) + (m - \ell) h(\gamma) + \log 2 = \log(4\sqrt{2}) + \frac{(m-\ell)\log \gamma}{2},$$

and by using inequality (12), we deduce that

$$h(\eta_1) < 2.7 \times 10^{10} (1 + \log(2m)). \tag{15}$$

On the other hand, we have

$$\eta_1 = \sqrt{\frac{8}{5}} \left(1 \mp \gamma^{\ell-m}\right)^{-1} < 2.2 \quad \text{and} \quad \eta_1^{-1} = \sqrt{\frac{5}{8}} \left(1 \mp \gamma^{\ell-m}\right) < 1.2. \tag{16}$$

Therefore, by inequalities (15) and (16), we get

$$A_1 := 1.1 \times 10^{11} \left(1 + \log(2m)\right) > \max\{4h(\eta_1), \left|\log \eta_1\right|, 0.16\}.$$

Finally, we can take $D = 2m$. One justifies that $\Gamma_2 \neq 0$ by using a similar argument used to show that $\Gamma_1 \neq 0$. Theorem 1 leads to

$$\log |\Gamma_2| > -1.4 \times 10^{21} (1 + \log(2m))^2.$$

Putting together the last inequality and inequality (13) yields

$$m < 1.6 \times 10^{21} \left(1 + \log(2m)\right)^2.$$

Thus, we conclude that

$$m < 5.6 \times 10^{24}. \tag{17}$$

The obtained upper bound (17) is too large. Next, we will reduce it by using Lemma 1. Let

$$\Lambda_1 := n\log\alpha - m\log\gamma + \log\left(\sqrt{8/5}\right) = \log\left(\Gamma_1 + 1\right),$$

where $\Gamma_1$ is defined by (11). Since $m - \ell \geq 1$, from inequality (10) we obtain $|\Gamma_1| < 0.92$. So, by taking $a := 0.92$ and $x := \Gamma_1$ in Lemma 2, we get

$$\left|n\log\alpha - m\log\gamma + \log\left(\sqrt{8/5}\right)\right| < 6.1\exp(-0.88\cdot(m-\ell)).$$

We apply Lemma 1 with the data

$$c := 6.1, \quad \rho := 0.88, \quad X_0 := 1.12\times 10^{25}, \quad \psi := -\frac{\log(\sqrt{8/5})}{\log\gamma},$$

$$\nu := \frac{\log\alpha}{\log\gamma}, \quad \nu_1 := \log\alpha, \quad \nu_2 := -\log\gamma, \quad \beta := \log(\sqrt{8/5}).$$

Let

$$\frac{p}{q} := \frac{3645100420041373928725600 1}{6676259944070120488821432 5}$$

be the 55th convergent of $\nu$. With this choice the hypotheses of Lemma 1 are satisfied. Furthermore, Lemma 1 tells us that

$$m - \ell \leq \frac{1}{0.88}\cdot\log\left(\frac{66762599440701204888214325^2\times 6.1}{\log\gamma\times 1.12\times 10^{25}}\right) \leq 72.$$

Taking $m - \ell \leq 72$, we let

$$\Lambda_2 := n\log\alpha - m\log\gamma + \log\left(\sqrt{8/5}(1\mp\gamma^{\ell-m})^{-1}\right) = \log\left(\Gamma_2 + 1\right).$$

Since $m \geq 100$, then by inequalities (14) and (13), we have $|\Gamma_2| < 0.01$. Thus, by applying Lemma 2 with $a := 0.01$ and $x := \Gamma_2$ we obtain

$$\left|n\log\alpha - m\log\gamma + \log\left(\sqrt{8/5}(1\mp\gamma^{\ell-m})^{-1}\right)\right| < 6.1\exp(-0.88\cdot m).$$

We apply Lemma 1 with the data

$$c := 6.1,\ \rho := 0.88,\ X_0 := 1.12\times 10^{25},\ \nu := \frac{\log\alpha}{\log\gamma},\ \nu_1 := -\log\alpha,\ \nu_2 : - = \log\gamma,$$

$$\psi_k := \frac{\log\left(\sqrt{8/5}(1\mp\gamma^{-k})^{-1}\right)}{\log\gamma}, \quad \beta_k := \log\left(\sqrt{8/5}(1\mp\gamma^{-k})^{-1}\right),\ k = 1,\dots 72.$$

We find that

$$\frac{p}{q} := \frac{1382144685309391596913895831}{2531496018245707230917451088}$$

the 59 convergent satisfies the hypotheses of Lemma 1 for $k = 1, \ldots, 72$. Applying Lemma 1 we get

$$m \leq \frac{1}{0.88} \cdot \log\left(\frac{253149601824570723091745108^2 \times 6.1}{\log\gamma \times 1.12 \times 10^{25}}\right) \leq 81,$$

which is a contradiction. Therefore, the proof of Theorem 2 is complete. □

**Theorem 3.** *The solutions of the Diophantine equation* (1) *in positive integers* $n, m$ *and* $\ell$ *with* $m \geq \ell$ *are given by*

$$\begin{aligned}(n, m, \ell) \in \quad &\{(1,3,1),(1,3,2),(1,4,3),(2,4,1),(2,4,2),(2,5,3),(3,6,4),(3,7,6),\\ &(4,7,1),(4,7,2),(5,9,5),(9,16,3),(2,2,1),(3,4,3),(5,8,6)\}.\end{aligned}$$

*Namely,*

$$\begin{aligned} P_1 = F_3 - F_1 = F_3 - F_2 = F_4 - F_3, &\quad P_2 = F_4 - F_1 = F_4 - F_2 = F_5 - F_3,\\ P_3 = F_6 - F_4 = F_7 - F_6, &\quad P_4 = F_7 - F_1 = F_7 - F_2,\\ P_5 = F_9 - F_5 &\quad P_9 = F_{16} - F_3, \end{aligned}$$

*and*

$$P_2 = F_2 + F_1, \quad P_3 = F_4 + F_3, \quad P_5 = F_8 + F_6.$$

*Proof.* Assume throughout that Equation (2) holds. Observe that if $\ell = m$, then Equation (2) becomes $P_n = 0$ or $P_n = 2F_m = F_{m+1} + F_{m-2}$, so we can assume that $m > \ell$.

A brute force search with `Maple` in the range $1 \leq \ell < m \leq 200$ gives the solution given in Theorem 3. Thus, for the rest of the proof we assume that $m > 200$.

The inequalities (4) and (6), imply

$$\gamma^{n-2} \leq P_n = F_m \pm F_\ell \leq 2F_m \leq \alpha^{m+1},$$

and since $m > 200$, we get $n < m$. So, it suffices to bound $m$. To get this bound, we examine Equation (2) in two different ways.

By Binet's formulas Equation (3) and Equation (5), the Equation (2) can be expressed in the form

$$\frac{\gamma^n}{2\sqrt{2}} - \frac{\alpha^m}{\sqrt{5}} = \frac{\delta^n}{2\sqrt{2}} - \frac{\beta^m}{\sqrt{5}} \pm F_\ell.$$

It follows that

$$\left|\frac{\gamma^n}{2\sqrt{2}} - \frac{\alpha^m}{\sqrt{5}}\right| \leq \alpha^{\ell-1} + 0.81.$$

Multiplying through by $\sqrt{5}\alpha^{-m}$, we get

$$\left|\sqrt{\frac{5}{8}}\gamma^n\alpha^{-m} - 1\right| < \sqrt{5}\alpha^{\ell-m}(\alpha^{-1} + 0.81\alpha^{-\ell}) < \frac{2.6}{\alpha^{m-\ell}}. \tag{18}$$

In order to apply Theorem 1, we take

$$\Gamma_3 := \sqrt{\frac{5}{8}}\gamma^n\alpha^{-m} - 1, \tag{19}$$

and

$$s := 3, \quad (\eta_1, b_1) := \left(\sqrt{\frac{5}{8}}, 1\right), \quad (\eta_2, b_2) := (\gamma, n), \quad (\eta_3, b_3) := (\alpha, -m).$$

We have $\eta_1, \eta_2, \eta_3 \in \mathbb{L} := \mathbb{Q}(\sqrt{2}, \sqrt{5})$, so $d_{\mathbb{L}} = 4$. Since

$$h(\eta_1) = \frac{\log 8}{2}, \quad h(\eta_2) = \frac{\log\gamma}{2}, \quad \text{and} \quad h(\eta_3) = \frac{\log\alpha}{2},$$

then, we take

$$A_1 := 2\log 8 = \max\{4h(\eta_1), |\log\eta_1|, 0.16\},$$
$$A_2 := 2\log\gamma = \max\{4h(\eta_2), |\log\eta_2|, 0.16\},$$

and

$$A_3 := 2\log\alpha = \max\{4h(\eta_3), |\log\eta_3|, 0.16\}.$$

Finally, we can take $D := m > \max\{1, n, m\}$. Note that $\Gamma_3 \neq 0$, since if we assume the contrary, we would obtain that $\gamma^n\alpha^{-m} = \sqrt{\frac{5}{8}}$. But, the left-hand side of the above relation is a unit in $\mathbb{L}$, whereas the right-hand side is not as its norm over $\mathbb{L}$ is $\frac{25}{64}$. Thus, $\Gamma_3 \neq 0$. By applying Theorem 1 to $\Gamma_3$ we get

$$\log|\Gamma_3| > -1.4 \times 30^6 \times 3^{4.5} \times 4^2(1+\log 4)(1+\log m)(2\log\alpha)(2\log\gamma)(2\log 8).$$

Using the above inequality and inequality (18), we obtain

$$(m-\ell)\log\alpha < 5.3 \times 10^{10}(1+\log m). \tag{20}$$

Returning to Equation (2) and rewrite it in the form

$$\frac{\gamma^n}{2\sqrt{2}} - \frac{\alpha^m \mp \alpha^\ell}{\sqrt{5}} = \frac{\delta^n}{2\sqrt{2}} - \frac{\beta^m \pm \beta^\ell}{\sqrt{5}},$$

which leads us to

$$\left|\frac{\gamma^n}{2\sqrt{2}} - \frac{\alpha^m\left(1 \mp \alpha^{\ell-m}\right)}{\sqrt{5}}\right| < 1.3.$$

Multiplying both sides of the above inequality by $\sqrt{5}\alpha^{-m}(1 \mp \alpha^{\ell-m})^{-1}$ and using the fact that $\left(1 \mp \alpha^{\ell-m}\right)^{-1} < 2.62$ give

$$|\Gamma_4| < \frac{1.3 \times \sqrt{5}\left(1 \mp \alpha^{\ell-m}\right)^{-1}}{\alpha^m} < \frac{7.7}{\alpha^m}, \tag{21}$$

where

$$\Gamma_4 := \sqrt{\frac{5}{8}}\left(1 \mp \alpha^{\ell-m}\right)^{-1}\gamma^n\alpha^{-m} - 1. \tag{22}$$

Now we will apply Theorem 1 on $\Gamma_4$ by choosing $s := 3$,

$$(\eta_1, b_1) := \left(\sqrt{\frac{5}{8}}\left(1 \mp \alpha^{\ell-m}\right)^{-1}, 1\right), \ (\eta_2, b_2) := (\gamma, n), \text{ and } (\eta_3, b_3) := (\alpha, -m).$$

We again take $\mathbb{L} = \mathbb{Q}(\sqrt{2}, \sqrt{5})$, and $d_{\mathbb{L}} = 4$. As before, we can take

$$A_2 = 2\log\alpha \quad \text{and} \quad A_3 = 2\log\gamma.$$

Let us estimate $h(\eta_1)$. By the properties of the absolute logarithmic height, we have

$$h(\eta_1) \le h(\sqrt{5/8}) + (m-\ell)h(\alpha) + \log 2 = \log(4\sqrt{2}) + \frac{(m-\ell)\log\alpha}{2}.$$

Thus, by inequality (20), we deduce that

$$h(\eta_1) < 2.7 \times 10^{10}(1 + \log m). \tag{23}$$

In addition, we have

$$\eta_1 = \sqrt{\frac{5}{8}}\left(1 \mp \alpha^{\ell-m}\right)^{-1} < 2.1 \quad \text{and} \quad \eta_1^{-1} = \sqrt{\frac{8}{5}}\left(1 \mp \alpha^{\ell-m}\right) < 2.1. \tag{24}$$

Therefore, by inequalities (23) and (24), we get

$$A_1 := 1.1 \times 10^{11}\left(1 + \log m\right) > \max\{4h(\eta_1), |\log\eta_1|, 0.16\}.$$

Finally, we can take $D = m$. Using a similar argument used to prove that $\Gamma_3 \neq 0$, we can show that $\Gamma_4 \neq 0$. Theorem 1 gives

$$\log|\Gamma_2| > -1.4 \times 10^{21}(1 + \log m)^2.$$

Now, we combine the last inequality with inequality (21) to get

$$m < 3 \times 10^{21}\left(1 + \log m\right)^2,$$

which gives

$$m < 1.1 \times 10^{25}. \tag{25}$$

Now, we will use Lemma 1 to lower the obtained bound (25). Let

$$\Lambda_3 := n\log\gamma - m\log\alpha + \log\left(\sqrt{5/8}\right) = \log\left(\Gamma_3 + 1\right),$$

where $\Gamma_3$ is defined by Equation (19). Assume that $m-\ell \geq 1$. Then using inequality (18), we have $|\Gamma_3| < 0.45$. Applying Lemma 2 by taking $a := 0.45$ and $x := \Gamma_3$ one gets

$$\left|n\log\gamma - m\log\alpha + \log\left(\sqrt{5/8}\right)\right| < 3.5\exp(-0.48\cdot(m-\ell)).$$

We apply Lemma 1 with the data

$$c := 3.5, \quad \rho := 0.48, \quad X_0 := 1.1\times 10^{25}, \quad \psi := -\frac{\log(\sqrt{5/8})}{\log\gamma},$$

$$\nu := \frac{\log\alpha}{\log\gamma}, \quad \nu_1 := -\log\alpha, \quad \nu_2 := \log\gamma, \quad \beta := \log(\sqrt{5/8}).$$

Let

$$\frac{p}{q} := \frac{364510042004137392872560\,01}{6676259944070120488821432\,5}$$

be the 55th convergent of $\nu$. With this choice the hypotheses of Lemma 1 are satisfied. Furthermore, Lemma 1 gives that

$$m-\ell \leq \frac{1}{0.48}\cdot\log\left(\frac{66762599440701204888214325^2\times 3.5}{\log\gamma\times 1.1\times 10^{25}}\right) \leq 130.$$

Taking $1\leq m-\ell\leq 130$, we let

$$\Lambda_4 := n\log\gamma - m\log\alpha + \log\left(\sqrt{5/8}(1\mp\alpha^{\ell-m})^{-1}\right) = \log\left(\Gamma_4+1\right).$$

Since $m\geq 200$, then by inequalities (22) and (21), we have $|\Gamma_2| < 0.01$. Hence, by choosing $a := 0.01$ and $x := \Lambda_4$ in Lemma 2, we obtain

$$\left|n\log\gamma - m\log\alpha + \log\left(\sqrt{5/8}(1\mp\alpha^{\ell-m})^{-1}\right)\right| < 7.8\exp(-0.48\cdot m).$$

Now, we apply Lemma 1 by taking $c := 7.8$,

$$\rho := 0.48,\ X_0 := 1.1\times 10^{25},\ \psi_k := \frac{\log\left(\sqrt{5/8}(1\mp\alpha^{-k})^{-1}\right)}{\log\gamma},\ k = 1,\ldots 72,$$

$$\nu := \frac{\log\alpha}{\log\gamma},\ \nu_1 := -\log\alpha,\ \nu_2 :- = \log\gamma,\ \beta_k := \log\left(\sqrt{8/5}(1\mp\alpha^{-k})^{-1}\right).$$

We find that the 60th convergent

$$\frac{p}{q} := \frac{712055714704067871184182805\,2}{130418054325401294820446929\,67}$$

satisfies the hypotheses of Lemma 1 for $k = 1,\ldots,130$. Applying Lemma 1 we get

$$m \leq \frac{1}{0.48}\cdot\log\left(\frac{1304180543254012948204469296 7^2\times 7.8}{\log\gamma\times 1.1\times 10^{25}}\right) < 154,$$

which is a contradiction. Therefore, the proof of Theorem 3 is complete. $\square$

**Acknowledgements.** We thank the referee for comments which improved the quality of this paper.

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