WEIGHTED FREE LATTICE PATHS AND LEGENDRE POLYNOMIALS

Teresa X.S. Li
School of Mathematics and Statistics, Southwest University, Chongqing, PR China
pmgb@swu.edu.cn

Alina F.Y. Zhao[1]
School of Mathematical Sciences, Nanjing Normal University, Nanjing, PR China
alinazhao@njnu.edu.cn

Received: 8/13/21, Accepted: 11/2/22, Published: 11/30/22

Abstract

In this paper, we present combinatorial proofs of some known representations of Legendre polynomials based on weighted enumeration of free lattice paths. We also obtain a new identity involving Legendre polynomials, and provide a combinatorial proof of MacMahon's identity.

1. Introduction

Orthogonal polynomials occur frequently in various areas of mathematics (such as group representation theory, combinatorics, functional analysis, approximation theory, stochastic processes) and physics, engineering and computer science as well; see [1, 2, 6, 14, 21]. In particular, combinatorialists have paid much attention to orthogonal polynomials due to their broad and diverse role in combinatorics; see [7, 8, 15, 16, 17, 18, 27, 30] for example. In this paper, we consider the Legendre polynomials, a special case of the Jacobi polynomials.

The n-th Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ of type (α, β) is defined by the generating function

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n = 2^{\alpha+\beta}\rho^{-1}(1-t+\rho)^{-\alpha}(1+t+\rho)^{-\beta}, \tag{1}$$

where $\rho = (1-2xt+t^2)^{\frac{1}{2}}$. In [9], Foata and Leroux gave a combinatorial interpretation of the Jacobi polynomials by introducing the structures "Jacobi-endofunctions". This combinatorial model was further extended by Leroux and Strehl [22] to prove identities involving Jacobi polynomials, and Strehl [29] used it to express a generating function for the generalized Jacobi polynomials $P_n^{(\alpha-\lambda n,\beta-\mu n)}(x)$ in closed form.

[1] Corresponding author.

In addition to Jacobi-endofunctions, other interesting combinatorial objects (such as complete oriented matchings [28] and weighted paths[10, 13, 31]) also appeared in combinatorial study of Jacobi polynomials. By setting $\alpha = \beta = 0$, the Legendre polynomial $P_n(x)$ can be viewed as a special Jacobi polynomial $P_n^{(0,0)}(x)$, and we could use these combinatorial structures to deduce formulas for the Legendre polynomials. The main object of this paper is to interpret various classic forms of Legendre polynomials via free weighted lattice paths. Based on the following known formula of Legendre polynomial:

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k}^2 (x+1)^k (x-1)^{n-k},$$

we give a weighted lattice path interpretation for the Legendre polynomial $P_n(x)$, and then use this interpretation to verify other classic representations of $P_n(x)$. By using the weighted path enumeration model, we also provide a combinatorial proof of MacMahon's identity:

$$\sum_{k=0}^{n} \binom{n}{k}^3 x^k y^{n-k} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k}\binom{2k}{k}\binom{n+k}{k} x^k y^k (x+y)^{n-2k},$$

and give a new identity involving Legendre polynomials.

The organization of this paper is as follows. In Section 2, we recall some basic definitions and enumerative results of free lattice paths. We then obtain some identities involving the weight-enumerators for those paths. In particular, a combinatorial proof of MacMahon's identity is provided. Section 3 verifies various classic formulas of Legendre polynomials by using combinatorial identities established in Section 2. In Section 4, we generalize a formula of Rainville [23] to get a new formula which expresses $P_n(X)$ as a sum of a series of $P_k(Y)$. We also derive a new identity involving Legendre polynomials by applying the weighted lattice path model.

2. Weighted Enumeration of Free Lattice Paths

A *free Dyck path* of semilength n is a lattice path from $(0, 0)$ to $(2n, 0)$, with possible *up steps* $u = (1, 1)$ and *down steps* $d = (1, -1)$. A *peak* in a free Dyck path consists of an up step immediately followed by a down step. Let $\mathcal{D}_n$ denote the set of free Dyck paths of semilength n. A *free Motzkin path* of length n is a lattice path from $(0, 0)$ to $(n, 0)$ using up steps $u = (1, 1)$, down steps $d = (1, -1)$ and *level steps* $l = (1, 0)$. A *free 2-Motzkin path* is a free Motzkin path where each level step can be colored by one of two colors (say, red and blue). Denote by $\mathcal{M}_n$ (resp. $\mathcal{M}_n^2$) the set of free Motzkin paths (resp. free 2-Motzkin paths) of length n. It is known that $|\mathcal{D}_n| = |\mathcal{M}_n^2| = \binom{2n}{n}$. A *free Schröder path* of semilength n is a lattice path from

$(0,0)$ to $(2n,0)$ using up steps $u=(1,1)$, down steps $d=(1,-1)$ and *horizontal steps* $h=(2,0)$. The set of free Schröder paths of semilength n is denoted by $\mathcal{S}_n$. For simplicity, these paths are all referred to as free lattice paths and written as sequences of steps from $\{u,d,l,l^r,l^b,h\}$, where l^r and l^b denote the red level step and the blue level step in a free 2-Motzkin path, respectively.

Let $\mathbb{C}$ denote the field of complex numbers and let a,b be two commuting variables. A weighted path is a path where each step is endowed with a weight in the polynomial algebra $\mathbb{C}[a,b]$. The weight of a weighted path is the product of the weights of the steps in it, and the weight-enumerator (or weight generating function) of a set of paths means the sum of the weights of the paths in it. Two sets of weighted paths are *weight-equivalent* if they have the same weight-enumerator.

Weighted paths have been used to give combinatorial interpretations of identities; see, for example [4, 5]. For convenience, we assume that the weight of the empty path is 1. If the weight of some step in a path is not specified, we assume the weight of the step is 1.

2.1. Weighted Free Dyck Paths

For a free Dyck path $p\in\mathcal{D}_n$, we consider two different ways to assign its weight. The first way is to assign the weight a to up steps in the peaks, and the weight b to other up steps. Denote by $\mathcal{D}_n(a,b)$ (resp. $D_n(a,b)$) the set (resp. weight-enumerator) of weighted free Dyck paths in $\mathcal{D}_n$ with respect to this weight assignment. To introduce another weight assignment, we need the notions of even ascents and even descents. Suppose that $p=p_1p_2\cdots p_{2n}\in\mathcal{D}_n$. For $1\le i\le n$, if $p_{2i}=u$ (resp. $p_{2i}=d$), we call this step an even ascent (resp. even descent). Now we assign the weight a to even ascents and the weight b to even descents in p. Denote by $\mathcal{D}^*_n(a,b)$ (resp. $D^*_n(a,b)$) the corresponding set (resp. weight-enumerator) of weighted free Dyck paths of semilength n.

It is not hard to see that $\mathcal{D}_n(a,b)$ and $\mathcal{D}^*_n(a,b)$ are weight-equivalent.

Proposition 1. *We have*

$$D_n(a,b)=D^*_n(a,b)=\sum_{k=0}^{n}\binom{n}{k}^2a^kb^{n-k}. \tag{2}$$

Proof. Suppose that $p=p_1p_2\cdots p_{2n}\in\mathcal{D}_n$ has k peaks. If p_ip_{i+1} is a peak, namely $p_i=u$ and $p_{i+1}=d$, then we define the location of this peak to be the pair (x_i,y_i) of integers, where x_i (resp. y_i) denotes the number of up (resp. down) steps p_j in p with $j\le i+1$. We get two sets of positive integers of size k:

$$X(p)=\{x_i\mid p_ip_{i+1}\text{ is a peak in }p\}\subseteq\{1,2,\ldots,n\},$$
$$Y(p)=\{y_i\mid p_ip_{i+1}\text{ is a peak in }p\}\subseteq\{1,2,\ldots,n\}.$$

It is easy to see that the path p is uniquely determined by the pair $(X(p), Y(p))$. Thus we deduce that the number of free Dyck paths in $\mathcal{D}_n$ with k peaks is $\binom{n}{k}^2$, and

$$D_n(a,b) = \sum_{k=0}^{n} \binom{n}{k}^2 a^k b^{n-k}.$$

Similarly, it is not hard to show that a free Dyck path $p \in \mathcal{D}_n$ with k even ascents is uniquely determined by the pair $(X^*(p), Y^*(p))$ of k-subsets of $\{1, 2, \ldots, n\}$ defined by

$$X^*(p) = \{i \mid p_{2i} = u, 1 \le i \le n\}$$

and

$$Y^*(p) = \{i \mid p_{2i-1} = d, 1 \le i \le n\}.$$

This completes the proof. □

2.2. Weighted Free Motzkin Paths and Free 2-Motzkin Paths

Now we continue to introduce the weighted free Motzkin path and weighted free 2-Motzkin path. In a free Motzkin path, all up steps are given the weight a and all level steps are given the weight b. In a free 2-Motzkin path, all up steps and red level steps are given the weight a, whereas the down steps and blue level steps are given the weight b. Denote by $\mathcal{M}_n(a,b)$ (resp. $\mathcal{M}_n^2(a,b)$) the set of weighted free Motzkin (resp. free 2-Motzkin) paths of length n, and by $M_n(a,b)$ (resp. $M_n^2(a,b)$) the weight-enumerator of weighted free Motzkin (resp. free 2-Motzkin) paths of length n.

Proposition 2. *We have*

$$M_n^2(a,b) = M_n(ab, a+b) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k}\binom{2k}{k} a^k b^k (a+b)^{n-2k}. \tag{3}$$

Proof. Note that up steps and down steps appear in pairs in a free Motzkin path, so the polynomial $M_n(ab, a+b)$ can be regarded as the weight-enumerator of $\mathcal{M}_n$ with respect to the new weight assignment by assigning weight a to up steps, weight b to down steps and weight $a+b$ to level steps. On the other hand, when we ignore the colors of level steps, we obtain a map from $\mathcal{M}_n^2$ to $\mathcal{M}_n$. Clearly, this map is surjective. Moreover, if we collect all free 2-Motzkin paths which correspond to a given $p \in \mathcal{M}_n$, then the weight-enumerator of this collection is just equal to the weight of the path p with respect to the new weight assignment mentioned above. Hence the set $\mathcal{M}_n^2(a,b)$ is weight-equivalent to $\mathcal{M}_n(ab, a+b)$, namely, $M_n^2(a,b) = M_n(ab, a+b)$.

For a free Motzkin path of length n, if there are k up steps, then there are k down steps and $n-2k$ level steps. The number of such paths is $\binom{n}{2k}\binom{2k}{k}$ since we can first choose $n-2k$ positions for the level steps, and the $2k$ steps left can be seen as a free Dyck path of semilength k. Then we have

$$M_n(a,b) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k}\binom{2k}{k} a^k b^{n-2k}$$

and so

$$M_n(ab, a+b) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k}\binom{2k}{k} a^k b^k (a+b)^{n-2k}.$$

This completes the proof. □

By counting the number of steps with some required weight, we have the following weight enumeration.

Theorem 1. *We have*

$$M_n^2(a,b) = \sum_{k=0}^{n} \binom{n}{k}\binom{2k}{k} (\sqrt{ab})^k (a+b-2\sqrt{ab})^{n-k}. \tag{4}$$

Proof. Since up and down steps appear in pairs in a free 2-Motzkin path, and $a = (a-\sqrt{ab}) + \sqrt{ab}$, $b = (b-\sqrt{ab}) + \sqrt{ab}$, we can easily deduce that $\mathcal{M}_n^2(a,b)$ is weight-equivalent to the set of all weighted free 2-Motzkin paths of length n with the following weight assignment:

- the red level steps are given the weight $a-\sqrt{ab}$ or $\sqrt{ab}$;
- the blue level steps are given the weight $b-\sqrt{ab}$ or $\sqrt{ab}$;
- the up steps and down steps are given the weight $\sqrt{ab}$.

We now enumerate such paths according to the number k of steps with weight $\sqrt{ab}$. To obtain such a path, we first choose $n-k$ positions for the level steps whose weights are given by $a-\sqrt{ab}$ or $b-\sqrt{ab}$. Then the remaining k steps can be seen as a free 2-Motzkin path in which all steps are given the weight $\sqrt{ab}$. Summing up the possible k, we then deduce Equation (4). □

Theorem 2. *We have*

$$M_n(ab, a+b) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{k}\binom{2n-2k}{n} \left(\frac{a-b}{2}\right)^{2k} \left(\frac{a+b}{2}\right)^{n-2k}. \tag{5}$$

Proof. Recall that $\mathcal{M}_n(ab, a+b)$ consists of all weighted free Motzkin paths, where up steps have weight ab and level steps have weight $a+b$. Since $a+b = \frac{a+b}{2} + \frac{a+b}{2}$ and $ab = (\frac{a+b}{2})^2 - (\frac{a-b}{2})^2$, we find that $\mathcal{M}_n(ab, a+b)$ is weight-equivalent to the set of weighted free 2-Motzkin paths with the following weight assignment:

- red level steps have weight $\frac{a+b}{2}$;
- blue level steps have weight $\frac{a+b}{2}$;
- up steps have weight $(\frac{a+b}{2})^2$ or $-(\frac{a-b}{2})^2$.

Suppose there are k up steps with weight $-(\frac{a-b}{2})^2$ in such a path p. We can first choose k positions in $\binom{n}{k}$ ways to place these k up steps with weight $-(\frac{a-b}{2})^2$. The remaining $n-k$ steps can be viewed as a weighted free 2-Motzkin path from $(0,0)$ to $(n-k, -k)$ with weight $\left(\frac{a+b}{2}\right)^{n-2k}$. From [32, Proposition 2.2], we find that the number of such paths is $\binom{2n-2k}{n}$. Summing up all possible k from 0 to $\lfloor \frac{n}{2} \rfloor$ yields the desired expression. □

We can also derive another formula for $M_n^2(a,b)$ by using generating function method.

Theorem 3. *We have*

$$M_n^2(a,b) = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} (a+b+2\sqrt{ab})^k (a+b-2\sqrt{ab})^{n-k}. \qquad (6)$$

Proof. Define $G(a,b,t) = \sum_{n \geq 0} M_n^2(a,b) t^n$. To compute G, multiply Equation (3) by t^n, sum over $n \geq 0$ and interchange the sum to obtain

$$\begin{aligned} G(a,b,t) &= \sum_{k \geq 0} \binom{2k}{k} \frac{(ab)^k}{(a+b)^{2k}} \sum_{n \geq 2k} \binom{n}{2k} [(a+b)t]^n \\ &= \frac{1}{1-(a+b)t} \sum_{k \geq 0} \binom{2k}{k} \left[\frac{abt^2}{(1-(a+b)t)^2} \right]^k \\ &= \frac{1}{\sqrt{1-2(a+b)t+(a-b)^2t^2}}. \end{aligned}$$

Here we use the known generating function

$$\frac{1}{\sqrt{1-4t}} = \sum_{n \geq 0} \binom{2n}{n} t^n.$$

On the other hand,

$$\sum_{n\geq 0}\frac{t^n}{2^{2n}}\sum_{k\geq 0}\binom{2k}{k}\binom{2n-2k}{n-k}(a+b+2\sqrt{ab})^k(a+b-2\sqrt{ab})^{n-k}$$

$$=\left\{\sum_{n\geq 0}\binom{2n}{n}\left(\frac{a+b+2\sqrt{ab}}{4}t\right)^n\right\}\cdot\left\{\sum_{n\geq 0}\binom{2n}{n}\left(\frac{a+b-2\sqrt{ab}}{4}t\right)^n\right\}$$

$$=\frac{1}{\sqrt{1-(a+b+2\sqrt{ab})t}}\cdot\frac{1}{\sqrt{1-(a+b-2\sqrt{ab})t}}$$

$$=\frac{1}{\sqrt{1-2(a+b)t+(a-b)^2t^2}}.$$

Thus the theorem follows. □

2.3. Weighted Free Schröder Paths

We now consider the weighted free Schröder paths defined by assigning the weight a to horizontal steps and the weight b to up steps. Denote by $\mathcal{S}_n(a,b)$ (resp. $S_n(a,b)$) the set (resp. weight-enumerator) of weighted free Schröder paths of semilength n.

Proposition 3. *We have*

$$S_n(a,b)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}a^{n-k}b^k. \tag{7}$$

Proof. Clearly, if there are k up steps in a free Schröder path, then there are k down steps and $n-k$ horizontal steps. To get a free Schröder path of semilength n with k up steps, we can first construct a free Dyck path $p=p_1p_2\cdots p_{2k}$ of semilength k in $\binom{2k}{k}$ ways. This free Dyck path p determines $2k+1$ positions: the position before the first step p_1, the $2k-1$ positions between the steps p_i and p_{i+1} for $1\leq i\leq 2k-1$, and the position after the last step p_{2k}. Then we choose $n-k$ positions with repetition from the $2k+1$ positions in $\binom{2k+1+n-k-1}{n-k}=\binom{n+k}{n-k}$ ways to place the $n-k$ horizontal steps. Since $\binom{2k}{k}\binom{n+k}{n-k}=\binom{n}{k}\binom{n+k}{k}$, the proof is completed. □

If there are k horizontal steps in a free Schröder paths of semilength n, then there are $n-k$ up steps. By the proof of Proposition 3, the number of free Schröder paths of semilength n with k horizontal steps equals $\binom{2n-k}{k}\binom{2n-2k}{n-k}$, which was first observed by Chen and Pang [5].

2.4. Identities Arising from Weight Enumeration of Paths

In this subsection, we study some relations among the weight-enumerators $D_n(a,b)$, $M_n^2(a,b)$ and $S_n(a,b)$ by constructing transformations among sets of weighted paths. Besides, we also obtain a combinatorial proof of MacMahon's identity.

Theorem 4. *For $a, b \in \mathbb{C}$, we have*

$$D_n^*(a,b) = M_n^2(a,b), \tag{8}$$

$$D_n(a,b) = S_n(a-b,b). \tag{9}$$

Proof. Formula (8) is established by finding a weight-preserving bijection α between the sets $\mathcal{M}_n^2(a,b)$ and $\mathcal{D}_n^*(a,b)$. For a path $p = p_1p_2\cdots p_n \in \mathcal{M}_n^2(a,b)$, let $\alpha(p) = q_1q_2\cdots q_{2n-1}q_{2n}$ be the path given as follows: for $1 \le i \le n$,

- $q_{2i-1}q_{2i} = uu$ if and only if $p_i = u$;
- $q_{2i-1}q_{2i} = dd$ if and only if $p_i = d$;
- $q_{2i-1}q_{2i} = du$ if and only if p_i is a red level step;
- $q_{2i-1}q_{2i} = ud$ if and only if p_i is a blue level step.

It is easy to see that $\alpha : \mathcal{M}_n^2(a,b) \to \mathcal{D}_n^*(a,b)$ is the desired weight-preserving bijection.

By definition, the set $\mathcal{S}_n(a-b,b)$ denotes the set of weighted free Schröder paths, where up steps and horizontal steps in each path have weight b and $a-b$, respectively. For a weighted path p in $\mathcal{S}_n(a-b,b)$, define a weighted free Dyck path $\beta(p)$ by changing every horizontal step h in p to a peak ud, assigning the weight $a-b$ to the up step in this new peak and keeping the weight of other steps invariant. We get a set of weighted free Dyck paths with the following weight assignment:

- the up step in each peak is given the weight $a-b$ or b;
- other up steps are given the weight b.

Since $a = a - b + b$, this set is weight-equivalent to $\mathcal{D}_n(a,b)$. Thus formula (9) follows. □

Since it is clear that $D_n(a,b) = D_n(b,a)$, we can deduce from Theorem 4 that

$$S_n(a-b,b) = S_n(b-a,a). \tag{10}$$

Let $a = 1+x$ and $b = x$. Formula (10) leads to the following well-known identity due to Simons [25]:

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} x^k = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k}\binom{n+k}{k} (1+x)^k.$$

From Theorem 4, we also have

$$S_n(a-b,b) = M_n^2(a,b). \tag{11}$$

Setting $a = \frac{1}{2}, b = -\frac{1}{2}$ in formula (11) and exploring Equation (3), we then get

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\left(-\frac{1}{2}\right)^k = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k}\binom{2k}{k}\left(-\frac{1}{4}\right)^k 0^{n-2k} = \begin{cases} 0 & n \text{ odd;} \\ \binom{n}{\frac{n}{2}}\frac{(-1)^{\frac{n}{2}}}{2^n} & n \text{ even.} \end{cases}$$

By using a double-counting technique on weighted lattice paths, we can give a combinatorial proof for MacMahon's identity [19].

Theorem 5. *We have*

$$\sum_{k=0}^{n} \binom{n}{k}^3 x^k y^{n-k} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k}\binom{2k}{k}\binom{n+k}{k} x^k y^k (x+y)^{n-2k}. \tag{12}$$

Proof. In order to interpret the right-hand side combinatorially, we define a restricted weighted free Schröder path to be a free Schröder path where up steps have weight x, down steps have weight y and horizontal steps have weight 1 or x or y satisfying that the number of horizontal steps with weight 1 equals the number of up steps. Given such a restricted weighted free Schröder path, if there are k up steps, then there are k down steps and k horizontal steps with weight 1. The remaining $n-2k$ steps are horizontal steps with weight x or y. Such a path can be obtained by first choosing an ordinary free Schröder path of semilength n with k up steps in $\binom{n}{k}\binom{n+k}{k}$ ways. Note that this ordinary free Schröder path must have $n-k$ horizontal steps. Then we choose k steps from these $n-k$ horizontal steps to assign the weight 1 in $\binom{n-k}{k}$ ways. Because of the identity $\binom{n}{k}\binom{n-k}{k} = \binom{n}{2k}\binom{2k}{k}$, the weight-enumerator of the set of all restricted weighted free Schröder paths of semilength n is just the right-hand side of formula (12).

On the other hand, we can count the weighted path by the total number k of steps with weight x. By the weight assignment, if there are k steps with weight x, we can suppose that there are i up steps and $k-i$ horizontal steps with weight x in such a path. Due to the restricted requirement, there are i horizontal steps with weight 1, i down steps and $n-k-i$ horizontal steps with weight y. Note that the total number of steps is equal to $n+i$. To obtain such a path, we first choose a free Schröder path by picking $2i$ positions to place the up and down steps in $\binom{n+i}{2i}\binom{2i}{i}$ ways and putting $n-i$ horizontal steps in the remaining positions. Then we can pick i steps from the $n-i$ horizontal steps to assign the weight 1 in $\binom{n-i}{i}$ ways and pick $k-i$ steps from the $n-2i$ horizontal steps left to assign the weight x in $\binom{n-2i}{k-i}$ ways, and the remaining horizontal steps are given the weight y. Summing over all possible i, we deduce that the total number of restricted weighted free Schröder paths of semilength n with weight $x^k y^{n-k}$ is

$$\sum_{i=0}^{k}\binom{n+i}{2i}\binom{2i}{i}\binom{n-i}{i}\binom{n-2i}{k-i}=\binom{n}{k}\sum_{i=0}^{k}\binom{k}{k-i}\binom{n-k}{i}\binom{n+i}{i}=\binom{n}{k}^3,$$

where the second equation follows from the following identity in [20]

$$\sum_{i=0}^{k}\binom{m-x+y}{i}\binom{k+x-y}{k-i}\binom{x+i}{m+k}=\binom{x}{m}\binom{y}{k}$$

by setting $m=n-k$ and $x=y=n$. This completes the proof. □

3. Different Expressions for Legendre Polynomials

Recall that one of the various explicit representations (see [11, formula 3.134]) of the Legendre polynomials is

$$P_n(x)=\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}^2(x+1)^k(x-1)^{n-k}. \tag{13}$$

By using this expression we can easily represent the Legendre polynomials via the weight-enumerators introduced in Section 2. From formulas (1) and (13), we have

$$P_n(x)=D_n\left(\frac{x+1}{2},\frac{x-1}{2}\right)=D_n^*\left(\frac{x+1}{2},\frac{x-1}{2}\right). \tag{14}$$

Moreover, by Theorem 4, we can also represent the Legendre polynomial $P_n(x)$ as:

$$P_n(x)=M_n^2\left(\frac{x+1}{2},\frac{x-1}{2}\right)=S_n\left(1,\frac{x-1}{2}\right). \tag{15}$$

By Equations (3), (4), (5), (6) and (7), we can obtain the following different formulas for $P_n(x)$ respectively.

Corollary 1. [11, formula 3.137] *We have*

$$P_n(x)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}\binom{2k}{k}2^{-2k}x^{n-2k}(x^2-1)^k.$$

Corollary 2. [11, formula 3.136] *We have*

$$P_n(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}2^{-k}(x^2-1)^{\frac{k}{2}}(x-(x^2-1)^{\frac{1}{2}})^{n-k}.$$

Corollary 3. [11, formula 3.133] *We have*

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k \binom{n}{k}\binom{2n-2k}{n} x^{n-2k}.$$

Corollary 4. *We have*

$$P_n(x) = \frac{1}{2^{2n}}\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}(x+\sqrt{x^2-1})^k(x-\sqrt{x^2-1})^{n-k}.$$

Corollary 5. [11, formula 3.135] *We have*

$$P_n(x) = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\left(\frac{x-1}{2}\right)^k.$$

Remark 1. The weight-enumerators $D_n(a,b)$, $M_n^2(a,b)$ and $S_n(a,b)$ can also be represented in terms of the Legendre polynomials. In fact, it follows from (2) and (13) that

$$D_n(a,b) = (a-b)^n P_n\left(\frac{a+b}{a-b}\right). \tag{16}$$

Then by Theorem 4, we have

$$M_n^2(a,b) = (a-b)^n P_n\left(\frac{a+b}{a-b}\right)$$

and

$$S_n(a,b) = D_n(a+b,b) = a^n P_n\left(\frac{a+2b}{a}\right). \tag{17}$$

The formula (17) is also equivalent to the formula in [13, Lemma 2.3].

Remark 2. Note that there is an obvious bijection between free Schröder paths of semilength n and the Delannoy paths (see [3, 12, 13] for example) from $(0,0)$ to (n,n). Then we get the well-known relation between the central Delannoy numbers and Legendre polynomials: $d_{n,n} = P_n(3)$. We can also express the number $M_n(1,1)$ of free Motzkin paths of length n as $(\sqrt{-3})^n P_n(\frac{1}{\sqrt{-3}})$; see the sequence A002426 of the On-Line Encyclopedia of Integer Sequences [26].

To conclude this section, we use the weighted free lattice path model to deduce another formula for $P_n(x)$.

Theorem 6. *We have*

$$P_n(x) = \sum_{r=0}^{n}\binom{n}{r}\frac{x^r}{2^r}\sum_{k=0}^{n-r}(-1)^k\binom{n-r}{k}\binom{n+k+r}{n}\frac{1}{2^k}. \tag{18}$$

Proof. To prove (18), it is more convenient to write it as

$$2^n P_n(x) = \sum_{r=0}^{n} \binom{n}{r} x^r \sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \binom{n+k+r}{n} 2^{n-r-k}. \tag{19}$$

By formula (15), we have

$$2^n P_n(x) = 2^n S_n\left(1, \frac{x-1}{2}\right) = S_n(2, x-1).$$

Note that $\mathcal{S}_n(2, x-1)$ is weight-equivalent to the set of weighted free Schröder paths where up steps have weight x or -1 and horizontal steps have weight 2. For such a path, if there are r up steps with weight x and k up steps with weight -1, then there are $n-r-k$ horizontal steps. First, the $k+r$ up steps and $k+r$ down steps can generate a free Dyck path in $\binom{2k+2r}{k+r}$ ways. Among these $k+r$ up steps, we can choose r up steps with weight x in $\binom{k+r}{r}$ ways. After the up and down steps are arranged, the remaining $n-r-k$ horizontal steps can be fitted in those up and down steps in

$$\binom{2k+2r+1+n-r-k-1}{n-r-k} = \binom{n+r+k}{n-r-k}$$

ways. From the above classification, we find

$$2^n P_n(x) = \sum_{r=0}^{n} x^r \sum_{k=0}^{n-r} \binom{2k+2r}{k+r} \binom{k+r}{r} (-1)^k \binom{n+r+k}{n-r-k} 2^{n-r-k}.$$

By simple calculation, we have

$$\binom{2k+2r}{k+r} \binom{k+r}{r} \binom{n+r+k}{n-r-k} = \binom{n}{r} \binom{n-r}{k} \binom{n+k+r}{n},$$

and the proof is completed. □

4. Identities Involving Legendre Polynomials

By making use of generating functions, Rainville [23] obtained the following expression for $P_n(\cos\alpha)$ as a series in $P_k(\cos\beta)$:

$$P_n(\cos\alpha) = \left(\frac{\sin\alpha}{\sin\beta}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left[\frac{\sin(\beta-\alpha)}{\sin\alpha}\right]^{n-k} P_k(\cos\beta), \tag{20}$$

where α and β are unrelated.

Based on weighted free 2-Motzkin paths, we now prove a formula that also relates $P_n(X)$ to a sum involving $P_k(Y)$, from which formula (20) and some other formulae appearing in [24, Chapter 10] follow as special cases.

Theorem 7. *Suppose that X, Y, f, g are variables satisfying*

$$(X^2-1)f^2 = (Y^2-1)g^2.$$

Then we have

$$f^n P_n(X) = \sum_{k=0}^{n} \binom{n}{k} (Xf - Yg)^{n-k} g^k P_k(Y),$$

where $P_n(\cdot)$ is the Legendre polynomial.

Proof. We first observe that

$$f^n P_n(X) = M_n^2\left(\frac{(X+1)f}{2}, \frac{(X-1)f}{2}\right) = M_n\left(\frac{(X^2-1)f^2}{4}, Xf\right)$$

and

$$g^k P_k(Y) = M_k^2\left(\frac{(Y+1)g}{2}, \frac{(Y-1)g}{2}\right) = M_k\left(\frac{(Y^2-1)g^2}{4}, Yg\right).$$

Since $Xf = Xf - Yg + Yg$, we then find easily that the set $\mathcal{M}_n\left(\frac{(X^2-1)f^2}{4}, Xf\right)$ is weight-equivalent to the set of weighted free Motzkin paths of length n where level steps have weight $Xf - Yg$ or Yg and up steps have weight $\frac{(X^2-1)f^2}{4}$. For such a path p, suppose there are $n-k$ level steps with weight $Xf - Yg$. We first choose $n-k$ positions in $\binom{n}{k}$ ways to place these $n-k$ level steps, and the weight of these level steps is $(Xf - Yg)^{n-k}$. The remaining part of the path can be regarded as a path in $\mathcal{M}_k\left(\frac{(X^2-1)f^2}{4}, Yg\right)$. By the assumption that $(X^2-1)f^2 = (Y^2-1)g^2$, the weight of the remaining part is $g^k P_k(Y)$. This completes the proof. □

We can get the identities of Rainville [24] by taking specific variables X, Y, f, g in Theorem 7.

- For $X = Y = x, f = 1$ and $g = -1$, we have

$$P_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2x)^{n-k} P_k(x);$$

- For $X = 1 - 2x^2, Y = x, f = 1$ and $g = -2x$, we have

$$P_n(1-2x^2) = \sum_{k=0}^{n} \binom{n}{k} (-2x)^k P_k(x);$$

- For $X = \sqrt{\frac{1+x}{2}}, Y = x, f = \sqrt{2(1+x)}$ and $g = 1$, we have

$$2^{\frac{n}{2}} (1+x)^{\frac{n}{2}} P_n\left(\sqrt{\frac{1+x}{2}}\right) = \sum_{k=0}^{n} \binom{n}{k} P_k(x);$$

- For $X = \frac{1-xt}{\rho}, Y = x, f = \rho$ and $g = -t$, we have

$$\rho^n P_n\left(\frac{1-xt}{\rho}\right) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} t^k P_k(x);$$

- For $X = \cos\alpha, Y = \cos\beta, f = \sin\beta$ and $g = \sin\alpha$, we have

$$\sin^n \beta P_n(\cos\alpha) = \sum_{k=0}^{n} \binom{n}{k} \sin^{n-k}(\beta-\alpha) \sin^k \alpha P_k(\cos\beta);$$

- For $X = \sin\beta, Y = \cos\beta, f = \sin\beta$ and $g = -\cos\beta$, we have

$$\sin^n \beta P_n(\sin\beta) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \cos^k \beta P_k(\cos\beta);$$

- For $X = \frac{1}{\sqrt{1-x^2}}, Y = \frac{1}{\sqrt{1-y^2}}, f = y\sqrt{1-x^2}$ and $g = x\sqrt{1-y^2}$, we have

$$y^n(1-x^2)^{\frac{n}{2}} P_n\left(\frac{1}{\sqrt{1-x^2}}\right) = \sum_{k=0}^{n} \binom{n}{k} (y-x)^{n-k} x^k (1-y^2)^{\frac{k}{2}} P_k\left(\frac{1}{\sqrt{1-y^2}}\right).$$

An *Appell sequence* is a polynomial sequence $\{f_n(x)\}_{n=0,1,2,\ldots}$ satisfying the identity $f_n'(x) = nf_{n-1}(x)$, where $f_0(x)$ is a non-zero constant. Another equivalent condition on an Appell sequence is that $f_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} f_k(x) y^{n-k}$ for $n \geq 0$. It is obvious that the Legendre polynomial sequence $\{P_n(x)\}_{n=0,1,2,\ldots}$ is not an Appell sequence. With the help of weighted free 2-Motzkin paths, we derive the following formula concerning the expansion of $P_n(x+y)$.

Theorem 8. *We have*

$$P_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} \left[\binom{2n-2k}{n-k} P_k(x) + Q_{n,k}(x)\right] \left(\frac{y}{2}\right)^{n-k}, \tag{21}$$

where

$$Q_{n,k}(x) = \sum_{i=1}^{n-k} \left\{ \binom{2n-2k}{n-k-i} \left[\left(\frac{x+1}{2}\right)^i + \left(\frac{x-1}{2}\right)^i\right] \times \sum_{j=0}^{k-i} \binom{k}{j} \binom{k-j}{i+j} \left(\frac{x^2-1}{4}\right)^j x^{k-i-2j} \right\}.$$

Proof. By formula (15), we have

$$P_n(x+y) = M_n^2\left(\frac{x+y+1}{2}, \frac{x+y-1}{2}\right).$$

Note that the set $\mathcal{M}_n^2\left(\frac{x+y+1}{2}, \frac{x+y-1}{2}\right)$ is weight-equivalent to the set of weighted free 2-Motzkin paths where up steps and red level steps are given the weight $\frac{x+1}{2}$ or $\frac{y}{2}$, whereas the down steps and blue level steps are given the weight $\frac{x-1}{2}$ or $\frac{y}{2}$. To generate such a path p, we assume that it has $n-k$ steps with weight $\frac{y}{2}$. We can first choose $n-k$ positions in $\binom{n}{k}$ ways to place these $n-k$ steps. Suppose that there are s up steps and t down steps in these $n-k$ steps, then we shall consider three cases according to the values of s and t.

For the case $s=t$, we can regard these $n-k$ steps as a weighted free 2-Motzkin path of length $n-k$ where every step has weight $\frac{y}{2}$. So the weight of the $n-k$ steps is $\binom{2n-2k}{n-k}(\frac{y}{2})^{n-k}$. The remaining k steps can also be regarded as a weighted free 2-Motzkin path of length k where up and red level steps are given by the weight $\frac{x+1}{2}$, while down and blue level steps are given by the weight $\frac{x-1}{2}$, the total weight is $P_k(x)$.

If $s=t-i$ $(i>0)$, then we can regard these $n-k$ steps as a lattice path from $(0,0)$ to $(n-k,-i)$ where each step has weight $\frac{y}{2}$. In [32, Proposition 2.2], it is shown that the number of such paths is $\binom{2n-2k}{n-k-i}$, so the weight of these $n-k$ steps is $\binom{2n-2k}{n-k-i}(\frac{y}{2})^{n-k}$. In the remaining k steps, suppose that there are $i+j$ up steps, j down steps and $k-i-2j$ level steps. We can first choose j positions from the k positions to place the j down steps and choose another $i+j$ positions to place the up steps, and the steps left can be either a red level step or a blue level step, thus the total weight of the remaining k steps is

$$\sum_{j=0}^{k-i}\binom{k}{j}\binom{k-j}{i+j}\left(\frac{x+1}{2}\right)^{i+j}\left(\frac{x-1}{2}\right)^{j}\left(\frac{x+1}{2}+\frac{x-1}{2}\right)^{k-i-2j}.$$

For the case $s=t+i$ $(i>0)$, we use similar argument to prove that the total weight of the remaining k steps is

$$\sum_{j=0}^{k-i}\binom{k}{j}\binom{k-j}{i+j}\left(\frac{x-1}{2}\right)^{i+j}\left(\frac{x+1}{2}\right)^{j}\left(\frac{x+1}{2}+\frac{x-1}{2}\right)^{k-i-2j}.$$

Summing up all the possible k yields the desired formula. □

Acknowledgments. We sincerely thank the referee for careful reading and invaluable suggestions which improve the quality of this manuscript. This work was supported by the National Science Foundation of China (11931006, 12071383).

References

[1] G.E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge University Press, 1999.

[2] R. Askey, *Orthogonal Polynomials and Special Functions*, SIAM, 1975.

[3] C. Banderier and S. Schwer, Why Delannoy numbers?, *J. Statist. Plann. Inference* **135** (2005), 40-54.

[4] W.Y.C. Chen, S.H.F. Yan and L.L.M. Yang, Identities from weighted 2-Motzkin paths, *Adv. in Appl. Math.* **41** (2008), 329-334.

[5] W.Y.C. Chen and S.X.M. Pang, On the combinatorics of the Pfaff identity, *Discrete Math.* **309** (2009), 2190-2196.

[6] W. Gautschi, Orthogonal polynomials and quadrature, *Electron. Trans. Numer. Anal.* **9** (1999) 65-76.

[7] D. Foata, *Combinatoire des Identités sur les Polynômes Orthogonaux*, International Congress of Mathematicians, Warsaw, 1983.

[8] D. Foata and J. Labelle, Modèles combinatoires pour les polynômes de Meixner, *European. J. Combin.* **4** (1983), 305-311.

[9] D. Foata and P. Leroux, Polynômes de Jacobi, interprétation combinatoire et fonction génératrice, *Proc. Amer. Math. Soc.* **87** (1983), 47-53.

[10] I.J. Good, Legendre polynomials and trinomial random walks, *Proc. Cambridge Philos. Soc.* **54** (1958), 39-42.

[11] H.W. Gould, *Combinatorial Identities, A Standardized Set of Tables Listing 500 Binomial Coefficient Summations*, Morgantown, W. Va., 1972.

[12] G. Hetyei, Central Delannoy numbers, Legendre polynomials, and a balanced join operation preserving the Cohen-Macaulay property, Proceedings of the Conference on Formal Power Series and Algebraic Combinatorics (FPSAC), 2006.

[13] G. Hetyei, Shifted Jacobi polynomials and Delannoy numbers, arXiv:0909.5512v2, 2009.

[14] M.E.H. Ismail, *Classical and Quantum Orthogonal Polynomials in one Variable*, Cambridge University Press, 2005.

[15] A. Joyal, Une théorie combinatoire des séries formelles, *Adv. in Math.* **42** (1981), 1-82.

[16] J. Labelle and Y.N. Yeh, Some combinatorics of the classical hypergeometric series, *European J. Combin.* **9** (1988), 593-605.

[17] J. Labelle and Y.N. Yeh, The combinatorics of Laguerre, Charlier and Hermite polynomials, *Stud. Appl. Math.* **80** (1989), 25-36.

[18] J. Labelle and Y.N. Yeh, Combinatorial proofs of some limit formulas involving orthogonal polynomials, *Discrete Math.* **79** (1990), 77-93.

[19] P.A. MacMahon, The sum of powers of the binomial coefficients, *Q. J. Math.* **33** (1902), 274-288.

[20] T.S. Nanjundiah, Remark on a note of P. Turán, *Amer. Math. Monthly* **65** (1958), 354.

[21] P. Nevai(ed.), *Orthogonal Polynomials: Theory and Practice*, Kluwer Acad. Publ., Dordrecht, 1989.

[22] P. Leroux and V. Strehl, Jacobi polynomials: Combinatorics of the basic identities, *Discrete Math.* **57** (1985), 167-187.

[23] E.D. Rainville, Notes on Legendre polynomials, *Bull. Amer. Math. Soc.* **51** (1945), 268-271.

[24] E.D. Rainville, *Special Functions*, Macmillan Company, New York, 1960.

[25] S. Simons, A curious identity, *Math. Gaz.* **85** (2001), 296-298.

[26] N.J.A. Sloane, On-Line Encyclopedia of Integer Sequences, http://oeis.org/

[27] D. Stanton, Orthogonal polynomials and combinatorics, in *Special Functions 2000: Current Perspective and Future Directions*, Kluwer, Dorchester, 2001.

[28] V. Strehl, Combinatorics of Jacobi configurations I: Complete oriented matchings, in *Combinnatoire Enumérative*, Montréal, Canada, 1985.

[29] V. Strehl, Combinatorics of Jacobi-Configurations III: The Srivastava-Singhal generating function revisited, *Discrete Math.* **73** (1988/89), 221-232.

[30] G. Viennot, *Une Théorie Combinatoire des Polynômes Orthogonaux Généraux*, Lecture Notes, UQAM, 1983.

[31] G. Viennot, A combinatorial theory for general orthogonal polynomials with extensions and applications, in *Orthogonal Polynomials and Applications*, Bar-le-Duc, Springer, 1984.

[32] A.F.Y. Zhao, A combinatorial proof of two equivalent identities by free 2-Motzkin paths, *Integers* **13** (2013),#A38.