



A DETERMINANT OF THE ARTIN-HASSE EXPONENTIAL COEFFICIENTS

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Abstract

We present a new determinant identity involving the coefficients of the Artin-Hasse exponential. In particular, if $E(x) = \exp(\sum_{k=0}^{\infty} \frac{x^{p^k}}{p^k}) = \sum_{n=0}^{\infty} u_n x^n$ is the Artin-Hasse exponential, we give, for any $\ell \geq 1$, a closed-form formula for the determinant $|u_{pi-j}|_{1 \leq i, j \leq \ell}$ and show it is a p -adic unit.

1. Introduction

Let p be a prime and define the Artin-Hasse exponential:

$$E(x) = \exp\left(\sum_{k=0}^{\infty} \frac{x^{p^k}}{p^k}\right) = \sum_{n=0}^{\infty} u_n x^n \in (\mathbb{Z}_p \cap \mathbb{Q})[[x]], \quad (1)$$

the coefficients of which can be shown to be p -adic integers by Dwork's Lemma (see §IV.2 of [4] for details). This exponential, and variations of it, play an essential role in the Dwork theory of exponential sums and the explicit computation of L -functions over finite fields. Its coefficients also have a rich combinatorial interpretation which will be the focus of this paper. Let h_n be the number of elements of non-negative p -power order in the symmetric group S_n . It can be shown (Lemma 2.11 in [5]) that

$$u_n = \frac{h_n}{n!},$$

where we take $h_0 = 1$ and $h_n = 0$ for negative n .

To date, there are few explicit results involving these coefficients. One can use the power series definition from Equation (1) and compute an explicit expansion of u_n (see Equation (2.9) in [5]). It is also known that the coefficients satisfy a recursive formula (Lemma 1 in [3]):

$$u_n = \frac{1}{n} \sum_{i=0}^{\infty} u_{n-p^i},$$

initially taking $u_0 = u_1 = 1$, and for negative n , defining u_n to be zero. More generally, Conrad has shown (see the proof of Theorem 2.10 in [2]) that the Artin-Hasse exponential mod p is not rational over \mathbb{F}_p , that is, $(E(x) \bmod p) \notin \mathbb{F}_p(x)$.

Other properties about the coefficients are less resolved, and the main mystery is how the coefficients behave p -adically. Conrad has conjectured (Section 5 of [2]) that they are uniformly distributed in \mathbb{Z}_p with respect to the Haar measure. Thakur [6] has also asked whether the Artin-Hasse exponential mod p is transcendental over $\mathbb{F}_p(x)$. In terms of the coefficients, Thakur’s question is whether or not the set of sequences of coefficients mod p ,

$$\{(u_{p^i n+j} \bmod p)_{n \geq 0} \mid i \geq 0 \text{ and } 0 \leq j < p^i\},$$

called the p -kernel of the sequence $(u_n \bmod p)_{n \geq 0}$, is finite. See Section 6.6 in [1] for further information on p -kernels and Section 12.2 for automatic sequences over finite fields in general.

In this paper, we present a new identity involving the Artin-Hasse exponential coefficients. Before we state our result, let us motivate it. Define the Cartier operator on $\mathbb{Z}_p[[x]]$ by:

$$U_p : \mathbb{Z}_p[[x]] \rightarrow \mathbb{Z}_p[[x]] \\ \sum_{n=0}^{\infty} a_n x^n \mapsto \sum_{n=0}^{\infty} a_{pn} x^n,$$

and construct an operator:

$$\phi : \mathbb{Z}_p[[x]] \rightarrow \mathbb{Z}_p[[x]] \\ f(x) \mapsto U_p(E(x) \cdot f(x)),$$

where the action inside the U_p is just multiplication. The Cartier operator appears in both the Dwork theory of exponential sums (§V.3 of [4]) and Christol’s theory of automatic sequences over finite fields (Section 12.2 of [1]). The matrix of this ϕ operator with respect to the basis $\{1, x, x^2, \dots\}$ is then:

$$\begin{bmatrix} 1 & u_p & u_{2p} & u_{3p} & & \\ 0 & u_{p-1} & u_{2p-1} & u_{3p-1} & \cdots & \\ 0 & u_{p-2} & u_{2p-2} & u_{3p-2} & & \\ & & \vdots & & & \end{bmatrix}.$$

Our main result is the explicit computation of the leading principal minors of the transpose of this matrix. Given the general difficulty in dealing with the coefficients u_n , especially when $n > p - 1$, it is surprising that a closed-form formula for these determinants exists. Explicitly,

Theorem 1. For any $1 \leq \ell$,

$$|u_{pi-j}|_{1 \leq i, j \leq \ell} = \prod_{k=1}^{\ell} \frac{k! p^k}{(pk)!} \in \mathbb{Z}_p^\times,$$

where we take $u_n = 0$ if $n < 0$.

To prove Theorem 1, we reduce the determinant $|u_{pi-j}|_{1 \leq i, j \leq \ell}$ to one involving only binomials. Then, using a result of Tonne [7], we are able to explicitly compute the determinant by counting certain types of Young tableaux.

This paper has three main sections. The first section is essentially independent from the rest of the paper, and in it we count the Young tableaux needed in order to get a closed-form formula from Tonne’s theorem. In the second section, we compute an expansion of h_n , Lemma 3, which we use in the third section after exploiting a determinant trick, Lemma 4, to finally prove Theorem 1.

2. Young Tableaux

We say a Young tableaux T has shape $(\lambda_1, \lambda_2, \dots, \lambda_n)$ if it has n rows and the k th row has λ_k blocks in it. All of our Young tableaux will contain integers in each block and we will use the notation $(T)_{i,j}$ to refer to the integer in the j th block in the i th row.

Example 1. Let T be the Young tableaux

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline 6 & & \\ \hline \end{array}.$$

Then T has shape $(3, 2, 1)$ and, for example, $(T)_{1,2} = 2$ and $(T)_{2,2} = 5$.

Definition 1. Let T_n be the set of all Young tableaux T such that:

1. T has shape $(n, n - 1, \dots, 1)$.
2. For each block, i, j :

$$1 \leq (T)_{i,j} \leq p + \sum_{k=1}^{i-1} ((T)_{k,j+1} - (T)_{k,j}).$$

For two tableaux in T_{n-1} and one $u \in \mathbb{Z}_{\geq 1}$, define a “gluing” operation $G(S, T, u)$, resulting in a new tableaux with shape $(n, n - 1, \dots, 1)$, in the following way:

1. Take the first column of S and glue it to the left side of T to generate a new young diagram with shape $(n + 1, n, \dots, 2)$.

2. Glue u to the bottom of the first column of this new diagram to create a diagram with shape $(n + 1, n, \dots, 1)$.

Example 2. Suppose S and T are both shape $(3, 2, 1)$:

$$G \left(\begin{array}{|c|c|c|} \hline a & b & c \\ \hline d & e & \\ \hline f & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline g & h & i \\ \hline j & k & \\ \hline l & & \\ \hline \end{array}, [m] \right) = \begin{array}{|c|c|c|c|} \hline a & g & h & i \\ \hline d & j & k & \\ \hline f & l & & \\ \hline m & & & \\ \hline \end{array}.$$

Definition 2. Suppose that $(S, T, u) \in T_{n-1} \times T_{n-1} \times \mathbb{Z}_{\geq 1}$ such that

$$1 \leq u \leq p + \sum_{k=1}^{n-1} ((T)_{k,1} - (S)_{k,1}).$$

We call such a triple n -admissible.

Lemma 1. For any $n \geq 1$, the number of n -admissible triples is $|T_{n-1}|^2 * p$.

Proof. Summing over each possibility of u for each pair of S, T yields:

$$\sum_{S, T \in T_{n-1}} p + \sum_{k=1}^{n-1} ((T)_{k,1} - (S)_{k,1}) = |T_{n-1}|^2 * p + \sum_{S, T \in T_{n-1}} \sum_{k=1}^{n-1} ((T)_{k,1} - (S)_{k,1}). \quad (2)$$

But then $\sum_{S, T \in T_{n-1}} \sum_{k=1}^{n-1} ((T)_{k,1} - (S)_{k,1}) = 0$ by symmetry and so Equation (2) is simply $|T_{n-1}|^2 * p$. □

Define two maps: $F : T_n \mapsto T_{n-1}$ and $L : T_n \mapsto T_{n-1}$ given by truncating the first block on each row and the last block on each row, respectively.

Example 3.

$$F \left(\begin{array}{|c|c|c|} \hline a & b & c \\ \hline d & e & \\ \hline f & & \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline b & c \\ \hline e & \\ \hline \end{array} \quad L \left(\begin{array}{|c|c|c|} \hline a & b & c \\ \hline d & e & \\ \hline f & & \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline a & b \\ \hline d & \\ \hline \end{array}$$

Clearly if (S, T, u) is an n -admissible triple with $F(S) = L(T)$, then $G(S, T, u) \in T_n$.

Proposition 1. The map $G(S, T, u)$ is a bijection between n -admissible triples (S, T, u) with $F(S) = L(T)$ and tableaux in T_n .

Proof. If $P \in T_n$, then $G(L(P), F(P), (P)_{n,1}) = P$, so G is surjective.

Suppose that (S, T, u) and (S', T', u') are n -admissible triples with $F(S) = L(T)$ and $F(S') = L(T')$ such that $G(S, T, u) = G(S', T', u')$. Immediately we see that $u = u'$ and $T = T'$ by the construction of G . For the same reason, the first columns of S and S' also coincide. But then $F(S') = L(T') = L(T) = F(S)$, and so the rest of the tableaux of S and S' coincide. Thus G is injective. □

Corollary 1. *For any $n \geq 3$:*

$$|T_n| = \frac{|T_{n-1}|^2}{|T_{n-2}|} \cdot p.$$

Proof. By the bijection in Proposition 1., each $P \in T_n$ corresponds to an n -admissible triple (S, T, u) with $F(S) = L(T)$. By Lemma 1, the total number of n -admissible triples is $|T_{n-1}|^2 * p$, and exactly $1/|T_{n-2}|$ many of these have the condition that $F(S) = L(T)$ since $L(T), F(S) \in T_{n-2}$. \square

Corollary 2. *For any $n \geq 1$, $|T_n| = p^{\sum_{k=1}^n k}$.*

Proof. If $n = 1$, then clearly $|T_1| = p$. For $n = 2$, suppose the following is a tableaux in T_2 :

$$\begin{array}{|c|c|} \hline a & b \\ \hline c & \\ \hline \end{array}$$

The only condition on a and b is that $1 \leq a, b \leq p$, and for each $(a, b) \in [1, p]^2$, there are $p + (b - a)$ choices for c . Thus in total there are:

$$\sum_{(a,b) \in [1,p]^2} (p + (b - a)) = \sum_{(a,b) \in [1,p]^2} p + \sum_{(a,b) \in [1,p]^2} (b - a)$$

tableaux in T_2 . But then $\sum_{(a,b) \in [1,p]^2} p = p^3$ and by symmetry $\sum_{(a,b) \in [1,p]^2} (b - a) = 0$. Hence $|T_1| = p$ and $|T_2| = p^3$. For general n we proceed by induction:

$$|T_n| = \frac{|T_{n-1}|^2}{|T_{n-2}|} \cdot p = p^{2 \sum_{k=1}^{n-1} k - \sum_{k=1}^{n-2} k + 1} = p^{n + \sum_{k=1}^{n-1} k} = p^{\sum_{k=1}^n k}.$$

\square

3. The Artin-Hasse Coefficients

Definition 3. For $n \geq 0$, let I_n be the set of all cycle types having non-negative p -power order in S_n . We do not include fixed points (represented by 1's) in our cycle types. If $t \in I_n$, denote by $|t|$ the length of the cycle type, again not counting fixed points, and observe that $0 \leq |t| \leq n$. Let I^n be the set of all cycle types t with p -power order and cycle length pn . For each cycle type t , let C_t^n be the number of elements of non-negative p -power order with cycle type t in S_n .

When $n = 0$, we take $I_0 = \{1\}$, $I^0 = \{1\}$ and $C_t^0 = 1$.

Example 4. Suppose $p = 3$. Then in S_{10} :

$$I_{10} = \{(3), (3, 3), (3, 3, 3), (9)\},$$

and if $t = (3, 3, 3)$, we see that $|t| = 3 + 3 + 3 = 9$. Further,

$$\begin{aligned} I^1 &= \{(3)\}, \\ I^2 &= \{(3, 3)\}, \\ I^3 &= \{(3, 3, 3), (9)\}. \end{aligned}$$

By definition of h_n :

$$h_n = 1 + \sum_{\substack{t \in I_n \\ t \neq 1}} C_t^n.$$

Definition 4. Let $n \in \mathbb{Z}_{\geq 2}$. Define a map:

$$r_n : S_{n-1} \times \{1, \dots, n-1\} \mapsto S_n,$$

where $r_n(\sigma, j)$ is given by replacing the element j in σ by n .

Lemma 2. *If $t \neq 1$ and $n > |t|$, then:*

$$C_t^n = \frac{n}{n - |t|} C_t^{n-1}.$$

Proof. Consider the multiset K :

$$K = \bigcup_{\sigma \in C_t^{n-1}} \{r_n(\sigma, 1), r_n(\sigma, 2), \dots, r_n(\sigma, n-1), \sigma\}.$$

Clearly $C_t^n \subset K$, so we will count and see with what multiplicity each element of C_t^n appears in K .

If j does not appear in some permutation $\sigma \in C_t^{n-1}$, then $r_n(\sigma, j) = \sigma$. So for each $\sigma \in C_t^{n-1}$, because there are $(n-1) - |t|$ numbers not appearing in σ , σ itself appears in the set K with multiplicity $(n-1) - |t| + 1 = n - |t|$. On the other hand, suppose $\sigma \in C_t^n - C_t^{n-1}$ so σ has an n in its cycle somewhere. By replacing n with all the numbers not appearing in σ , we can see there are $n - |t|$ many $\tau \in C_t^{n-1}$ such that $r_n(\tau, j) = \sigma$, some j .

Therefore since every element in K has multiplicity $n - |t|$ and K has cardinality nC_t^{n-1} , the claim follows. \square

Corollary 3. *If $t \neq 1$, then:*

$$C_t^n = \binom{n}{|t|} C_t^{|t|}.$$

Proof. Inductively applying Lemma 2 yields:

$$\begin{aligned} C_t^n &= \frac{n}{n - |t|} C_t^{n-1} = \frac{n}{n - |t|} \cdot \frac{n-1}{n-1 - |t|} C_t^{n-2} \\ &= \frac{n}{n - |t|} \cdot \frac{n-1}{n-1 - |t|} \cdot \frac{n-2}{n-2 - |t|} \cdots \frac{|t|+1}{1} C_t^{|t|}. \end{aligned}$$

\square

Lemma 3. For $n \geq 1$,

$$h_n = \sum_{k=0}^{\lfloor n/p \rfloor} \binom{n}{kp} C_k,$$

where $C_k = \sum_{t \in I^k} C_t^{kp}$.

Proof. The lemma follows from Corollary 3, and because:

$$h_n = 1 + \sum_{\substack{t \in I_n \\ t \neq 1}} C_t^n = \sum_{k=0}^{\lfloor n/p \rfloor} \sum_{t \in I^k} C_t^n = \sum_{k=0}^{\lfloor n/p \rfloor} \sum_{t \in I^k} \binom{n}{kp} C_t^{kp} = \sum_{k=0}^{\lfloor n/p \rfloor} \binom{n}{kp} \cdot \left(\sum_{t \in I^k} C_t^{kp} \right).$$

□

4. The Determinant

In this section we will piece together the results of the previous two sections and prove our main theorem. We start with a technical lemma which is key to proving Proposition 2.

Lemma 4. Let $\ell \geq 0$. For formal variables $E_{i,j}$ and $X_{i,j}$ indexed by $i, j \in \mathbb{Z}_{\geq 0}$,

$$\left| \sum_{k=0}^{\ell} E_{j,i-k} X_{i,k} \right|_{0 \leq i, j \leq \ell} = |E_{j,i}|_{0 \leq i, j \leq \ell} \cdot \prod_{i=0}^{\ell} X_{i,0},$$

where $E_{i,j}$ and $X_{i,j}$ are taken to be zero when $i < 0$ or $j < 0$.

Proof. Consider the matrix M :

$$M = |X_{i,i-j}|_{0 \leq i, j \leq \ell} \cdot |E_{j,i}|_{0 \leq i, j \leq \ell}.$$

The entry $(M)_{i,j}$ is given by:

$$(M)_{i,j} = \sum_{v=0}^{\ell} X_{i,i-v} E_{j,v} = \sum_{k=0}^{\ell} E_{j,i-k} X_{i,k},$$

where the last equality follows from the change of variables given by $k = i - v$. The claim then follows since $|X_{i,i-j}|_{0 \leq i, j \leq \ell}$ is lower triangular and equals $\prod_{i=0}^{\ell} X_{i,0}$. □

Proposition 2. Let $\ell \geq 0$. For formal variables α_i and β_i indexed by $i \in \mathbb{Z}_{\geq 0}$,

$$|\alpha_j \beta_i u_{pi-j}|_{0 \leq i, j \leq \ell} = \left| \frac{\alpha_j \beta_i}{(pi-j)!} \right|_{0 \leq i, j \leq \ell},$$

where we take the (i, j) th entry on the right to be zero when $pi - j < 0$.

Proof. Let $i, j \in \mathbb{Z}_{\geq 0}$ with $pi - j \geq 0$. By Lemma 3:

$$h_{pi-j} = \sum_{k=0}^{i-\lfloor j/p \rfloor} \binom{ip-j}{kp} C_k.$$

Hence:

$$\frac{\alpha_j \beta_i}{(pi-j)!} h_{pi-j} = \sum_{k=0}^{i-\lfloor j/p \rfloor} \left(\frac{\alpha_j}{(p(i-k)-j)!} \right) \left(\beta_i \frac{C_k}{(pk)!} \right). \tag{3}$$

Thus if

$$X_{i,k} = \begin{cases} \beta_i \frac{C_k}{(pk)!} & k \geq 0 \\ 0 & k < 0 \end{cases},$$

and

$$E_{j,k} = \begin{cases} \frac{\alpha_j}{(pk-j)!} & k \geq \lfloor j/p \rfloor \\ 0 & k < \lfloor j/p \rfloor \end{cases},$$

then Equation (3) becomes:

$$\frac{\alpha_j \beta_i}{(pi-j)!} h_{pi-j} = \sum_{k=0}^{\ell} E_{j,i-k} X_{i,k}. \tag{4}$$

So by Lemma 4,

$$\begin{aligned} |\alpha_j \beta_i u_{pi-j}|_{0 \leq i, j \leq \ell} &= \left| \frac{\alpha_j \beta_i}{(pi-j)!} h_{pi-j} \right|_{0 \leq i, j \leq \ell} = |E_{j,i}|_{0 \leq i, j \leq \ell} \cdot \prod_{i=0}^{\ell} X_{i,0} \\ &= \left| \frac{\alpha_j \beta_i}{(pi-j)!} \right|_{0 \leq i, j \leq \ell}. \end{aligned}$$

□

Corollary 4. For any $\ell \geq 1$,

$$\left| \binom{pi}{j} h_{pi-j} \right|_{1 \leq i, j \leq \ell} = \left| \binom{pi}{j} \right|_{1 \leq i, j \leq \ell}.$$

Proof. Taking $\alpha_j = 1/j!$ and $\beta_i = (pi)!$ in Proposition 2 yields:

$$\left| \frac{(pi)!}{j!} u_{pi-j} \right|_{0 \leq i, j \leq \ell} = \left| \binom{pi}{j} h_{pi-j} \right|_{0 \leq i, j \leq \ell} = \left| \frac{(pi)!}{j!(pi-j)!} \right|_{0 \leq i, j \leq \ell} = \left| \binom{pi}{j} \right|_{0 \leq i, j \leq \ell}.$$

Then, because $\alpha_0 = \beta_0 = 1$, we see that the first row of the determinant of both $\left| \frac{(pi)!}{j!} u_{pi-j} \right|_{0 \leq i, j \leq \ell}$ and $\left| \binom{pi}{j} \right|_{0 \leq i, j \leq \ell}$ is $[1, 0, 0, \dots]$, yielding the claim. □

Theorem 2. For any $\ell \geq 1$,

$$\left| \binom{pi}{j} h_{pi-j} \right|_{1 \leq i, j \leq \ell} = p^{\sum_{k=1}^{\ell} k}.$$

Proof. By Corollary 4 and Theorem 1 in [7],

$$\left| \binom{pi}{j} h_{pi-j} \right|_{1 \leq i, j \leq \ell} = \left| \binom{pi}{j} \right|_{1 \leq i, j \leq \ell} = p^{\ell} \cdot |T_{n-1}|,$$

and the theorem follows by Corollary 2. □

Corollary 5. For any $\ell \geq 1$,

$$\left| u_{pi-j} \right|_{1 \leq i, j \leq \ell} = \prod_{k=1}^{\ell} \frac{k! p^k}{(pk)!} \in \mathbb{Z}_p^{\times}.$$

Proof. Observe that

$$\binom{pi}{j} h_{pi-j} = \frac{(pi)!}{j!(pi-j)!} h_{pi-j} = \frac{(pi)!}{j!} u_{pi-j},$$

so that $u_{pi-j} = \frac{\binom{pi}{j} h_{pi-j}}{\frac{(pi)!}{j!}}$. Then, by Theorem 2,

$$\left| u_{pi-j} \right|_{1 \leq i, j \leq \ell} = \frac{\left| \binom{pi}{j} h_{pi-j} \right|_{1 \leq i, j \leq \ell}}{\prod_{i=1}^{\ell} \frac{(pi)!}{i!}} = \frac{p^{\sum_{k=1}^{\ell} k}}{\prod_{i=1}^{\ell} \frac{(pi)!}{i!}} = \prod_{k=1}^{\ell} \frac{k! p^k}{(pk)!},$$

and so since

$$\text{ord}_p \left(\frac{k! p^k}{(kp)!} \right) = \sum_{j=1}^{\infty} \left\lfloor \frac{k}{p^j} \right\rfloor + k - \sum_{j=1}^{\infty} \left\lfloor \frac{kp}{p^j} \right\rfloor = 0,$$

the claim follows. □

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