# DIRECT AND INVERSE PROBLEMS FOR SUBSET SUMS WITH CERTAIN RESTRICTIONS

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**Abstract**

Let $A$ be a finite nonempty subset of an additive abelian group $G$. For a nonnegative integer $\alpha$, let $\Sigma_{\geq\alpha}(A)$ denote the set of those elements of $G$ which can be represented as a sum of a subset with at least $\alpha$ elements. Substituting $\alpha = 0$ and $\alpha = 1$, we obtain the usual sets of subset sums. The direct problem for $\Sigma_{\geq\alpha}(A)$ is to find an optimal lower bound for the cardinality of $\Sigma_{\geq\alpha}(A)$ and the inverse problem is to characterize the sets $A$ for which the cardinality of $\Sigma_{\geq\alpha}(A)$ is minimal. Recently, Balandraud studied the direct problem for the set of subsums $\Sigma_{\geq\alpha}(A)$ in the finite prime field $\mathbb{F}_p$, and Bhanja and Pandey proved some direct theorems for $\Sigma_{\geq\alpha}(A)$ for an arbitrary finite nonempty set $A$ of integers. They also proved an inverse theorem for $\Sigma_{\geq\alpha}(A)$ in the case where the set $A$ contains only nonnegative integers. We prove some inverse theorems for the set of subsums $\Sigma_{\geq\alpha}(A)$ for arbitrary finite nonempty sets of integers. The idea involved in the proofs also enables us to give new proofs of several direct theorems for $\Sigma_{\geq\alpha}(A)$ due to Bhanja and Pandey.

## 1. Introduction and Main Results

Let $G$ be an additive abelian group. Given a subset $B$ of a set $A \subseteq G$, let $\sigma(B)$ denote the subset sum of $B$ which is the sum of all elements of the set $B$. Clearly, for the empty set $\phi$, we have $\sigma(\phi) = 0$. Let $|S|$ denote the cardinality of the set $S$. Following the commonly used notation in zero-sum problems or sumsets problems (see [12]), the set of all subset sums of the set $A$ is denoted by

$$\Sigma_0(A) = \{\sigma(B) : B \subseteq A\},$$

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and the set of all nontrivial subset sums of the set $A$ is denoted by

$$\Sigma(A) = \{\sigma(B) : B \subseteq A \text{ and } |B| \geq 1\}.$$

For a fixed integer $\alpha$ such that $0 \leq \alpha \leq |A|$, if we consider only those subsets of the set $A$ which have at least $\alpha$ elements, then the set of all subset sums of such subsets is denoted by

$$\Sigma_{\geq \alpha}(A) = \{\sigma(B) : B \subseteq A \text{ and } |B| \geq \alpha\}.$$

Obviously, we have $\Sigma_{\geq 0}(A) = \Sigma_0(A)$ and $\Sigma_{\geq 1}(A) = \Sigma(A)$.

The estimation of the optimal lower bound for the cardinality of the subset sums $\Sigma_{\geq \alpha}(A)$ is an important problem in additive combinatorics, called the direct problem. The problem of characterizing the sets for which the optimal lower bound is achieved for the cardinality of $\Sigma_{\geq \alpha}(A)$ is also an equally important problem, called the inverse problem. These are extremely important problems and have many applications related to the study of zero-sum constants such as Noether numbers, the Davenport constant, etc. (see [3, 4, 8, 12, 13, 17, 19, 20] and the references given therein).

Nathanson [17] proved the direct and inverse results for the sumset $\Sigma_{\geq \alpha}(A)$ in the case of $G = \mathbb{Z}$ and $\alpha = 1$. Balandraud studied the direct problems for $\Sigma_{\geq \alpha}(A)$ in the finite prime field $\mathbb{F}_p$ for $\alpha = 0$ and $\alpha = 1$ in [3], and for arbitrary $\alpha$ in [4]. Recently, Bhanja and Pandey [5, 6] proved the following direct theorems for $\Sigma_{\geq \alpha}(A)$ in the case of $G = \mathbb{Z}$ and arbitrary $\alpha$.

**Theorem 1** ([5, Theorem 2.1, Corollary 2.1])**.** *Let $k \geq 2$. Let $\alpha$ be an integer such that $1 \leq \alpha \leq k$. If $A$ is a set of $k$ nonnegative (nonpositive) integers and $0 \in A$, then*

$$|\Sigma_{\geq \alpha}(A)| \geq \frac{k(k-1)}{2} - \frac{\alpha(\alpha-1)}{2} + 1. \tag{1}$$

*If $A$ is a set of $k$ positive (negative) integers, then*

$$|\Sigma_{\geq \alpha}(A)| \geq \frac{k(k+1)}{2} - \frac{\alpha(\alpha+1)}{2} + 1. \tag{2}$$

*The lower bounds in Equation* (1) *and Equation* (2) *are best possible.*

**Theorem 2** ([6, Corollary 8])**.** *Let $A$ be a finite set containing $p$ positive integers, $n$ negative integers and zero, where $1 \leq n \leq p$. For any integer $\alpha \in [1, n+p+1]$, we have*

$$|\Sigma_{\geq \alpha}(A)| \geq \begin{cases} \frac{p(p+1)}{2} + \frac{n(n+1)}{2} + 1, & \text{if } 1 \leq \alpha \leq n \leq p; \\ \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\alpha-n)(\alpha-n-1)}{2} + 1, & \text{if } 1 \leq n < \alpha \leq p; \\ \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\alpha-p)(\alpha-p-1)}{2} \\ \qquad - \frac{(\alpha-n)(\alpha-n-1)}{2} + 1, & \text{if } 1 \leq n \leq p < \alpha. \end{cases} \tag{3}$$

*The lower bound in Equation* (3) *is best possible.*

**Theorem 3** ([6, Theorem 7]). *Let $A$ be a finite set containing $p$ positive integers and $n$ negative integers, where $1 \leq n \leq p$. For any integer $\alpha \in [1, n+p]$, we have*

$$|\Sigma_{\geq\alpha}(A)| \geq \begin{cases} \frac{p(p+1)}{2} + \frac{n(n+1)}{2} + 1, & \text{if } 1 \leq \alpha \leq n \leq p; \\ \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\alpha-n)(\alpha-n+1)}{2} + 1, & \text{if } 1 \leq n < \alpha \leq p; \\ \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\alpha-p)(\alpha-p+1)}{2} & \\ \qquad - \frac{(\alpha-n)(\alpha-n+1)}{2} + 1, & \text{if } 1 \leq n \leq p < \alpha. \end{cases} \tag{4}$$

*The lower bound in Equation (4) is best possible.*

**Remark 1.** The lower bounds in Theorem 2 and Theorem 3 are obtained under the assumption that $n \leq p$. If $n > p$, then we can find the corresponding lower bound by replacing the set $A$ by $-A = \{-a : a \in A\}$ and applying the above theorems.

For integers $x$ and $y$, where $x \leq y$, we denote the interval of integers $\{n \in \mathbb{Z} : x \leq n \leq y\}$ by $[x, y]$. For a nonzero integer $c$ and a set $A$, we write $c * A = \{ca : a \in A\}$. A $k$-term arithmetic progression in an additive abelian group $G$ is a set of the form $\{a, a+d, \ldots, a+(k-1)d\}$ for some $a \in G$ and $0 \neq d \in G$.

In [5], the authors also proved the inverse theorems for $\Sigma_{\geq\alpha}(A)$ for arbitrary $\alpha$ in the case where the set $A$ is a finite set of nonnegative integers including or excluding zero.

**Theorem 4** ([5, Theorem 2.2, Corollary 2.3, Remark 2.1 and Remark 2.2]). *Let $k \geq 3$ be an integer, and let $\alpha$ be an integer such that $1 \leq \alpha \leq k-2$. If $A$ is a set of $k$ nonnegative integers such that $0 \in A$ and*

$$|\Sigma_{\geq\alpha}(A)| = \frac{k(k-1)}{2} - \frac{\alpha(\alpha-1)}{2} + 1,$$

*then*

$$A = d * [0, k-1]$$

*for some positive integer $d$ except in the cases $k = 3$ and $k = 4$ when we have $A = \{0, a_1, a_2\}$ and $A = \{0, a_1, a_2, a_1 + a_2\}$, respectively, where $0 < a_1 < a_2$.*

*If $A$ is a set of $k$ positive integers such that*

$$|\Sigma_{\geq\alpha}(A)| = \frac{k(k+1)}{2} - \frac{\alpha(\alpha+1)}{2} + 1,$$

*then*

$$A = d * [1, k]$$

*for some positive integer $d$ except in the case $k = 3$ when we have $A = \{a_1, a_2, a_1 + a_2\}$, where $0 < a_1 < a_2$.*

**Remark 2.** For $\alpha = k-1$ and $\alpha = k$, we can find examples of sets $A$ which are not arithmetic progressions but $|\Sigma_{\geq\alpha}(A)|$ are minimal (see [5, Remark 2.1 and Remark 2.2]).

In this paper, our objective is to prove the inverse theorems for $\Sigma_{\geq\alpha}(A)$ for arbitrary finite sets of integers. We prove the following main results.

**Theorem 5.** *Let $A$ be a finite set containing $p$ positive integers, $n$ negative integers and zero, where $1 \leq n \leq p$. Let $\alpha \in [1, n+p-1]$, and let*

$$L = \begin{cases} \frac{p(p+1)}{2} + \frac{n(n+1)}{2} + 1, & \text{if } 1 \leq \alpha \leq n \leq p; \\ \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\alpha-n)(\alpha-n-1)}{2} + 1, & \text{if } 1 \leq n < \alpha \leq p; \\ \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\alpha-p)(\alpha-p-1)}{2} - \frac{(\alpha-n)(\alpha-n-1)}{2} + 1, & \text{if } 1 \leq n \leq p < \alpha. \end{cases} \tag{5}$$

*Then*

$$|\Sigma_{\geq\alpha}(A)| = L$$

*if and only if $A = d * [-n, p]$, where $d$ is the smallest positive element of the set $A$.*

**Remark 3.** For $\alpha = p+n$ and $\alpha = p+n+1$, it can be easily seen that $|\Sigma_{\geq\alpha}(A)|$ achieves the lower bound in Equation (5) for every set $A$ containing $p$ positive integers, $n$ negative integers and zero.

**Theorem 6.** *Let $A$ be a finite set containing $p$ positive integers and $n$ negative integers, where $1 \leq n \leq p$ and $n+p \geq 3$. Let $\alpha \in [1, n+p-2]$ and*

$$L = \begin{cases} \frac{p(p+1)}{2} + \frac{n(n+1)}{2} + 1, & \text{if } 1 \leq \alpha \leq n \leq p; \\ \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\alpha-n)(\alpha-n+1)}{2} + 1, & \text{if } 1 \leq n < \alpha \leq p; \\ \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\alpha-p)(\alpha-p+1)}{2} - \frac{(\alpha-n)(\alpha-n+1)}{2} + 1, & \text{if } 1 \leq n \leq p < \alpha. \end{cases} \tag{6}$$

*Then*

$$|\Sigma_{\geq\alpha}(A)| = L$$

*if and only if $A = d * \{-n, \ldots, -1, 1, \ldots, p\}$, where $d$ is the smallest positive element of the set $A$.*

**Remark 4.** For $\alpha = p+n-1$ and $\alpha = p+n$, it can be easily seen that $|\Sigma_{\geq\alpha}(A)|$ achieves the lower bound in Equation (6) for every set $A$ containing $p$ positive integers and $n$ negative integers with $n+p \geq 3$.

**Remark 5.** In Theorem 5 and Theorem 6, we have assumed that $n \leq p$. If $n > p$, then we can replace the set $A$ by $-A = \{-a : a \in A\}$ and apply the above theorems to establish the corresponding inverse theorems.

We prove Theorem 5 and Theorem 6 in Subsection 2.2. In Subsection 2.1, we discuss some auxiliary results required for the proofs.

## 2. Proof of the Main Results

We begin with a definition.

**Definition 1** (Generalized $h$-fold sumset [14])**.** Let $G$ be an additive abelian group. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a nonempty subset of $G$. Let $\bar{\mathbf{r}} = (r_0, r_1, \ldots, r_{k-1})$ be an ordered $k$-tuple of positive integers, where $r_i$ is the associated maximum repetitions of $a_i \in A$ for $i = 1, \ldots, k$. For a positive integer $h$, we define the generalized $h$-fold sumset

$$h^{(\bar{\mathbf{r}})}A = \left\{ \sum_{i=0}^{k-1} s_i a_i : s_i \in \mathbb{Z}, 0 \leq s_i \leq r_i \text{ for } i = 0, 1, \ldots, k-1 \text{ and } \sum_{i=0}^{k-1} s_i = h \right\}.$$

Note that whenever we write $h^{(\bar{\mathbf{r}})}A$, we assume that given the ordered $k$-tuple $\bar{\mathbf{r}}$, the elements of the set $A$ are in a fixed order. If $\bar{\mathbf{r}} = (r, \ldots, r)$, we simply write $h^{(r)}A$ in place of $h^{(\bar{\mathbf{r}})}A$. For $r = h$ and $r = 1$, we get the usual $h$-fold sumset $hA$ and the restricted $h$-fold sumset $h\hat{\ }A$, respectively. These particular sumsets have been studied extensively in literatures (see [1, 2, 7, 9, 10, 11, 18, 21] and the references given therein). The direct and inverse problems for the unified sumset $h^{(r)}A$ have been studied by Mistri and Pandey [14] in $\mathbb{Z}$, and by Monopoli [16] in $\mathbb{Z}_p$, where $p$ is a prime number (see [15] also). Yang and Chen [22] have studied the direct and inverse problems for the sumset $h^{(\bar{\mathbf{r}})}A$ in $\mathbb{Z}$.

### 2.1. Idea of the Proof and Auxiliary Results

The main idea of the proof is to express the set of subset sums $\Sigma_{\geq\alpha}(A)$ as a generalized $h$-fold sumset $h^{(\bar{\mathbf{r}})}A'$ for some specific values of $h$, $\bar{\mathbf{r}}$ and a specific set $A'$ (see Lemma 1 and Lemma 2 below) and apply the inverse results for the generalized $h$-fold sumset in $\mathbb{Z}$. We remark that this approach also enables us to give new proofs of the existing direct and inverse theorems for the sumset $\Sigma_{\geq\alpha}(A)$ (see Subsection 2.2 and Subsection 2.3).

It is easy to see that $\Sigma_{\geq 0}(A) = \Sigma(A) = \Sigma_{\geq 1}(A)$ if $0 \in \Sigma_{\geq 1}(A)$, and $\Sigma_{\geq 0}(A) = \Sigma_{\geq 1}(A) \cup \{0\}$ if $0 \notin \Sigma_{\geq 1}(A)$. Therefore, we consider only the positive values of $\alpha$. Whenever we say that $A$ is an ordered set, we mean that the elements of the set $A$ are kept in a fixed order. A simple set-theoretic argument yields the following results.

**Lemma 1.** *Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be an ordered finite nonempty subset of an additive abelian group $G$ with $a_0 = 0$, where $k$ is a positive integer. Let $\alpha$ be an integer such that $1 \leq \alpha \leq k$ and let $\bar{\mathbf{r}} = (k-\alpha+1, \underbrace{1, \ldots, 1}_{k-1 \text{ times}})$. Then $\Sigma_{\geq\alpha}(A) = k^{(\bar{\mathbf{r}})}A$.*

**Lemma 2.** *Let $k$ and $\alpha$ be integers such that $1 \leq \alpha \leq k$. Let $A = \{a_1, a_2, \ldots, a_k\}$ be an ordered finite nonempty subset of an additive abelian group $G$ with $0 \notin A$.*

*Consider the ordered set* $A_0 = \{a_0, a_1, a_2, \ldots, a_k\} \subseteq G$, *where* $a_0 = 0$. *Let* $\bar{\mathbf{r}} = (k-\alpha, \underbrace{1, \ldots, 1}_{k \text{ times}})$. *Then* $\Sigma_{\geq \alpha}(A) = k^{(\bar{\mathbf{r}})} A_0$.

The following results for the generalized $h$-fold sumset will also be crucial for the proofs. To state the results, we need to fix some notation which will be used throughout the paper. If $x$ and $y$ are integers such that $x > y$, then we take $\sum_{j=x}^{y} f(j) = 0$, where $f$ is a function. Given positive integers $h$ and $k$, and an ordered $k$-tuple $\bar{\mathbf{r}} = (r_0, r_1, \ldots, r_{k-1})$ of positive integers, define

$$L(\bar{\mathbf{r}}, h) = \left( \sum_{j=\eta+1}^{k-1} j r_j - \sum_{j=0}^{\mu-1} j r_j \right) + \eta\theta - \mu\delta + 1,$$

where $\mu = \mu(\bar{\mathbf{r}}, h)$ is the largest integer and $\eta = \eta(\bar{\mathbf{r}}, h)$ is the least integer such that

$$\sum_{j=0}^{\mu-1} r_j \leq h, \quad \sum_{j=\eta+1}^{k-1} r_j \leq h$$

and

$$\delta = \delta(\bar{\mathbf{r}}, h) = h - \sum_{j=0}^{\mu-1} r_j, \quad \theta = \theta(\bar{\mathbf{r}}, h) = h - \sum_{j=\eta+1}^{k-1} r_j.$$

These definitions of $\mu = \mu(\bar{\mathbf{r}}, h)$, $\eta = \eta(\bar{\mathbf{r}}, h)$ and $L(\bar{\mathbf{r}}, h)$ will be used throughout the paper.

**Theorem 7** ([22, Theorem 1]). *Let* $A = \{a_0, a_1, \ldots, a_{k-1}\}$ *be a set of integers with* $a_0 < a_1 < \cdots < a_{k-1}$, *where* $k$ *is a positive integer. Let* $\bar{\mathbf{r}} = (r_0, r_1, \ldots, r_{k-1})$ *be an ordered* $k$*-tuple of positive integers, and* $h$ *be an integer satisfying* $2 \leq h \leq \sum_{j=0}^{k-1} r_j$. *Then*

$$|h^{(\bar{\mathbf{r}})} A| \geq L(\bar{\mathbf{r}}, h).$$

*This lower bound is best possible.*

**Theorem 8** ([22, Theorem 2]). *Let* $k \geq 5$ *be an integer. Let* $\bar{\mathbf{r}} = (r_0, \ldots, r_{k-1})$ *be an ordered* $k$*-tuple of positive integers and* $h$ *be an integer satisfying*

$$2 \leq h \leq \sum_{j=0}^{k-1} r_j - 2.$$

*If* $A$ *is a set of* $k$ *integers, then*

$$|h^{(\bar{\mathbf{r}})} A| = L(\bar{\mathbf{r}}, h)$$

*if and only if* $A$ *is a* $k$*-term arithmetic progression.*

**Theorem 9** ([22, Theorem 3])**.** *Let $A = \{a_0, a_1, a_2\}$ be a set of integers with $a_0 < a_1 < a_2$ and $\bar{\mathbf{r}} = (r_0, r_1, r_2)$ be an ordered 3-tuple of positive integers. Suppose that $h$ is an integer with $2 \leq h \leq r_0 + r_1 + r_2 - 2$. Then*

*(i) for $r_1 = 1$, we have $|h^{(\bar{\mathbf{r}})}A| = L(\bar{\mathbf{r}}, h)$;*

*(ii) for $r_1 \geq 2$, we have $|h^{(\bar{\mathbf{r}})}A| = L(\bar{\mathbf{r}}, h)$ if and only if $A$ is a 3-term arithmetic progression.*

**Theorem 10** ([22, Theorem 4])**.** *Let $A = \{a_0, a_1, a_2, a_3\}$ be a set of integers with $a_0 < a_1 < a_2 < a_3$ and $\bar{\mathbf{r}} = (r_0, r_1, r_2, r_3)$ be an ordered 4-tuple of positive integers. Suppose that $h$ is an integer with $2 \leq h \leq r_0 + r_1 + r_2 + r_3 - 2$. Then*

*(i) for $r_1 = r_2 = 1$, we have $|h^{(\bar{\mathbf{r}})}A| = L(\bar{\mathbf{r}}, h)$ if and only if $a_1 - a_0 = a_3 - a_2$;*

*(ii) for $r_1 \geq 2$ or $r_2 \geq 2$, we have $|h^{(\bar{\mathbf{r}})}A| = L(\bar{\mathbf{r}}, h)$ if and only if $A$ is a 4-term arithmetic progression.*

**Remark 6.** When using the sumset $h^{(\bar{\mathbf{r}})}A$, the relative orders of elements listed in the set $A$ are considered. Therefore, from now onwards, whenever we use the sumset $h^{(\bar{\mathbf{r}})}A$ or $h^{(\bar{\mathbf{r}})}A_0$ in a proof, we write the elements of the sets $A$ and $A_0$ in a fixed order.

Let $\pi : [1, k] \rightarrow [1, k]$ be a permutation, where $k$ is a positive integer. For an ordered set $A = \{a_1, a_2, \ldots, a_k\}$ and an ordered $k$-tuple $\bar{\mathbf{r}} = (r_1, r_2, \ldots, r_k)$ of positive integers, denote the ordered set $A_\pi = \{a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(k)}\}$ and $\bar{\mathbf{r}}_\pi = (r_{\pi(1)}, r_{\pi(2)}, \ldots, r_{\pi(k)})$. Since the order of the elements in the set $A$ is assumed to be fixed in the definition of $h^{(\bar{\mathbf{r}})}A$, the following lemma will be useful in expressing the generalized $h$-fold sumset $h^{(\bar{\mathbf{r}})}A$ as a similar generalized $h$-fold sumset, if we change the order of the elements in the set $A$.

**Lemma 3.** *Let $A = \{a_1, a_2, \ldots, a_k\}$ be an ordered finite nonempty subset of an additive abelian group $G$, where $k$ is a positive integer. Let $\bar{\mathbf{r}} = (r_1, r_2, \ldots, r_k)$ be an ordered $k$-tuple of positive integers. Let $h \geq 2$ be an integer and let $\pi$ be a permutation of $[1, k]$. Then*

$$h^{(\bar{\mathbf{r}})}A = h^{(\bar{\mathbf{r}}_\pi)}A_\pi.$$

*Proof.* The proof is easy, and is left to the reader. □

### 2.2. Proof of Theorem 5 and Theorem 6

*Proof of Theorem 5.* Let $A = \{a_0, a_1, \ldots, a_{n-1}, a_n, a_{n+1}, \ldots, a_{n+p}\}$, where $a_0 < a_1 < \cdots < a_n = 0 < a_{n+1} < \cdots < a_{n+p}$. Then $k = |A| = p + n + 1$. Let

$$\bar{\mathbf{r}} = (r_0, r_1, \ldots, r_{n-1}, r_n, r_{n+1}, \ldots, r_{n+p}),$$

where $r_0 = r_1 = \cdots = r_{n-1} = r_{n+1} = \cdots = r_{n+p} = 1$ and $r_n = k - \alpha + 1$. It follows from Lemma 1 and Lemma 3 that $\Sigma_{\geq\alpha}(A) = k^{(\bar{\mathbf{r}})}A$. We can verify that $L = L(\bar{\mathbf{r}}, k)$. If $k = 3$, then clearly, $p = n = 1$. Hence $A = \{a_0, a_1, a_2\}$ with $a_0 < 0 = a_1 < a_2$ and $\bar{\mathbf{r}} = (r_0, r_1, r_2) = (1, k - \alpha + 1, 1)$. Since $r_1 = k - \alpha + 1 \geq 3 > 2$, it follows from Theorem 9 that $|k^{(\bar{\mathbf{r}})}A| = |\Sigma_{\geq\alpha}(A)| = L = L(\bar{\mathbf{r}}, k)$ if and only if $A$ is an arithmetic progression. Hence

$$a_1 - a_0 = a_2 - a_1$$

which implies $a_0 = -a_2$, and so $A = \{-a_2, 0, a_2\} = a_2 * [-1, 1]$.

If $k = 4$, then clearly we have $n = 1$ and $p = 2$. Hence $A = \{a_0, a_1, a_2, a_3\}$ with $a_0 < 0 = a_1 < a_2 < a_3$ and $\bar{\mathbf{r}} = (r_0, r_1, r_2, r_3) = (1, k - \alpha + 1, 1, 1)$. Since $r_1 = k - \alpha + 1 \geq 3 > 2$, it follows from Theorem 10 that $|k^{(\bar{\mathbf{r}})}A| = |\Sigma_{\geq\alpha}(A)| = L = L(\bar{\mathbf{r}}, k)$ if and only if $A$ is an arithmetic progression. Hence

$$a_1 - a_0 = a_2 - a_1 = a_3 - a_2$$

which implies $a_0 = -a_2$ and $a_3 = 2a_2$, and so $A = \{-a_2, 0, a_2, 2a_2\} = a_2 * [-1, 2]$.

If $k \geq 5$, then it follows from Theorem 8 that $|k^{(\bar{\mathbf{r}})}A| = |\Sigma_{\geq\alpha}(A)| = L = L(\bar{\mathbf{r}}, k)$ if and only if $A$ is an arithmetic progression. Hence

$$\begin{aligned} a_1 - a_0 = a_2 - a_1 = \cdots = a_{n-1} - a_{n-2} = a_n - a_{n-1} \\ = a_{n+1} - a_n = a_{n+2} - a_{n+1} = \cdots = a_{n+p} - a_{n+p-1}, \end{aligned}$$

which implies $a_{n-j} = -ja_{n+1}$ for $j = 1, \dots, n$ and $a_{n+j} = ja_{n+1}$ for $j = 2, \dots, p$, and so $A = a_{n+1} * [-n, p]$. Thus in all cases, we have $|\Sigma_{\geq\alpha}(A)| = L(\bar{\mathbf{r}}, h)$ if and only if $A = a_{n+1} * [-n, p]$. This completes the proof. □

*Proof of Theorem 6.* Let $A = \{a_0, a_1, \dots, a_{n-1}, a_{n+1}, \dots, a_{n+p}\}$ and $A_0 = \{a_0, a_1, \dots, a_{n-1}, a_n, a_{n+1}, \dots, a_{n+p}\}$ with $a_0 < a_1 < \cdots < a_{n-1} < 0 = a_n < a_{n+1} < \cdots < a_{n+p}$. Let $k = |A| = p+n$, $k_0 = |A_0| = p+n+1$, and let $\bar{\mathbf{r}} = (r_0, r_1, \dots, r_{n-1}, r_n, r_{n+1}, \dots, r_{n+p})$, where $r_0 = r_1 = \cdots = r_{n-1} = r_{n+1} = \cdots = r_{n+p} = 1$ and $r_n = k - \alpha$. Then by Lemma 2 and Lemma 3, we have $\Sigma_{\geq\alpha}(A) = k^{(\bar{\mathbf{r}})}A_0$. Therefore, $|\Sigma_{\geq\alpha}(A)| = |k^{(\bar{\mathbf{r}})}A_0|$. We can verify that $L = L(\bar{\mathbf{r}}, k)$.

If $k = 3$, then clearly we have $n = 1$ and $p = 2$. Hence $A = \{a_0, a_2, a_3\}$ and $A_0 = \{a_0, a_1, a_2, a_3\}$ with $a_0 < 0 = a_1 < a_2 < a_3$ and $\bar{\mathbf{r}} = (r_0, r_1, r_2, r_3) = (1, k - \alpha, 1, 1)$. Since $r_1 = k - \alpha \geq 2$, it follows from Theorem 10 that $|k^{(\bar{\mathbf{r}})}A_0| = |\Sigma_{\geq\alpha}(A)| = L = L(\bar{\mathbf{r}}, k)$ if and only if $A_0$ is an arithmetic progression. Hence

$$a_1 - a_0 = a_2 - a_1 = a_3 - a_2$$

which implies $a_0 = -a_2$ and $a_3 = 2a_2$, and so $A_0 = \{-a_2, 0, a_2, 2a_2\}$. Hence $A = \{-a_2, a_2, 2a_2\} = a_2 * \{-1, 1, 2\}$.

If $k \geq 4$, then it follows from Theorem 8 that $|k^{(\bar{\mathbf{r}})}A_0| = |\Sigma_{\geq\alpha}(A)| = L = L(\bar{\mathbf{r}}, k)$ if and only if $A_0$ is an arithmetic progression. Hence

$$\begin{aligned} a_1 - a_0 = a_2 - a_1 = \cdots = a_{n-1} - a_{n-2} = a_n - a_{n-1} \\ = a_{n+1} - a_n = a_{n+2} - a_{n+1} = \cdots = a_{n+p} - a_{n+p-1} \end{aligned}$$

which implies $a_{n-j} = -ja_{n+1}$ for $j = 1, \ldots, n$ and $a_{n+j} = ja_{n+1}$ for $j = 2, \ldots, p$, and so $A_0 = a_{n+1} * [-n, p]$. Therefore, $A = a_{n+1} * \{-n, -(n-1), \ldots, -1, 1, 2, \ldots, p\}$.

Thus in all cases, we have $|\Sigma_{\geq\alpha}(A)| = L$ if and only if

$$A = d * \{-n, -(n-1), \ldots, -1, 1, 2, \ldots, p\},$$

where $d$ is the smallest element of the set $A$. This completes the proof. □

**Remark 7.** Theorem 4 also can be proved by a similar approach. We omit the details.

### 2.3. New Proofs of Direct Theorems

The idea used to prove Theorem 5 and Theorem 6 also enables us to give new proofs of the direct theorems for $\Sigma_{\geq\alpha}(A)$. Here we give proofs of Theorem 1 and Theorem 2. The proof of Theorem 3 is similar to the proof of Theorem 2.

*Proof of Theorem 1.* First assume that $A$ is a set of $k \geq 2$ nonnegative integers with $0 \in A$. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$, where $0 = a_0 < a_1 < \cdots < a_{k-1}$. If $\alpha = k$, then $\Sigma_{\geq\alpha}(A) = \{a_0 + a_1 + \cdots + a_{k-1}\}$, and thus the theorem holds in this case. Therefore, we can assume that $1 \leq \alpha \leq k-1$. Let $h = k$, and $\bar{\mathbf{r}} = (r_0, r_1, \ldots, r_{k-1})$, where $r_0 = k - \alpha + 1$ and $r_1 = r_2 = \cdots = r_{k-1} = 1$. It follows from Lemma 1 that $\Sigma_{\geq\alpha}(A) = h^{(\bar{\mathbf{r}})}A$.

Clearly, the integers $h, k$ and $k$-tuple $\bar{\mathbf{r}}$ satisfy the assumptions of Theorem 7. It is easy to see that $\mu = \alpha$ and $\eta = 0$. Therefore, $\delta = h - \sum_{j=0}^{\alpha-1} = k - k = 0$ and $\theta = h - \sum_{j=\eta+1}^{k-1} = k - (k-1) = 1$. Hence

$$L(\bar{\mathbf{r}}, h) = \left( \sum_{j=1}^{k-1} j - \sum_{j=1}^{\alpha-1} j \right) + 1 = \frac{k(k-1)}{2} - \frac{\alpha(\alpha-1)}{2} + 1.$$

Therefore, it follows from Theorem 7 that

$$|\Sigma_{\geq\alpha}(A)| = |h^{(\bar{\mathbf{r}})}A| \geq L(\bar{\mathbf{r}}, h) = \frac{k(k-1)}{2} - \frac{\alpha(\alpha-1)}{2} + 1.$$

We can see that the lower bound in Equation (1) is best possible by taking the set $A = [0, k-1]$, where $k \geq 2$. This proves the first part of the theorem.

A similar argument proves the theorem in the case where the set $A$ is a set of $k \geq 2$ positive integers. □

*Proof of Theorem 2.* Let $A = \{-b_n, -b_{n-1}, \ldots, -b_1, 0, a_1, \ldots, a_p\}$, where

$$-b_n < -b_{n-1} < \ldots < -b_1 < 0 < a_1 < \ldots < a_p.$$

Let $k = |A| = p + n + 1$ and $\bar{\mathbf{r}} = (r_0, r_1, \ldots, r_{n-1}, r_n, r_{n+1}, \ldots, r_{n+p})$, where $r_0 = r_1 = \cdots = r_{n-1} = r_{n+1} = \cdots = r_{n+p} = 1$ and $r_n = k - \alpha + 1$. Then by Lemma 1 and Lemma 3, we have $\Sigma_{\geq\alpha}(A) = k^{(\bar{\mathbf{r}})}A$. Therefore, $|\Sigma_{\geq\alpha}(A)| = |k^{(\bar{\mathbf{r}})}A| \geq L(\bar{\mathbf{r}}, k)$.

`Case 1` ($1 \leq \alpha \leq n \leq p$). In this case, we can easily determine that $\mu = \eta = n, \delta = p + 1$ and $\theta = n + 1$. Hence

$$\begin{aligned} L(\bar{\mathbf{r}}, k) &= \left( \sum_{j=n+1}^{p+n} j r_j - \sum_{j=0}^{n-1} j r_j \right) + n(n+1) - n(p+1) + 1 \\ &= \left( \sum_{j=n+1}^{p+n} j - \sum_{j=0}^{n-1} j \right) + n^2 - pn + 1 \\ &= \frac{p(p+1)}{2} + \frac{n(n+1)}{2} + 1. \end{aligned}$$

`Case 2` ($1 \leq n < \alpha \leq p$). In this case, we can determine that $\mu = \alpha, \eta = n, \delta = 0$ and $\theta = n + 1$. Hence

$$\begin{aligned} L(\bar{\mathbf{r}}, k) &= \left( \sum_{j=n+1}^{p+n} j r_j - \sum_{j=0}^{\alpha-1} j r_j \right) + n(n+1) - 0 + 1 \\ &= \left( \sum_{j=n+1}^{p+n} j - \sum_{j=0}^{n-1} j - n(k - \alpha + 1) - \sum_{j=n+1}^{\alpha-1} j \right) + n^2 + n + 1 \\ &= \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\alpha - n)(\alpha - n - 1)}{2} + 1. \end{aligned}$$

`Case 3` ($1 \leq n \leq p < \alpha \leq p + n$). Write $\alpha = p + x$. Then $1 \leq x \leq n$. By considering the subcases $x = 1$ and $2 \leq x \leq n$, we can see that in both the subcases, we have

$\mu = \alpha = p + x, \eta = n - x, \delta = 0$ and $\theta = 0$. Hence for $x = 1$, we have $\alpha = p + 1$ and

$$\begin{aligned}
L(\bar{\mathbf{r}}, k) &= \left( \sum_{j=n}^{p+n} j r_j - \sum_{j=0}^{\alpha-1} j r_j \right) + 0 - 0 + 1 \\
&= n(k - \alpha + 1) + \sum_{j=n+1}^{p+n} j - \sum_{j=0}^{n-1} j - n(k - \alpha + 1) - \sum_{j=n+1}^{\alpha-1} j + 1 \\
&= \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\alpha - n)(\alpha - n - 1)}{2} + 1. \\
&= \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\alpha - p)(\alpha - p - 1)}{2} - \frac{(\alpha - n)(\alpha - n - 1)}{2} + 1.
\end{aligned}$$

Now for $2 \leq x \leq n$, we have

$$\begin{aligned}
L(\bar{\mathbf{r}}, k) &= \left( \sum_{j=n-x-1}^{p+n} j r_j - \sum_{j=0}^{\alpha-1} j r_j \right) + 0 - 0 + 1 \\
&= \left( \sum_{j=n-x+1}^{n-1} j + n(k - \alpha + 1) + \sum_{j=n+1}^{p+n} j \right) \\
&\qquad - \left( \sum_{j=0}^{n-1} j + n(k - \alpha + 1) - \sum_{j=n+1}^{\alpha-1} j \right) + 1 \\
&= \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\alpha - p)(\alpha - p - 1)}{2} - \frac{(\alpha - n)(\alpha - n - 1)}{2} + 1.
\end{aligned}$$

Thus in both the subcases, we have

$$L(\bar{\mathbf{r}}, k) = \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\alpha - p)(\alpha - p - 1)}{2} - \frac{(\alpha - n)(\alpha - n - 1)}{2} + 1.$$

`Case 4` ($\alpha = p + n + 1$). In this case, clearly we have

$$|\Sigma_{\geq \alpha}(A)| = 1 = \frac{p(p+1)}{2} + \frac{n(n+1)}{2} - \frac{(\alpha - p)(\alpha - p - 1)}{2} - \frac{(\alpha - n)(\alpha - n - 1)}{2} + 1.$$

Combining all the cases, we get the desired lower bound in Equation (3). We can see that the lower bound in Equation (3) is best possible by taking the set $A = [-n, p]$. This completes the proof. □

## 3. Concluding Remarks

We have shown the connection between the set of subsums $\Sigma_{\geq \alpha}(A)$ and the generalized $h$-fold sumset $h^{(\bar{\mathbf{r}})}A$ in an additive abelian group $G$. Thus the problem of studying the set of subsums $\Sigma_{\geq\alpha}(A)$ in arbitrary abelian groups reduces to the study of the generalized $h$-fold sumset $h^{(\bar{\mathbf{r}})}A$. Therefore, it would be an important and interesting problem to investigate this sumset in general abelian groups.

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