# ON THE $x$-COORDINATES OF PELL EQUATIONS WHICH ARE NARAYANA NUMBERS

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**Abstract**

The sequence of Narayana numbers $\{N_n\}_{n\geq 0}$ is recursively given by $N_{n+3} = N_{n+2} + N_n$ with the initial conditions $N_0 = 0, N_1 = N_2 = 1$. In this work it is shown that there exists at most one value of $x > 0$ satisfying the Pell equation $x^2 - dy^2 = \pm 1$, which is a Narayana number except for $d = 2$. Baker's theory of nonzero linear forms in logarithms of algebraic numbers, the Baker-Davenport reduction procedure, as well as the elementary properties of Narayana's sequence, are used to prove the main result. All computations are done with the help of a computer program in *Mathematica*.

## 1. Introduction

Consider the Pell equation

$$x^2 - dy^2 = \pm 1 \tag{1}$$

with a square free integer $d > 1$. All the positive integer solutions $(x, y)$ of Equation (1) are given by

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$$

for some $n \geq 1$, with the fundamental solution $(x_1, y_1)$. The sequence $\{x_n\}_{n\geq 1}$ forms a binary recurrent sequence and the formula

$$x_n = \frac{(x_1 + y_1\sqrt{d})^n + (x_1 - y_1\sqrt{d})^n}{2}$$

holds for all $n \geq 1$.

Many authors studied various Diophantine equations involving members of the sequence $\{x_n\}_{n\geq 1}$ or $\{y_n\}_{n\geq 1}$ which belong to some interesting sequences such as

Fibonacci sequence, Lucas sequence, Tribonacci sequence, Padovan sequence, etc. For instance, Luca and Togbé [20] considered the Diophantine equation

$$x_n = F_m, \tag{2}$$

where $x_n$ are the $x$-coordinates of the solutions of Equation (1) and $\{F_m\}_{m\geq 0}$ denotes the sequence of Fibonacci numbers. The authors proved that Equation (2) has at most one solution $n$ for all $d$ except $d = 2$, in which it has two solutions. Kafle et al. [16] solved the Diophantine equation

$$x_n = L_m,$$

where $\{L_m\}_{m\geq 0}$ is the sequence of Lucas numbers and proved that it has at most one solution $(n, m)$ except the case $d = 2$, in which case it has four solutions, namely, $(1, 1), (1, 2), (2, 2), (3, 4)$. In [19], Luca et al. studied a similar problem but with the sequence of Tribonacci numbers given by $T_{m+3} = T_{m+2} + T_{m+1} + T_m$, with $T_0 = 0, T_1 = T_2 = 1$. They showed that the equation

$$x_n = T_m$$

has at most one solution $(n, m)$ except when $d = 2$ and $d = 3$, in which cases it has solutions $(n, m) = (1, 1), (1, 2), (3, 5)$ and $(n, m) = (1, 3), (2, 5)$, respectively. Later, Bravo et al.[3] proved that for each square free integer $d$, there is at most one positive integer $k$ such that $x_k$ admits a representation as

$$x_k = T_m + T_n$$

for some nonnegative integers $0 \leq m \leq n$, except for $d \in \{2, 3, 5, 15, 26\}$. In [7], Ddamulira found all the solutions to the $x$-coordinates of Pell equations that are sums of two Padovan numbers. Several researchers have looked into related problems involving the intersection of the sequence $\{x_n\}_{n\geq 1}$ with linear recurrence sequences (see [4], [6], [8], [10], [11], [12], [14], [15], [17],[22]).

For most of the sequences it is expected that, the solution to the $x$- coordinate of Pell Equation (1) has at most one positive integer solution for any value of $d$ except for finitely many. In this study, we consider the sequence of Narayana numbers and show that the solution to the $x$-coordinate of Pell Equation (1) has at most one positive integer solution for any value $d$ except for $d = 2$.

Narayana numbers originated from a herd of cows and calves problem which was proposed by the Indian mathematician Narayana Pandit [1]. Narayana's cows sequence $\{N_n\}_{n\geq 0}$ is recursively defined as

$$N_{n+3} = N_{n+2} + N_n,$$

for $n \geq 0$ with initial condition $(N_0, N_1, N_2) = (0, 1, 1)$. It is the sequence A000930 in the OEIS (Online Encyclopedia of Integer Sequences). The first few terms are

$$0, 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, \cdots .$$

The characteristic polynomial of this sequence is given by $f(x) = x^3 - x^2 - 1$ which has zeros $\alpha$ ($\approx 1.46557$) and two conjugate complex zeros $\beta$ and $\gamma$ with $|\beta| = |\gamma| < 1$. Narayana's cows sequence has Binet's formula

$$N_n = a\alpha^n + b\beta^n + c\gamma^n \quad \text{for all} \quad n \geq 0,$$

with

$$a = \frac{\alpha}{(\alpha - \beta)(\alpha - \gamma)}, \; b = \frac{\beta}{(\beta - \alpha)(\beta - \gamma)}, \; c = \frac{\gamma}{(\gamma - \alpha)(\gamma - \beta)}.$$

This can also be rewritten as $N_n = C_\alpha \alpha^{n+2} + C_\beta \beta^{n+2} + C_\gamma \gamma^{n+2}$ for all $n \geq 0$ where $C_x = \frac{1}{x^3+2}$ for $x \in \{\alpha, \beta, \gamma\}$. The minimal polynomial of $C_\alpha$ is $31x^3 - 31x^2 + 10x - 1$ and all of its zeros lie inside the unit circle. One can estimate the following for $\alpha, C_\alpha$ and $C_\beta \beta^{n+2} + C_\gamma \gamma^{n+2}$:

$$1.46 < \alpha < 1.47; \quad 5.14 < C_\alpha^{-1} < 5.15; \quad |C_\beta \beta^{n+2} + C_\gamma \gamma^{n+2}| < 1/2 \quad \text{for} \quad \text{all} \quad n \geq 1.$$

By induction it is easy to prove that

$$\alpha^{n-2} \leq N_n \leq \alpha^{n-1} \text{ for } \text{ all } \; n \geq 1. \tag{3}$$

Our main theorem will be the following.

**Theorem 1.** *Let $d \geq 2$ be a square free integer. Then the Diophantine equation*

$$x_n = N_m \tag{4}$$

*has at most one positive integer solution $n$ for any value of $d$ except for $d = 2$.*

## 2. Preliminaries

Baker's theory plays an important role in reducing the bounds concerning linear forms in logarithms of algebraic numbers. Let $\eta$ be an algebraic number with minimal primitive polynomial

$$f(X) = a_0 x^d + a_1 x^{d-1} + \ldots + a_d = a_0 \prod_{i=1}^{d} (X - \eta^{(i)}) \in \mathbb{Z}[X],$$

where the leading coefficient $a_0 > 0$, and $\eta^{(i)}$'s are conjugates of $\eta$. Then the *logarithmic height* of $\eta$ is given by

$$h(\eta) = \frac{1}{d} \left( \log a_0 + \sum_{j=1}^{d} \max\{0, \log |\eta^{(j)}|\} \right).$$

The following are some properties of the logarithmic height function (see [23, Property 3.3]) which will be used to calculate the heights of the algebraic integers:

$$h(\eta+\gamma) \le h(\eta)+h(\gamma)+\log 2,$$
$$h(\eta\gamma^{\pm 1}) \le h(\eta)+h(\gamma),$$
$$h(\eta^k) = |k|h(\eta), k \in \mathbb{Z}.$$

With these notation, Matveev (see [21] or [5, Theorem 9.4]) proved the following result.

**Theorem 2.** *Let $\eta_1, \eta_2, \ldots, \eta_l$ be positive real algebraic integers in a real algebraic number field $\mathbb{L}$ of degree $d_{\mathbb{L}}$ and $b_1, b_2, \ldots, b_l$ be nonzero integers. If $\Gamma = \prod_{i=1}^{l} \eta_i^{b_i} - 1$ is not zero, then*

$$\log|\Gamma| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1+\log d_{\mathbb{L}})(1+\log D) A_1 A_2 \ldots A_l,$$

*where $D = max\{|b_1|, |b_2|, \ldots, |b_l|\}$ and $A_1, A_2, \ldots, A_l$ are positive real numbers such that*

$$A_j \ge max\{d_{\mathbb{L}} h(\eta_j), |\log \eta_j|, 0.16\} \quad \text{for } j = 1, \ldots, l.$$

When $l = 2$ and $\eta_1, \eta_2$ are positive and multiplicatively independent, we can use a result of Laurent, Mignotte and Nesterenko [18]. Let $B_1, B_2$ be real numbers greater than 1 such that

$$\log B_i = \max\left\{h(\eta_i), \frac{|\log \eta_i|}{d_{\mathbb{L}}}, \frac{1}{d_{\mathbb{L}}}\right\} \quad \text{for} \quad i = 1, 2$$

and

$$b' = \frac{|b_1|}{d_{\mathbb{L}} \log B_2} + \frac{|b_2|}{d_{\mathbb{L}} \log B_1}.$$

We put

$$\Lambda := b_1 \log \eta_1 + b_2 \log \eta_2.$$

Note that $\Lambda \neq 0$ when $\eta_1$ and $\eta_2$ are multiplicatively independent. With the above notation, we have the following result.

**Theorem 3.** *If $\eta_1, \eta_2$ are positive and multiplicatively independent, then*

$$\log|\Lambda| > -24.34 \cdot d_{\mathbb{L}}^4 \left(\max\left\{\log b' + 0.14, \frac{21}{d_{\mathbb{L}}}, \frac{1}{2}\right\}\right)^2 \log B_1 \log B_2.$$

The following Baker-Davenport reduction method due to Dujella and Pethő [9, Lemma 5] is used to reduce the bounds of the variables that are too large.

**Lemma 1.** *Let $M$ be a positive integer and $p/q$ be a convergent of the continued fraction of the irrational number $\tau$ such that $q > 6M$. Let $A$, $B$, $\mu$ be real numbers with $A > 0$ and $B > 1$. Let $\varepsilon := \|\mu q\| - M\|\tau q\|$, where $\|.\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there exists no solution to the inequality*

$$0 < |u\tau - v + \mu| < AB^{-w}$$

*in positive integers $u, v, w$ with*

$$u \leq M \text{ and } w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

## 3. Proof of Theorem 1

Let $(x_1, y_1)$ be the fundamental solution of the Pell Equation (1). Putting

$$\delta := x_1 + y_1\sqrt{d} \quad \text{and} \quad \eta := x_1 - y_1\sqrt{d},$$

we get

$$\delta \cdot \eta = x_1^2 - dy_1^2 = \pm 1.$$

Then

$$x_n = \frac{\delta^n + \eta^n}{2}.$$

Since $\delta \geq 1 + \sqrt{2} > \alpha^2$, it follows that

$$\frac{\delta^n}{\alpha^2} < x_n < \frac{\delta^n}{\alpha}. \tag{5}$$

Using inequalities (3) and (5), we get from Equation (4) that

$$\alpha^{m-2} \leq N_m = X_n < \frac{\delta^n}{\alpha}.$$

Taking the logarithm on both sides gives

$$(m-2)\log\alpha < n\log\delta - \log\alpha,$$

which leads to

$$m\log\alpha < n\log\delta + \log\alpha.$$

Thus,

$$m < n\left(\frac{\log\delta}{\log\alpha}\right) + 1.$$

Similarly, $\frac{\delta^n}{\alpha^2} < X_n = N_m \leq \alpha^{m-1}$ gives

$$n \log \delta - 2 \log \alpha < (m-1) \log \alpha.$$

Thus,

$$n \left( \frac{\log \delta}{\log \alpha} \right) - 1 < m.$$

Hence, we obtain

$$n \left( \frac{\log \delta}{\log \alpha} \right) - 1 < m < n \left( \frac{\log \delta}{\log \alpha} \right) + 1. \tag{6}$$

Using Binet's formula of Narayana's cows sequence in Equation (4), we get

$$\frac{1}{2}(\delta^n + \eta^n) = C_\alpha \alpha^{m+2} + C_\beta \beta^{m+2} + C_\gamma \gamma^{m+2}. \tag{7}$$

We can write Equation (7) as

$$\frac{1}{2}\delta^n - C_\alpha \alpha^{m+2} = \left( C_\beta \beta^{m+2} + C_\gamma \gamma^{m+2} \right) - \frac{1}{2}\eta^n.$$

Taking absolute values on both sides and dividing by $C_\alpha \alpha^{m+2}$, we get

$$\begin{aligned} \left| \delta^n (2C_\alpha)^{-1} \alpha^{-(m+2)} - 1 \right| &\leq \left| \frac{1}{2C_\alpha \delta^n \alpha^{m+2}} \right| + \left| \frac{C_\beta \beta^{m+2} + C_\gamma \gamma^{m+2}}{C_\alpha \alpha^{m+2}} \right| \\ &< \frac{1}{2C_\alpha \alpha^{2m+1}} + \frac{1}{2C_\alpha \alpha^{m+2}} \\ &< \frac{1}{2C_\alpha} \frac{2}{\alpha^{m+2}} \\ &< \frac{3}{\alpha^m}. \end{aligned} \tag{8}$$

Put

$$\Gamma = \delta^n (2C_\alpha)^{-1} \alpha^{-(m+2)} - 1. \tag{9}$$

We need to show $\Gamma \neq 0$. Suppose $\Gamma = 0$; then

$$\delta^n = 2C_\alpha \alpha^{m+2}.$$

Note that $\delta^n \in \mathbb{Q}(\sqrt{d})$, a quadratic field, while $2C_\alpha \alpha^{m+2} \in \mathbb{Q}(\alpha)$, a cubic field. The intersection of these two fields is $\mathbb{Q}$. Thus, $\delta^n \in \mathbb{Q}$. Since $\delta^n$ is an algebraic integer and $n \geq 1$, it follows that $\delta^n \in \mathbb{Z}$. Since $\delta$ is a unit, we get $\delta^n = 1$, so $n = 0$, a contradiction. Therefore, $\Gamma \neq 0$. To apply Theorem 2 in Equation (9), let

$$\eta_1 = \delta,\ \eta_2 = 2C_\alpha,\ \eta_3 = \alpha,\ b_1 = n,\ b_2 = -1,\ b_3 = -(m+2),\ l = 3,$$

where $\eta_1, \eta_2, \eta_3 \in \mathbb{Q}(\sqrt{d}, \alpha)$ and $b_1, b_2, b_3 \in \mathbb{Z}$. The degree $d_{\mathbb{L}} = [\mathbb{Q}(\sqrt{d}, \alpha) : \mathbb{Q}]$ is 6.

Since $\delta \geq 1 + \sqrt{2} > \alpha^2$, we have $n < m + 2$. Therefore, $D = \max\{1, n, m+2\} = m+2$. To estimate the parameters $A_1, A_2, A_3$, we calculate the logarithmic heights of $\eta_1, \eta_2, \eta_3$ as follows

$$h(\eta_1) = \frac{\log \delta}{2},\ h(\eta_2) = h(2C_\alpha) \leq h(2) + h(C_\alpha) < 1.84,\ h(\eta_3) = \frac{\log \alpha}{3} < 0.13.$$

Thus, we can take

$$A_1 = 3 \log \delta,\ A_2 = 11.04,\ A_3 = 0.78.$$

By virtue of Theorem 2, we find

$$\begin{aligned} \log |\Gamma| &> -1.4 \cdot 30^6 3^{4.5} 3^2 (1 + \log 6)(1 + \log(m+2))(3 \log \delta)(11.04)(0.78) \\ &> -9.2 \cdot 10^{13} \log \delta (1 + \log(m+2)). \end{aligned}$$

Comparing the above inequality with (8) gives

$$m \log \alpha - \log 3 < 9.2 \cdot 10^{13} (\log \delta)(1 + \log(m+2)).$$

Then we get

$$m \log \alpha < 9.3 \cdot 10^{13} (\log \delta)(1 + \log(m+2)),$$

and further

$$m < 24.33 \cdot 10^{13} (\log \delta)(1 + \log(m+2)). \tag{10}$$

Using (6) we get

$$n \log \delta < m \log \alpha + \log \alpha,$$

which implies

$$n < 9.4 \cdot 10^{13} (1 + \log(m+2)). \tag{11}$$

Now, we summarize what we have proved so far.

**Lemma 2.** *All solutions to Equation (4) satisfy*

$$m < 24.33 \cdot 10^{13} (\log \delta)(1 + \log(m+2)) \quad \textit{and} \quad n < 9.4 \cdot 10^{13} (1 + \log(m+2)).$$

Now, define a linear form in two logarithms

$$\Lambda := n \log \delta - \log(2C_\alpha) - (m+2) \log \alpha.$$

Since $|e^\Lambda - 1| < 1/2$ for $m \geq 5$, it follows that

$$|\Lambda| < 2|e^\Lambda - 1| < \frac{6}{\alpha^m}.$$

Let $(n_1, m_1)$ and $(n_2, m_2)$ be two pairs of positive integers such that

$$x_{n_1} = N_{m_1} \text{ and } x_{n_2} = N_{m_2}.$$

Also, assume that $n_1 < n_2$, so $m_1 < m_2$. Thus, for $m_1 \geq 5$ we have

$$|n_i \log \delta - \log(2C_\alpha) - (m_i + 2)\log \alpha| < \frac{6}{\alpha^{m_i}} \tag{12}$$

for both $i = 1, 2$. Then we multiply the inequality (12) for $i = 1$ with $n_2$ and for $i = 2$ with $n_1$, subtract them and apply the triangle inequality to get that

$$\begin{aligned}
&|(n_2 - n_1)\log(2C_\alpha) + (n_2(m_1 + 2) - n_1(m_2 + 2))\log \alpha| \\
&\quad = \quad |n_2(n_1 \log \delta - \log(2C_\alpha) - (m_1 + 2)\log \alpha) \\
&\qquad\qquad -n_1(n_2 \log \delta - \log(2C_\alpha) - (m_2 + 2)\log \alpha)| \\
&\quad \leq \quad n_2|n_1 \log \delta - \log(2C_\alpha) - (m_1 + 2)\log \alpha| \\
&\qquad\qquad +n_1|n_2 \log \delta - \log(2C_\alpha) - (m_2 + 2)\log \alpha| \\
&\quad \leq \quad \frac{6n_2}{\alpha^{m_1}} + \frac{6n_1}{\alpha^{m_2}} < \frac{12n_2}{\alpha^{m_1}}.
\end{aligned} \tag{13}$$

Now, we are ready to apply Theorem 3 with

$$\eta_1 = 2C_\alpha,\ \eta_2 = \alpha,\ \ b_1 = n_2 - n_1,\ b_2 = n_2(m_1 + 2) - n_1(m_2 + 2).$$

Observe that $n_2 - n_1 < n_2$. Thus, we have

$$|n_2(m_1 + 2) - n_1(m_2 + 2)| \leq (n_2 - n_1)\frac{\log(2C_\alpha)}{\log \alpha} + \frac{12n_2}{\alpha^{m_1}\log \alpha} < 2.5n_2.$$

Since $\mathbb{L} = \mathbb{Q}(\alpha) = 3$, we can take

$$\log B_1 = \max\left\{h(\eta_1), \frac{|\log \eta_1|}{3}, \frac{1}{3}\right\} = h(2C_\alpha)$$

and

$$\log B_2 = \max\left\{h(\eta_2), \frac{|\log \eta_2|}{3}, \frac{1}{3}\right\} = \frac{1}{3}.$$

Thus,

$$b' = \frac{|(n_2 - n_1)|}{3 \cdot \frac{1}{3}} + \frac{n_2(m_1 + 2) - n_1(m_2 + 2)}{3 \cdot 1.84} < 2n_2.$$

We put

$$\Lambda := (n_2 - n_1)\log 2C_\alpha + (n_2(m_1 + 2) - n_1(m_2 + 2))\log \alpha.$$

By virtue of Theorem 3, we get

$$\log|\Lambda| > -24.34 \cdot 3^4 (\max\{\log 2n_2 + 0.14, 7, 0.5\})^2 (1.84)(1/3),$$

which further implies

$$\log|\Lambda| > -1210 (\max\{\log 2n_2 + 0.14, 7, 0.5\})^2.$$

Combining this with (13), we get

$$m_1 \log\alpha - \log(12n_2) < 1210(\max\{\log 2n_2 + 0.14, 7, 0.5\})^2. \tag{14}$$

If $\log 2n_2 + 0.14 < 7$, then $n_2 < 473$. Then (14) gives

$$m_1 \log\alpha < \log(12 \cdot 473) + 1210 \cdot 7^2.$$

Thus, $m_1 < 59300$. Hence, $n_1 < n_2 < 473$ and $m_1 < 59300$ in this case. If $n_2 \geq 473$, then

$$m_1 \log\alpha < \log(12n_2) + 1210(\log 2n_2 + 0.14)^2,$$

which implies

$$m_1 < 2.62 \log(12n_2) + 3165.6(\log 2n_2 + 0.14)^2. \tag{15}$$

Since $\alpha^{m_1+1} > \delta^{n_1} \geq \delta$, we get

$$\log\delta < m_1 \log\alpha + \log\alpha < 1.1\log(12n_2) + 1210.1(\log 2n_2 + 0.14)^2 + \log\alpha.$$

Combining the above inequality with (10) and since $n_2 < m_2 + 2$, we have

$$m_2 < 24.33 \cdot 10^{13}(1 + \log(m_2+2))[1.1\log(12(m_2+2)) + 1210.1(\log 2(m_2+2) + 0.14)^2 + \log\alpha],$$

which gives

$$m_2 < 4.4 \cdot 10^{22}. \tag{16}$$

Using (16) in (11) gives

$$n_2 < 5 \cdot 10^{15}, \tag{17}$$

which together with (15) gives

$$m_1 < 4.33 \cdot 10^6.$$

This was when $m_1 \geq 5$. But, if $m_1 < 5$, then from inequality (6), we have

$$\frac{\delta^{n_1}}{\alpha^2} < X_{n_1} = N_{m_1} \leq \alpha^{m_1-1}.$$

Thus, $n_1 \log\delta < (m_1+1)\log\alpha$, which gives $n_1 < 3$. Since $\delta \leq \alpha^{m_1+1} \leq \alpha^5 < 7$, it follows from (10) that

$$m_2 < 24.33 \cdot 10^{13}(\log 7)(1+\log(m_2+2)),$$

which implies

$$m_2 < 1.9 \cdot 10^{16}.$$

Using the above inequality in (11), we obtain that

$$n_2 < 3.7 \cdot 10^{15}.$$

Comparing these bounds with (16) and (17), obtained in the case $m \geq 5$, we conclude that (16) and (17) hold for all $m \geq 1$. Now, we summarize what we have proved so far.

**Lemma 3.** *All solutions to $x_{n_i} = N_{m_i}$ for $i = 1, 2$ with $1 \leq m_1 < m_2$ and $n_1 < n_2$ satisfy*

$$m_1 < 4.33 \cdot 10^6, m_2 < 4.4 \cdot 10^{22} \quad \textit{and} \quad n_2 < 5 \cdot 10^{15}.$$

To reduce the bounds obtained above we use continued fractions and the Baker-Davenport reduction method. Put $\chi := -\frac{\log 2C_\alpha}{\log\alpha}$. Assume $m_1 > 230$. Inequality (13) implies

$$|(n_2-n_1)\chi - (n_2(m_1+2) - n_1(m_2+2))| < \frac{12n_2}{\alpha^{m_1}\log\alpha}.$$

Using Lemma 3, we have

$$\frac{24}{\log\alpha}(n_2-n_1)n_2 < 63(n_2-n_1)n_2 < 63n_2^2 < 1.6 \cdot 10^{33} < \alpha^{m_1},$$

which leads to

$$\frac{12n_2}{\alpha^{m_1}\log\alpha} < \frac{1}{2(n_2-n_1)}.$$

Thus, we get

$$\left|\chi - \frac{n_2(m_1+2) - n_1(m_2+2)}{n_2-n_1}\right| < \frac{12n_2}{(n_2-n_1)\alpha^{m_1}\log\alpha}.$$

So, $\frac{n_2(m_1+2)-n_1(m_2+2)}{n_2-n_1}$ is a convergent of $-\frac{\log 2C_\alpha}{\log \alpha}$. Obviously $(n_2 - n_1) < n_2 < 5{\cdot}10^{15}$. Let $[a_0, a_1, a_2, \ldots] = [2, 2, 8, 1, 8, 1, 1, 4, 1, 2, 1, 3, 2, 1, 16, 23, 5, 1, 1, 1, 1, 13, \ldots]$ be the continued fraction expansion of $\chi$. Let $p_k/q_k$ be the $k$th convergent. We recall an estimate (see [13, Theorem 163]) that

$$\frac{1}{(a_k + 2)q_k^2} < \left|\chi - \frac{p_k}{q_k}\right|,$$

where $a_{k+1}$ is the $(k + 1)$th partial quotient of $\chi$ defined as above. Using *Mathematica*, we find that

$$q_{37} < 5 \cdot 10^{15} < q_{38}.$$

Further, the maximum of $a_i(i = 0, 1, ..., 38)$ is $23 = a_{15}$. Thus, we have

$$\frac{1}{25(n_2 - n_1)^2} < \left|\chi - \frac{n_2(m_1 + 2) - n_1(m_2 + 2)}{n_2 - n_1}\right| < \frac{12n_2}{(n_2 - n_1)\alpha^{m_1} \log \alpha},$$

which implies

$$\alpha^{m_1} \log \alpha < (12n_2) \cdot (25(n_2 - n_1)) < 300n_2^2 < 7.5 \cdot 10^{33}, \text{ by Lemma 3.}$$

Thus, we get $m_1 < 207$. Since we assumed $m_1 > 230$ it follows in fact that $m_1 \leq 230$. Then (6) gives $n_1 < 91$. These upper bounds (on $n_1$ and $m_1$) make it possible to compute all $n_1$ and $m_1$. Define

$$P_n^+(x) = \frac{(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n}{2}$$

and

$$P_n^-(x) = \frac{(x + \sqrt{x^2 + 1}^n + (x - \sqrt{x^2 + 1})^n}{2}.$$

We use the fact that $x_n = P_n^\epsilon(x_1)$ for $\epsilon \in \pm 1$. All the $d$s can be calculated from $x_1^2 - \epsilon = dy_1^2$ for some integers $d$ and $y_1$ with $d$ square free. Using *Mathematica* on the equations

$$P_{n_1}^+(x_1) = N_{m_1} \quad \text{and} \quad P_{n_1}^-(x_1) = N_{m_1},$$

with $1 \leq m_1 \leq 230$ and $1 \leq n_1 < 91$, where $n_1 \leq m_1$ we get that, besides the trivial case $n_1 = 1$ (for both equations), which implies $x_1 = N_{m_1}$, $(n_1, m_1, x_1) = \{(2, 8, 2), (2, 10, 3), (2, 15, 8)\}$ are the only solutions in $-1$ case. Now, applying (12) and Lemma 1, we verify the following new solutions to Equation (4). Observe that

$$\left|n_2 \frac{\log \delta}{\log \alpha} - (m_2 + 2) + \chi\right| < 16 \cdot \alpha^{-m_2}.$$

Put $\delta_1 = 2+\sqrt{5}, \delta_2 = 3+\sqrt{10}$ and $\delta_3 = 8+\sqrt{65}$. The convergents of the continued fraction expansions of $\frac{\log \delta_i}{\log \alpha}, (i = 1, 2, 3)$ whose denominator exceeds $6 \cdot 5 \cdot 10^{15}$ and corresponding $\epsilon_i$ are positive, are

$$q_{1,31} = 61214844407183281$$
$$q_{2,32} = 282910657514169795 \quad \text{and}$$
$$q_{3,37} = 61968676817803112$$

for $i = 1, 2, 3$, respectively. Note that $\frac{\log \delta}{\log \alpha}$ is transcendental by Gelfond-Schneider Theorem [2, Theorem 2.1] and hence irrational. Now, we apply Lemma 1 with $u = n_2, v = m_2 + 2, \tau = \frac{\log \delta}{\log \alpha}$ and $\mu = -\frac{\log 2C_\alpha}{\log \alpha}$. Further, corresponding to the three cases $q = q_{1,31}, q = q_{2,32}$, and $q = q_{3,37}$, we get $\epsilon_1 > 0.2794, \epsilon_2 > 0.0670$ and $\epsilon_3 > 0.2528$. Consequently, $m_2 \leq 111, n_2 \leq 48$ in the first case, $m_2 \leq 119, n_2 \leq 52$ in the second case and $m_2 \leq 118, n_2 \leq 51$ in the third case. However, we assumed $m_2 > 230$, which is a contradiction. So, $m_2 \leq 230$. With the help of *Mathematica*, we find no values of $d$ that lead to atleast two positive integer solutions to Equation (4) except for $d = 2$ and hence proved.

It can be easily verified that the solutions to the Diophantine Equation (4) for $d = 2$ are in the form $(n, m) = (1, 1), (1, 2), (1, 3), (5, 12)$.

## References

[1] J.P. Allouche and T. Johnson, Narayana's cows and delayed morphisms, in *Articles of 3rd Computer Music Conference* JIM96, France (1996).

[2] A.Baker and G. Wüstholz, *Logarithmic Forms and Diophantine Geometry*, Cambridge University Press, Cambridge, 2008.

[3] E. F. Bravo, C. A. Goméz and F. Luca, $x$-coordinates of Pell equations as sums of two tribonacci numbers, *Period. Math. Hungar*, **77**(2) (2018), 175-190.

[4] E. F. Bravo, C. A. Goméz and F. Luca, Correction to: $x$-coordinates of Pell equations as sums of two tribonacci numbers, *Period. Math. Hungar*, **80**(1) (2020), 145-146.

[5] Y. Bugeaud, M. Mignotte and S.Siksek, Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas perfect powers, *Ann. of Math. (2)* **163** (2006), 969-1018.

[6] M. Ddamulira and F. Luca, On the $x$-coordinates of Pell equations which are $k$-generalized Fibonacci numbers, *J. Number Theory*, **207** (2020), 156-195.

[7] M. Ddamulira, On the $x$-coordinates of Pell equations that are sums of two Padovan numbers, *Bol. Soc. Mat. Mex.*, **27**(4) (2021), 1-23.

[8] A. Dossavi-Yovo, F. Luca and A. Togbé, On the $x$-coordinates of Pell equations which are repdigits, *Publ. Math. Debrecen*, **88** (2016), 381-399.

[9] A. Dujella and A. Pethő, A generalization of a theorem of Baker and Davenport, *Quart. J. Math. Oxford Ser.*, **49** (1998), 291-306.

[10] B. Faye and F. Luca, On $x$-coordinates of Pell equations that are repdigits, *Fibonacci Quart.*, **56**(1) (2018), 52-62.

[11] B. Faye and F. Luca, On $y$-coordinates of Pell equations which are base 2 repdigits, *Glasnik Matematicki*, **55**(75) (2020), 1-12.

[12] B. Faye and F. Luca, $y$-coordinates of Pell equations which are members of a fixed binary recurrence sequence, *New York J. Math.*, **26** (2020), 184-206.

[13] G. H. Hardy and E. M. Wright, *An Introduction to Theory of Numbers*, Fifth Ed., Clarendon Press, New york, 1979.

[14] S. E. Harold and C. A. Goméz, An exponential Diophantine equation related to the sum of powers of two consecutive terms of a Lucas sequence and $x$-coordinates of Pell equations, *Period. Math. Hungar*, **83** (2021), 165-184.

[15] B. Kafle, F. Luca, A. Montejano, L. Szalay, and A. Togbé, On the x-coordinates of Pell equations which are products of two Fibonacci numbers, *J. Number Theory*, **203** (2019), 310-333.

[16] B. Kafle, F. Luca and A. Togbé, $x$-coordinates of Pell equations which are Lucas numbers, *Bol. Soc. Mat. Mex.*, **25** (2019), 481-493.

[17] B. Kafle, F. Luca and A. Togbé, On the $x$-coordinates of Pell equations which are Fibonacci numbers II, *Colloq. Math.*, **149** (2017), 75-85.

[18] M. Laurent, M. Mignotte and Yu. Nesterenko, Formes linéaires en deux logarithmes et déterminants d'interpolation, *J. Number Theory*, **55** (1995), 285-321.

[19] F. Luca, A. Montejano, L. Szalay, and A. Togbé, On the $x$-coordinates of Pell equations which are Tribonacci numbers, *Acta Arith.*, **179** (2017), 25-35.

[20] F. Luca and A. Togbé, On the $x$-coordinates of Pell equations which are Fibonacci numbers, *Math. Scand.*, **122** (2018), 18-30.

[21] E. M. Matveev, An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers II, *Izv. Ross. Akad. Nauk Ser. Mat.*, **64** (2000), 125-180. Translation in *Izv. Math.*, **64** (2000), 1217-1269.

[22] S. E. Rihane, M. O. Hernane and A. Togbé, The $x$-coordinates of Pell equations and Padovan numbers, *Turkish J. of Math.*, **43** (2019), 207-223.

[23] M. Waldshmidt, *Diophantine Approximation on Linear Algebraic Groups: Transcendence Properties of the Exponential Function in Several Variables*, Springer-verlag, Berlin Heidelberg, Berlin, 2000.