# SUMS OF $\omega(n)$ AND $\Omega(n)$ OVER THE $k$-FREE PARTS AND $k$-FULL PARTS OF SOME PARTICULAR SEQUENCES

**Rafael Jakimczuk**
*División Matemática, Universidad Nacional de Luján, Buenos Aires, Argentina*
jakimczu@mail.unlu.edu.ar

**Matilde Lalín**
*Département de mathématiques et de statistique, Université de Montréal, Montréal, Québec, Canada*
matilde.lalin@umontreal.ca

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**Abstract**

The $k$-free part of a positive integer $n$ is the product of the prime powers dividing $n$ that have exponent less than $k$ in the factorization, while the $k$-full part of $n$ is the product of the prime powers that have exponent at least $k$. We consider sums of the prime factor counting functions $\omega$ and $\Omega$ going over the $k$-free parts and $k$-full parts of some particular number sequences.

## 1. Introduction

For a positive integer with prime factorization

$$n = q_1^{s_1} \cdots q_r^{s_r}, \tag{1}$$

where the $q_j$ are the prime factors and the $s_j \geq 1$ are their respective exponents, the prime factor counting functions are defined by $\omega(n) = r$ and $\Omega(n) = s_1 + \cdots + s_r$.

For $k \geq 1$, and $n$ as above, let

$$L_k(n) = \prod_{\substack{1 \leq j \leq r \\ s_j < k}} q_j^{s_j} \qquad \text{and} \qquad U_k(n) = \prod_{\substack{1 \leq j \leq r \\ k \leq s_j}} q_j^{s_j}.$$

We say that $L_k(n)$ is the *$k$-free part* of $n$ and that $U_k(n)$ is the *$k$-full part* of $n$. By convention, $L_1(n) = 1$, while naturally $U_1(n) = n$. Similarly, when $k > \max_j s_j$, we have $L_k(n) = n$ and $U_k(n) = 1$. We remark that $n = L_k(n)U_k(n)$ for any $k$ and that $L_k(n)$ and $U_k(n)$ are coprime. The case of $k = 2$ was considered by Cloutier, De Koninck, and Doyon [2].

The aim of this article is to consider sums of $\omega$ and $\Omega$ composed with $U_k$ and $L_k$ evaluated in certain sequences of positive integer numbers.

To begin, we consider the evaluation in the whole sequence of positive integer numbers.

**Theorem 1.** *Let $k \geq 1$ be an integer. We have that*

$$\sum_{n\leq x} \omega(U_k(n)) = \left(\sum_p \frac{1}{p^k}\right) x + O_k\left(\frac{x^{\frac{1}{k}}}{\log x}\right), \tag{2}$$

*and*

$$\sum_{n\leq x} \Omega(U_k(n)) = \left(\sum_p \frac{1-k+kp}{p^{k+1}-p^k}\right) x + O_k\left(\frac{x^{\frac{1}{k}}}{\log x}\right), \tag{3}$$

*where the sums over $p$ indicate that the sums are taken over all prime numbers.*

For the rest of this article we will continue to use the convention that sums and products over $p$ indicate over all the primes, unless stated otherwise.

**Corollary 1.** *Let $k \geq 1$ be an integer. We have that*

$$\sum_{n\leq x} \omega(L_k(n)) = x\log\log x + \left(B_1 - \sum_p \frac{1}{p^k}\right) x + O\left(\frac{x}{\log x}\right), \tag{4}$$

*where $B_1$ is the Mertens constant given by*

$$B_1 = \gamma + \sum_p \left(\log\left(1-\frac{1}{p}\right) + \frac{1}{p}\right), \tag{5}$$

*and $\gamma = 0.57721\ldots$ is the Euler–Mascheroni constant.*

*We have that*

$$\sum_{n\leq x} \Omega(L_k(n)) = x\log\log x + \left(B_2 - \sum_p \frac{1-k+kp}{p^{k+1}-p^k}\right) x + O\left(\frac{x}{\log x}\right), \tag{6}$$

*where*

$$B_2 = B_1 + \sum_p \frac{1}{p(p-1)}. \tag{7}$$

Let $h \geq 1$ be an integer. A positive integer $n$ is said to be $h$-free if all its prime factors have exponents less than $h$. In other words, if $n$ has prime factorization (1), then $s_j \leq h-1$ for all $j$. In particular, $n$ is square-free if all $s_j = 1$. We denote by $\mathcal{S}_h$ the set of $h$-free positive integers.

We have the following result.

**Theorem 2.** *Let $h > k > 1$ be integers. Then we have*

$$\sum_{\substack{n\in\mathcal{S}_h\\ n\le x}} \Omega(U_k(n)) =\frac{1}{\zeta(h)} D_{\Omega,k,h} x + O_h\left(x^{\frac{2k-1}{k^2}} \log\log x\right), \tag{8}$$

$$\sum_{\substack{n\in\mathcal{S}_h\\ n\le x}} \omega(U_k(n)) =\frac{1}{\zeta(h)} D_{\omega,k,h} x + O_h\left(x^{\frac{2k-1}{k^2}} \log\log x\right). \tag{9}$$

*where*

$$D_{\Omega,k,h} = \sum_p \frac{h-1-(k-1)p^{h-k} - hp + kp^{h-k+1}}{(p-1)\left(p^h-1\right)}, \tag{10}$$

*and*

$$D_{\omega,k,h} = \sum_p \frac{p^{h-k}-1}{p^h-1}. \tag{11}$$

**Corollary 2.** *Let $h > k > 1$ be integers. Then we have*

$$\sum_{\substack{n\in\mathcal{S}_h\\ n\le x}} \Omega(L_k(n)) =\frac{1}{\zeta(h)} x\log\log x + O(x), \tag{12}$$

$$\sum_{\substack{n\in\mathcal{S}_h\\ n\le x}} \omega(L_k(n)) =\frac{1}{\zeta(h)} x\log\log x + O(x). \tag{13}$$

Let $h \ge 1$ be an integer. A positive integer $n$ is said to be $h$-full if all its prime factors have exponents greater or equal than $h$. In other words, if $n$ has prime factorization (1), then $s_j \ge h$ for all $j$. (This definition is trivial for $h = 1$.) We denote by $\mathcal{N}_h$ the set of $h$-full positive integers.

We prove the following estimates.

**Theorem 3.** *Let $k > h > 0$ be integers. Then we have*

$$\sum_{\substack{n\in\mathcal{N}_h\\ n\le x}} \Omega(U_k(n)) =\gamma_{0,h} E_{\Omega,k,h} x^{\frac{1}{h}} + O\left(x^{\frac{1}{h}-\left(\frac{k}{h}-1\right)\frac{1}{k+2h(h+1)}+\varepsilon} \log\log x\right), \tag{14}$$

$$\sum_{\substack{n\in\mathcal{N}_h\\ n\le x}} \omega(U_k(n)) =\gamma_{0,h} E_{\omega,k,h} x^{\frac{1}{h}} + O\left(x^{\frac{1}{h}-\left(\frac{k}{h}-1\right)\frac{1}{k+2h(h+1)}+\varepsilon} \log\log x\right), \tag{15}$$

*where*

$$\gamma_{0,h} = \prod_p \left(1 + \frac{p-p^{\frac{1}{h}}}{p^2\left(p^{\frac{1}{h}}-1\right)}\right), \tag{16}$$

$$E_{\Omega,k,h} = \sum_p \frac{kp^{\frac{1}{h}} - k + 1}{p^{\frac{k-h-1}{h}} \left(p^{\frac{1}{h}} - 1\right)\left(p^{1+\frac{1}{h}} + p^{\frac{1}{h}} - p\right)}, \tag{17}$$

*and*

$$E_{\omega,k,h} = \sum_p \frac{1}{p^{\frac{k-h-1}{h}} \left(p^{1+\frac{1}{h}} + p^{\frac{1}{h}} - p\right)}. \tag{18}$$

**Corollary 3.** *Let $k > h > 0$ be integers. The following formula holds:*

$$\sum_{\substack{n\in\mathcal{N}_h \\ n\le x}} \Omega(L_k(n)) = h\gamma_{0,h} x^{\frac{1}{h}} \log\log x + \gamma_{0,h}\left(C_{\Omega,h} - E_{\omega,k,h}\right) x^{\frac{1}{h}} + O_h\left(\frac{x^{\frac{1}{h}}}{\sqrt{\log x}}\right), \tag{19}$$

*where*

$$C_{\Omega,h} = h(B_2 - \log h) + \sum_p \frac{(h+1)p^{1+\frac{1}{h}} - hp - 2hp^{\frac{2}{h}} + (2h-1)p^{\frac{1}{h}}}{(p-1)\left(p^{\frac{1}{h}} - 1\right)\left(p^{1+\frac{1}{h}} + p^{\frac{1}{h}} - p\right)}. \tag{20}$$

Corollary 3 is deduced from an estimate for the first moment of $\Omega(n)$ over $h$-full numbers that was computed in [8, Theorem 2]. It would be interesting to obtain an analogous result for $\omega(n)$. To do this, we would need to use different techniques than the ones employed in the proof of [8, Theorem 2], which rely in the total multiplicativity of $\Omega(n)$. See [9, Section 6] for a discussion of this issue in the function field case.

This article is organized as follows. Section 2 includes the proof of Theorem 1 and Corollary 1 by elementary counting, as well as a corollary considering the sum going over $h$-powers. Theorem 2 is proven in Section 3. This is achieved by counting first the $h$-free integers that are coprime to certain fixed number. Corollary 2 is obtained as a consequence of known results for the count over all $h$-free numbers. Finally, Section 4 contains a proof of Theorem 3, which follows from counting integers that are simultaneously $h$-free and $k$-full, while Corollary 3 is obtained as a consequence of known results for the count over all $h$-full numbers.

## 2. Sums over Integers

In this section we prove Theorem 1. We start by recalling the following results involving sums of primes.

**Lemma 1.** *[1, Lemma 1.2] If $s > 1$,*

$$\sum_{p\ge x} \frac{1}{p^s} = \frac{1}{(s-1)x^{s-1}\log x} + O\left(\frac{1}{x^{s-1}\log^2 x}\right).$$

**Lemma 2.** *[1, Lemma 1.4] If* $r, s \geq 0$,

$$\sum_{p \leq x} \frac{p^s}{\log^r p} = \frac{x^{s+1}}{(s+1)\log^{r+1} x} + O\left(\frac{x^{s+1}}{\log^{r+2} x}\right).$$

*Proof of Theorem 1.* We consider Equation (2). Notice that summing over all the numbers of the form $\omega(U_k(n))$ is equivalent to counting the number of powers $p^\ell \leq x$ such that $\ell \geq k$, and each power must be counted with multiplicity equal to the number of $n \leq x$ such that $p^\ell \mid n$. But this is equivalent to counting the multiples of $p^k$ that are less than or equal to $x$. In other words, we have

$$\sum_{n \leq x} \omega(U_k(n)) = \sum_{p^k \leq x} \left\lfloor \frac{x}{p^k} \right\rfloor = \sum_{p \leq x^{\frac{1}{k}}} \frac{x}{p^k} - \sum_{p \leq x^{\frac{1}{k}}} \left\{ \frac{x}{p^k} \right\}.$$

Applying the Prime Number Theorem as well as Lemma 1, we have

$$\begin{aligned}\sum_{n \leq x} \omega(U_k(n)) =& x \sum_p \frac{1}{p^k} - x \sum_{p > x^{\frac{1}{k}}} \frac{1}{p^k} + O_k\left(\frac{x^{\frac{1}{k}}}{\log x}\right) \\ =& x \sum_p \frac{1}{p^k} + O_k\left(\frac{x^{\frac{1}{k}}}{\log x}\right).\end{aligned}$$

Equation (3) is proven similarly. Summing over all the numbers of the form $\Omega(U_k(n))$ is equivalent to counting the number of powers $p^\ell \leq x$ such that $\ell \geq k$, and each power must be counted with multiplicity equal to the number of $n \leq x$ such that $p^\ell \mid n$ but $p^{\ell+1} \nmid n$, multiplied by $\ell$. Set $t = \lfloor \log_p x \rfloor$. We have

$$\begin{aligned}\sum_{n \leq x} \Omega(U_k(n)) &= \sum_{p^k \leq x} \sum_{\ell=k}^{t} \ell \left( \left\lfloor \frac{x}{p^\ell} \right\rfloor - \left\lfloor \frac{x}{p^{\ell+1}} \right\rfloor \right) \\ &= \sum_{p^k \leq x} \left( k \left\lfloor \frac{x}{p^k} \right\rfloor + \left\lfloor \frac{x}{p^{k+1}} \right\rfloor + \cdots + \left\lfloor \frac{x}{p^t} \right\rfloor \right) \\ &= x \sum_{p^k \leq x} \left( \frac{k}{p^k} + \frac{1}{p^{k+1}} + \cdots + \frac{1}{p^t} \right) \\ &\quad - \sum_{p^k \leq x} \left( k \left\{ \frac{x}{p^k} \right\} + \left\{ \frac{x}{p^{k+1}} \right\} + \cdots + \left\{ \frac{x}{p^t} \right\} \right) \\ &= x \sum_{p^k \leq x} \left( \frac{\frac{1}{p^{k+1}} - \frac{1}{p^{t+1}}}{1 - \frac{1}{p}} + \frac{k}{p^k} \right) + O\left( \sum_{p \leq x^{\frac{1}{k}}} t \right) \\ &= x \sum_{p^k \leq x} \frac{\frac{1-k}{p^{k+1}} - \frac{1}{p^{t+1}} + \frac{k}{p^k}}{1 - \frac{1}{p}} + O\left( \log x \sum_{p \leq x^{\frac{1}{k}}} \frac{1}{\log p} \right).\end{aligned}$$

Now we use the Prime Number Theorem to estimate

$$x\sum_{p^k\le x}\frac{1}{p^{t+1}(1-\frac{1}{p})}\ll x\sum_{p\le x^{\frac{1}{k}}}\frac{1}{x}\ll_k\frac{x^{\frac{1}{k}}}{\log x}.$$

By applying the above estimate as well as Lemmas 1 and 2 (with $r=1, s=0$), we obtain

$$\begin{aligned}\sum_{n\le x}\Omega(U_k(n)) =& x\sum_{p}\frac{\frac{1-k}{p^{k+1}}+\frac{k}{p^k}}{1-\frac{1}{p}}-x\sum_{p>x^{\frac{1}{k}}}\frac{\frac{1-k}{p^{k+1}}+\frac{k}{p^k}}{1-\frac{1}{p}}+O_k\left(\frac{x^{\frac{1}{k}}}{\log x}\right)\\ =& x\sum_{p}\frac{1-k+kp}{p^{k+1}-p^k}+O_k\left(\frac{x^{\frac{1}{k}}}{\log x}\right).\end{aligned}$$

This concludes the proof of Theorem 1. □

*Proof of Corollary 1.* To prove Equations (4) and (6) we use the well-known identities [5, Theorem 430] and [4, Section 1.4.4]) for $x\ge 2$:

$$\sum_{n\le x}\omega(n) = x\log\log x + B_1x + O\left(\frac{x}{\log x}\right), \tag{21}$$

$$\sum_{n\le x}\Omega(n) = x\log\log x + B_2x + O\left(\frac{x}{\log x}\right), \tag{22}$$

where $B_1$ and $B_2$ are given by Equations (5) and (7) respectively.

Notice that $\Omega(n)=\Omega(L_k(n))+\Omega(U_k(n))$ and, since $L_k(n)$ and $U_k(n)$ are coprime, $\omega(n)=\omega(L_k(n))+\omega(U_k(n))$ as well. Combining Equations (2) and (3) with Equations (21) and (22), we get Equations (4) and (6).

□

A perfect power is a number of the form $n^h$, where $h\ge 2$ and $n$ are positive integers. We can immediately deduce the following result from Theorem 1.

**Corollary 4.** *Let $k\ge 2$ be an integer. The following formulas hold:*

$$\begin{aligned}\sum_{n^h\le x}\Omega(U_k(n^h)) =& h\left(\sum_{p}\frac{1-k+kp}{p^{k+1}-p^k}\right)x^{\frac{1}{h}}+O_{k,h}\left(\frac{x^{\frac{1}{hk}}}{\log x}\right),\\ \sum_{n^h\le x}\omega(U_k(n^h)) =& \left(\sum_{p}\frac{1}{p^k}\right)x^{\frac{1}{h}}+O_{k,h}\left(\frac{x^{\frac{1}{hk}}}{\log x}\right).\end{aligned}$$

*In addition, the following formulas hold:*

$$\sum_{n^h\le x}\Omega(L_k(n^h)) = hx^{\frac{1}{h}}\log\log x + h\left(B_2 - \log h - \sum_p \frac{1-k+kp}{p^{k+1}-p^k}\right)x^{\frac{1}{h}} + O_h\left(\frac{x^{\frac{1}{h}}}{\log x}\right),$$

$$\sum_{n^h\le x}\omega(L_k(n^h)) = x^{\frac{1}{h}}\log\log x + \left(B_1 - \log h - \sum_p \frac{1}{p^k}\right)x^{\frac{1}{h}} + O_h\left(\frac{x^{\frac{1}{h}}}{\log x}\right).$$

Let $\omega_k(n)$ be the number of primes with exponent $k$ in the prime factorization of $n$.

**Corollary 5.** *Let $k \ge 1$ be an integer. We have the asymptotic formula*

$$\sum_{n\le x}\omega_k(n) = \left(\sum_p \frac{p-1}{p^{k+1}}\right)x + O_k\left(\frac{x^{\frac{1}{k}}}{\log x}\right).$$

This recovers a result of Elma and Liu [3], who also studied the second moment of $\omega_k$.

*Proof.* By Equation (2), we have

$$\sum_{n\le x}\omega_k(n) = \sum_{n\le x}\omega(U_k(n)) - \omega(U_{k+1}(n)) = x\left(\sum_p \frac{1}{p^k} - \sum_p \frac{1}{p^{k+1}}\right) + O_k\left(\frac{x^{\frac{1}{k}}}{\log x}\right),$$

and the result follows. □

**Remark 1.** It is interesting to consider the quotient of the sums appearing in Equations (2) and (3). We get

$$\frac{\sum_{n\le x}\Omega(U_k(n))}{\sum_{n\le x}\omega(U_k(n))} \to \frac{\sum_p \frac{1-k+kp}{p^{k+1}-p^k}}{\sum_p \frac{1}{p^k}}. \tag{23}$$

Since we have that

$$\frac{k}{p^k} = \frac{k(p-1)}{p^k(p-1)} < \frac{kp-(k-1)}{p^k(p-1)} \le \frac{(k+1)(p-1)}{p^k(p-1)} = \frac{k+1}{p^k},$$

and the second inequality is strict for $p > 2$, we conclude that the limit (23) belongs to the interval $(k, k+1)$.

**Remark 2.** The constants appearing in Equations (2) and (3) can also be expressed as

$$\sum_p \frac{1}{p^k} = \frac{1}{\zeta(k)}\sum_{U\in\mathcal{N}_k}\prod_{q|U}\left(\frac{q^k-q^{k-1}}{q^k-1}\right)\frac{\omega(U)}{U} \tag{24}$$

and
$$\sum_p \frac{1-k+kp}{p^{k+1}-p^k} = \frac{1}{\zeta(k)} \sum_{U\in\mathcal{N}_k} \prod_{p|n} \left(\frac{q^k-q^{k-1}}{q^k-1}\right) \frac{\Omega(U)}{U}. \tag{25}$$

This can be seen by working with the generating functions, in a method that will be employed to find the constants in Theorems 2 and 3. In fact, Equations (24) and (25) can be obtained from $D_{\omega,k,h}$ and Equation (30) as well as $D_{\Omega,k,h}$ and Equation (29) by letting $h \to \infty$ and therefore removing the condition $h$-free.

## 3. Sums over $h$-Free Numbers

In this section we prove Theorem 2. We start with the following estimate for the number of $k$-free positive integers that are not divisible by some fixed primes.

**Lemma 3.** *Let $q_1, \ldots, q_r$ be prime numbers, and let $\mathfrak{Q}_{k,q_1\cdots q_r}(x)$ be the number of $k$-free positive integers not exceeding $x$ such that they are relatively prime to $q_1 \cdots q_r$. The following formula holds:*

$$\mathfrak{Q}_{k,q_1\cdots q_r}(x) = \frac{1}{\zeta(k)} \prod_{j=1}^{r} \frac{\left(1-\frac{1}{q_j}\right)}{\left(1-\frac{1}{q_j^k}\right)} x + O_k\left(2^r x^{\frac{1}{k}}\right).$$

We remark that the above formula generalizes the classical estimate giving

$$Q_k(x) = \frac{x}{\zeta(k)} + O\left(x^{\frac{1}{k}}\right),$$

where $Q_k(x)$ is the number of $k$-free numbers not exceeding $x$.

*Proof.* Consider the modified Möbius function defined as

$$\mu_{q_1\cdots q_r}(d) = \begin{cases} \mu(d) & (d, q_1\cdots q_r) = 1, \\ 0 & \text{otherwise.} \end{cases}.$$

By Möbius inversion, we have

$$\begin{aligned} \mathfrak{Q}_{k,q_1\cdots q_r}(x) &= \sum_{\substack{n\in\mathcal{S}_k \\ n\le x \\ (n,q_1\cdots q_r)=1}} 1 = \sum_{\substack{n\le x \\ (n,q_1\cdots q_r)=1}} \sum_{\substack{d^k|n \\ (d,q_1\cdots q_r)=1}} \mu(d) \\ &= \sum_{\substack{n\le x \\ (n,q_1\cdots q_r)=1}} \sum_{d^k|n} \mu_{q_1\cdots q_r}(d). \end{aligned}$$

Writing $n = d^k e$, we have

$$\mathfrak{Q}_{k,q_1\cdots q_r}(x) = \sum_{d^k \le x} \mu_{q_1\cdots q_r}(d) \sum_{\substack{e \le x/d^k \\ (e, q_1\cdots q_r)=1}} 1.$$

Estimating the inner sum with inclusion-exclusion, we obtain

$$\begin{aligned}
\mathfrak{Q}_{k,q_1\cdots q_r}(x) &= \sum_{d^k \le x} \mu_{q_1\cdots q_r}(d) \left( \left\lfloor \frac{x}{d^k} \right\rfloor - \left\lfloor \frac{x}{q_i d^k} \right\rfloor + \left\lfloor \frac{x}{q_i q_j d^k} \right\rfloor + \cdots \right) \\
&= \sum_{d^k \le x} \mu_{q_1\cdots q_r}(d) \frac{x}{d^k} \prod_{j=1}^{r} \left(1 - \frac{1}{q_j}\right) \\
&\quad + O\left( \sum_{d^k \le x} \mu_{q_1\cdots q_r}(d) \left( \left\{ \frac{x}{d^k} \right\} - \left\{ \frac{x}{q_i d^k} \right\} + \left\{ \frac{x}{q_i q_j d^k} \right\} + \cdots \right) \right) \\
&= \sum_{d^k \le x} \mu_{q_1\cdots q_r}(d) \frac{x}{d^k} \prod_{j=1}^{r} \left(1 - \frac{1}{q_j}\right) + O\left( 2^r \sum_{d^k \le x} 1 \right).
\end{aligned}$$

After using the full sum to estimate, the above becomes,

$$\begin{aligned}
&\sum_{d} \mu_{q_1\cdots q_r}(d) \frac{x}{d^k} \prod_{j=1}^{r} \left(1 - \frac{1}{q_j}\right) - \sum_{d^k > x} \mu_{q_1\cdots q_r}(d) \frac{x}{d^k} \prod_{j=1}^{r} \left(1 - \frac{1}{q_j}\right) + O\left(2^r x^{\frac{1}{k}}\right) \\
=& x \prod_{p \ne q_j} \left(1 - \frac{1}{p^k}\right) \prod_{j=1}^{r} \left(1 - \frac{1}{q_j}\right) + O\left( x \sum_{d^k > x} \frac{1}{d^k} \right) + O\left(2^r x^{\frac{1}{k}}\right).
\end{aligned}$$

Estimating the first big-$O$ term by approximating with an integral, we obtain $O_k(x^{\frac{1}{k}})$, and this yields

$$\begin{aligned}
\mathfrak{Q}_{k,q_1\cdots q_r}(x) =& x \prod_{p} \left(1 - \frac{1}{p^k}\right) \prod_{j=1}^{r} \frac{\left(1 - \frac{1}{q_j}\right)}{\left(1 - \frac{1}{q_j^k}\right)} + O_k\left(2^r x^{\frac{1}{k}}\right) \\
=& \frac{1}{\zeta(k)} \prod_{j=1}^{r} \frac{\left(1 - \frac{1}{q_j}\right)}{\left(1 - \frac{1}{q_j^k}\right)} x + O_k\left(2^r x^{\frac{1}{k}}\right).
\end{aligned}$$

□

We now state some results involving sums of prime factor counting functions over $h$-full numbers that will be needed for the proofs of Theorem 2 and Corollary 2.

**Theorem 4.** *Let $h \geq 1$ be an integer. We have*

$$\sum_{\substack{n \in \mathcal{N}_h \\ n \leq x}} \Omega(n) = h\gamma_{0,h} x^{\frac{1}{h}} \log\log x + \gamma_{0,h} C_{\Omega,h} x^{\frac{1}{h}} + O_h\left(\frac{x^{\frac{1}{h}}}{\sqrt{\log x}}\right), \tag{26}$$

*where $\gamma_{0,h}$ is given by Equation (16) and $C_{\Omega,h}$ is given by Equation (20).*

We omit the proof, since Equation (26) was proven in [8, Theorem 2].

**Lemma 4.** *Let $\alpha \in \mathbb{R}$. Then, we have*

$$\sum_{\substack{n \in \mathcal{N}_h \\ x < n \leq y}} \Omega(n) n^{\alpha} = O_h\left(y^{\frac{1}{h}+\alpha} \log\log y\right) + O_h\left(x^{\frac{1}{h}+\alpha} \log\log x\right).$$

*and*

$$\sum_{\substack{n \in \mathcal{N}_h \\ x < n \leq y}} \omega(n) n^{\alpha} = O_h\left(y^{\frac{1}{h}+\alpha} \log\log y\right) + O_h\left(x^{\frac{1}{h}+\alpha} \log\log x\right).$$

*Proof.* Denote

$$\mathscr{N}_h(x) = \sum_{\substack{n \in \mathcal{N}_h \\ n \leq x}} \Omega(n),$$

and remark that the asymptotics for $\mathscr{N}_h(x)$ is given by Equation (26).

By Abel's summation formula,

$$\begin{aligned}
\sum_{\substack{n \in \mathcal{N}_h \\ x < n \leq y}} \Omega(n) n^{\alpha} =& \mathscr{N}_h(y) y^{\alpha} - \mathscr{N}_h(x) x^{\alpha} - \alpha \int_x^y \mathscr{N}_h(t) t^{\alpha-1} dt \\
=& h\gamma_{0,h} y^{\frac{1}{h}+\alpha} \log\log y - h\gamma_{0,h} x^{\frac{1}{h}+\alpha} \log\log x \\
&+ O\left(y^{\frac{1}{h}+\alpha}\right) + O\left(x^{\frac{1}{h}+\alpha}\right) + O\left(\int_x^y t^{\alpha - \frac{h-1}{h}} \log\log t dt\right) \\
=& O_h\left(y^{\frac{1}{h}+\alpha} \log\log y\right) + O_h\left(x^{\frac{1}{h}+\alpha} \log\log x\right).
\end{aligned}$$

The estimate for the sum over $\omega(n)$ can be deduced from the fact that $\omega(n) \leq \Omega(n)$. □

*Proof of Theorem 2.* We prove Equations (8) and (10). Fix $0 < B \leq x$ (to be determined later) and suppose that $U = U_k(n)$ is such that $U \leq B$. We start by counting all the possible values of $L = L_k(n)$ satisfying $L \leq x/U$. By Lemma 3, the number of possible values of $L$ is given by

$$\mathfrak{Q}_{k,q_1\cdots q_r}\left(\frac{x}{U}\right) = \frac{1}{\zeta(k)} \prod_{j=1}^{r} \left(\frac{q_j^k - q_j^{k-1}}{q_j^k - 1}\right) \frac{x}{U} + O\left(2^r \frac{x^{\frac{1}{k}}}{U^{\frac{1}{k}}}\right),$$

where $q_1, \ldots, q_r$ are the primes in the factorization of $U$. Thus we have

$$\begin{aligned}\sum_{\substack{n\in\mathcal{S}_h\\ n\le x}} \Omega(U_k(n)) =& \sum_{\substack{n\in\mathcal{S}_h\\ n\le x\\ U_k(n)\le B}} \Omega(U_k(n)) + \sum_{\substack{n\in\mathcal{S}_h\\ n\le x\\ B<U_k(n)\le x}} \Omega(U_k(n))\\ =&\frac{x}{\zeta(k)} \sum_{\substack{U\in\mathcal{N}_k\cap\mathcal{S}_h\\ U\le B}} \prod_{q|U} \left(\frac{q^k-q^{k-1}}{q^k-1}\right) \frac{\Omega(U)}{U}\\ &+ O\left(\sum_{\substack{U\in\mathcal{N}_k\cap\mathcal{S}_h\\ U\le B}} \Omega(U) 2^{\omega(U)} \frac{x^{\frac{1}{k}}}{U^{\frac{1}{k}}}\right) + \sum_{\substack{n\in\mathcal{S}_h\\ n\le x\\ B<U_k(n)\le x}} \Omega(U_k(n)).\end{aligned}$$

Notice that for $U \in \mathcal{N}_k$, we have $2^{\omega(U)} \le q_1 \cdots q_r \le U^{\frac{1}{k}}$. Using this to bound the error term gives

$$\begin{aligned}\sum_{\substack{n\in\mathcal{S}_h\\ n\le x}} \Omega(U_k(n)) =&\frac{x}{\zeta(k)} \sum_{U\in\mathcal{N}_k\cap\mathcal{S}_h} \prod_{q|U} \left(\frac{q^k-q^{k-1}}{q^k-1}\right) \frac{\Omega(U)}{U} + O\left(x^{\frac{1}{k}} \sum_{\substack{U\in\mathcal{N}_k\cap\mathcal{S}_h\\ U\le B}} \Omega(U)\right)\\ &+ \sum_{\substack{n\in\mathcal{S}_h\\ n\le x\\ B<U_k(n)\le x}} \Omega(U_k(n)) - \frac{x}{\zeta(k)} \sum_{\substack{U\in\mathcal{N}_k\cap\mathcal{S}_h\\ B<U}} \prod_{q|U} \left(\frac{q^k-q^{k-1}}{q^k-1}\right) \frac{\Omega(U)}{U}.\end{aligned} \tag{27}$$

We have the following estimate

$$\sum_{\substack{n\in\mathcal{S}_h\\ n\le x\\ B<U_k(n)\le x}} \Omega(U_k(n)) \le \sum_{\substack{U\in\mathcal{N}_k\cap\mathcal{S}_h\\ B<U\le x}} \left\lfloor\frac{x}{U}\right\rfloor \Omega(U) \le \sum_{\substack{U\in\mathcal{N}_k\\ U\le x}} \frac{x}{U}\Omega(U). \tag{28}$$

Applying Lemma 4 to Equations (27) and (28), we have

$$\begin{aligned}\sum_{\substack{n\in\mathcal{S}_h\\ n\le x}} \Omega(U_k(n)) =&\frac{x}{\zeta(k)} \sum_{U\in\mathcal{N}_k\cap\mathcal{S}_h} \prod_{q|U} \left(\frac{q^k-q^{k-1}}{q^k-1}\right) \frac{\Omega(U)}{U} + O_h\left(x^{\frac{1}{k}} B^{\frac{1}{k}} \log\log B\right)\\ &+ O_h\left(x^{\frac{1}{k}} \log\log x\right) + O_h\left(x B^{\frac{1}{k}-1} \log\log B\right).\end{aligned}$$

Let $B = x^{1-\frac{1}{k}}$. We get

$$\sum_{\substack{n\in\mathcal{S}_h\\ n\le x}} \Omega(U_k(n)) =\frac{x}{\zeta(k)} \sum_{U\in\mathcal{N}_k\cap\mathcal{S}_h} \prod_{q|U} \left(\frac{q^k-q^{k-1}}{q^k-1}\right) \frac{\Omega(U)}{U} + O_h\left(x^{\frac{2k-1}{k^2}} \log\log x\right).$$

We now proceed to find a closed expression for

$$\frac{1}{\zeta(k)} \sum_{U \in \mathcal{N}_k \cap \mathcal{S}_h} \prod_{q|U} \left( \frac{q^k - q^{k-1}}{q^k - 1} \right) \frac{\Omega(U)}{U}. \tag{29}$$

We consider a generating function given by

$$\begin{aligned}
\mathcal{D}_{\Omega,k,h}(z) = & \sum_{n \in \mathcal{N}_k \cap \mathcal{S}_h} \frac{z^{\Omega(n)}}{n} \prod_{q|n} \frac{q^k - q^{k-1}}{q^k - 1} \\
= & \prod_p \left( 1 + \left( \frac{p^k - p^{k-1}}{p^k - 1} \right) \frac{z^k}{p^k} \left( 1 + \frac{z}{p} + \cdots + \frac{z^{h-k-1}}{p^{h-k-1}} \right) \right) \\
= & \prod_p \left( 1 + \left( \frac{p^k - p^{k-1}}{p^k - 1} \right) \frac{\frac{z^h}{p^h} - \frac{z^k}{p^k}}{\frac{z}{p} - 1} \right),
\end{aligned}$$

which is absolutely convergent over compact sets.

We will recover our term of interest from considering $\mathcal{D}'_{\Omega,k,h}(1)$. In order to find this term, we consider the logarithmic derivative of $\mathcal{D}_{\Omega,k,h}(z)$:

$$\frac{\mathcal{D}'_{\Omega,k,h}(z)}{\mathcal{D}_{\Omega,k,h}(z)} = \sum_p \frac{\left( \frac{p^k - p^{k-1}}{p^k - 1} \right) \left( (h-1) \frac{z^h}{p^{h+1}} - (k-1) \frac{z^k}{p^{k+1}} - h \frac{z^{h-1}}{p^h} + k \frac{z^{k-1}}{p^k} \right)}{\left( \frac{z}{p} - 1 \right)^2 \left( 1 + \left( \frac{p^k - p^{k-1}}{p^k - 1} \right) \frac{\frac{z^h}{p^h} - \frac{z^k}{p^k}}{\frac{z}{p} - 1} \right)}.$$

Evaluating at $z = 1$, we obtain,

$$\left. \frac{\mathcal{D}'_{\Omega,k,h}(z)}{\mathcal{D}_{\Omega,k,h}(z)} \right|_{z=1} = \sum_p \frac{\left( \frac{p^k}{p^k - 1} \right) \left( \frac{h-1}{p^{h+1}} - \frac{k-1}{p^{k+1}} - \frac{h}{p^h} + \frac{k}{p^k} \right)}{\left( 1 - \frac{1}{p} \right) \left( 1 - \frac{p^k}{p^k - 1} \left( \frac{1}{p^h} - \frac{1}{p^k} \right) \right)}.$$

Multiplying the above by $\mathcal{D}_{\Omega,k,h}(1)$ and by the coefficient $\frac{1}{\zeta(k)} = \prod_p \left( 1 - \frac{1}{p^k} \right)$ provides the coefficient for the main term of (8):

$$\begin{aligned}
\frac{\mathcal{D}'_{\Omega,k,h}(1)}{\zeta(k)} = & \frac{1}{\zeta(k)} \sum_p \frac{\left( \frac{p^k}{p^k - 1} \right) \left( \frac{h-1}{p^{h+1}} - \frac{k-1}{p^{k+1}} - \frac{h}{p^h} + \frac{k}{p^k} \right)}{\left( 1 - \frac{1}{p} \right) \left( 1 - \frac{p^k}{p^k - 1} \left( \frac{1}{p^h} - \frac{1}{p^k} \right) \right)} \\
& \times \prod_p \left( 1 - \left( \frac{p^k}{p^k - 1} \right) \left( \frac{1}{p^h} - \frac{1}{p^k} \right) \right) \\
= & \sum_p \frac{\frac{h-1}{p^{h+1}} - \frac{k-1}{p^{k+1}} - \frac{h}{p^h} + \frac{k}{p^k}}{\left( 1 - \frac{1}{p} \right) \left( 1 - \frac{1}{p^h} \right)} \prod_p \left( 1 - \frac{1}{p^h} \right) \\
= & \frac{1}{\zeta(h)} \sum_p \frac{h - 1 - (k-1)p^{h-k} - hp + kp^{h-k+1}}{(p-1)(p^h - 1)}.
\end{aligned}$$

Equations (9) and (11) are proven analogously. Here the difference is that we must consider instead

$$\frac{1}{\zeta(k)}\sum_{U\in\mathcal{N}_k\cap\mathcal{S}_h}\prod_{q|U}\left(\frac{q^k-q^{k-1}}{q^k-1}\right)\frac{\omega(U)}{U}, \tag{30}$$

while the error term can be bounded as in the $\Omega$ case, using the fact that $\omega(n)\leq\Omega(n)$.

In this case the generating function is given by

$$\begin{aligned}\mathcal{D}_{\omega,k,h}(z)=&\sum_{n\in\mathcal{N}_k\cap\mathcal{S}_h}\frac{z^{\omega(n)}}{n}\prod_{q|n}\frac{q^k-q^{k-1}}{q^k-1}\\ =&\prod_p\left(1+\left(\frac{p^k-p^{k-1}}{p^k-1}\right)\frac{z}{p^k}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{h-k-1}}\right)\right)\\ =&\prod_p\left(1+\left(\frac{p^k-p^{k-1}}{p^k-1}\right)\frac{z\left(\frac{1}{p^h}-\frac{1}{p^k}\right)}{\frac{1}{p}-1}\right),\end{aligned}$$

which is absolutely convergent.

In order to find $\mathcal{D}'_{\omega,k,h}(1)$, we consider the logarithmic derivative:

$$\frac{\mathcal{D}'_{\omega,k,h}(z)}{\mathcal{D}_{\omega,k,h}(z)}=\sum_p\frac{\left(\frac{p^k-p^{k-1}}{p^k-1}\right)\frac{\left(\frac{1}{p^h}-\frac{1}{p^k}\right)}{\frac{1}{p}-1}}{1+\left(\frac{p^k-p^{k-1}}{p^k-1}\right)\frac{z\left(\frac{1}{p^h}-\frac{1}{p^k}\right)}{\frac{1}{p}-1}}.$$

Therefore,

$$\left.\frac{\mathcal{D}'_{\omega,k,h}(z)}{\mathcal{D}_{\omega,k,h}(z)}\right|_{z=1}=\sum_p\frac{p^{h-k}-1}{p^h-1}.$$

Multiplying the above by $\mathcal{D}_{\omega,k,h}(1)$ and by the coefficient $\frac{1}{\zeta(k)}=\prod_p\left(1-\frac{1}{p^k}\right)$ yields the coefficient for the main term of Equation (9):

$$\begin{aligned}\frac{\mathcal{D}'_{\omega,k,h}(1)}{\zeta(k)}=&\frac{1}{\zeta(k)}\sum_p\frac{p^{h-k}-1}{p^h-1}\prod_p\left(1-\left(\frac{p^k}{p^k-1}\right)\left(\frac{1}{p^h}-\frac{1}{p^k}\right)\right)\\ =&\sum_p\frac{p^{h-k}-1}{p^h-1}\prod_p\left(1-\frac{1}{p^h}\right)\\ =&\frac{1}{\zeta(h)}\sum_p\frac{p^{h-k}-1}{p^h-1}.\end{aligned}$$

This concludes the proof of Theorem 2. □

**Theorem 5.** *The following asymptotic formulas hold:*

$$\sum_{\substack{n\in\mathcal{S}_h\\ n\le x}} \Omega(n) = \frac{1}{\zeta(h)} x\log\log x + O(x), \tag{31}$$

*and*

$$\sum_{\substack{n\in\mathcal{S}_h\\ n\le x}} \omega(n) = \frac{1}{\zeta(h)} x\log\log x + O(x). \tag{32}$$

We omit the proof, since Equation (31) was proven in [8, Theorem 1] and Equation (32) can be proven similarly.

*Proof of Corollary 2.* Since $n = L_k(n)U_k(n)$, we have $\Omega(n) = \Omega(L_k(n)) + \Omega(U_k(n))$, and similarly with $\omega$ (since $L_k(n)$ and $U_k(n)$ are coprime). Combining Equations (8) and (31), we immediately obtain Equation (12). Equation (13) follows by combining Equations (9) and (32). □

## 4. Sums over $h$-Full Numbers

In this section we prove Theorem 3. Before proceeding to the proof, we need the following generalization of Lemma 3.

**Lemma 5.** *Let $q_1, \ldots, q_r$ be prime numbers and let $k > h$ be integers. We define $\mathfrak{Q}_{k,h,q_1\cdots q_r}(x)$ as the number of $k$-free, $h$-full positive integers not exceeding $x$ such that they are relatively prime to $q_1 \cdots q_r$. The following formula holds:*

$$\mathfrak{Q}_{k,h,q_1\cdots q_r}(x) = \prod_{j=1}^{r}\left(1 + \frac{\frac{1}{q_j} - \frac{1}{q_j^{\frac{k}{h}}}}{1 - \frac{1}{q_j^{\frac{1}{h}}}}\right)^{-1} \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \frac{\frac{1}{p} - \frac{1}{p^{\frac{k}{h}}}}{1 - \frac{1}{p^{\frac{1}{h}}}}\right) x^{\frac{1}{h}}$$
$$+ O\left(2^r x^{\frac{2h+1}{2h(h+1)}+\varepsilon}\right),$$

*where $\varepsilon > 0$ is arbitrarily small.*

*Proof.* Consider the generating function

$$\sum_{\substack{n\in\mathcal{N}_h\cap\mathcal{S}_k\\(n,q_1\cdots q_r)=1}}\frac{1}{n^s}=\prod_{p\neq q_j}\left(1+\frac{1}{p^{sh}}+\cdots+\frac{1}{p^{s(k-1)}}\right)$$

$$=\prod_{p\neq q_j}\left(1+\frac{\frac{1}{p^{sh}}-\frac{1}{p^{sk}}}{1-\frac{1}{p^s}}\right)$$

$$=\prod_{j=1}^{r}\left(1+\frac{1}{q_j^{sh}}\right)^{-1}\prod_{p}\left(1+\frac{1}{p^{sh}}\right)\prod_{p\neq q_j}\left(1+\frac{\frac{1}{p^{s(h+1)}}-\frac{1}{p^{sk}}}{\left(1-\frac{1}{p^s}\right)\left(1+\frac{1}{p^{sh}}\right)}\right)$$

$$=\prod_{j=1}^{r}\left(1+\frac{1}{q_j^{sh}}\right)^{-1}\frac{\zeta(sh)}{\zeta(2sh)}\mathcal{H}_{q_1\cdots q_r}(s).$$

Notice that for $\mathrm{Re}(s)>\frac{1}{h+1}$,

$$|\mathcal{H}_{q_1\cdots q_r}(s)|\leq\prod_{p\neq q_j}\left(1+\left|\frac{\frac{1}{p^{s(h+1)}}-\frac{1}{p^{sk}}}{\left(1-\frac{1}{p^s}\right)\left(1+\frac{1}{p^{sh}}\right)}\right|\right)$$

$$\leq\prod_{p}\left(1+\left|\frac{\frac{1}{p^{s(h+1)}}-\frac{1}{p^{sk}}}{\left(1-\frac{1}{p^s}\right)\left(1+\frac{1}{p^{sh}}\right)}\right|\right), \tag{33}$$

which is convergent for $\mathrm{Re}(s)\geq\frac{1}{h+1}+\varepsilon$, and therefore $\mathcal{H}_{q_1\cdots q_r}(s)$ is convergent for $\mathrm{Re}(s)>\frac{1}{h+1}$. Now we use Perron's formula ([10, Section 5.1], [11, Section 4.4], more precisely, Problems 4.4.15-4.4.17). Take $\sigma_0=\frac{1}{h}+\varepsilon$. As $T\to\infty$,

$$\mathfrak{Q}_{k,h,q_1\cdots q_r}(x)=\sum_{\substack{n\in\mathcal{N}_h\cap\mathcal{S}_k\\n\leq x\\(n,q_1\cdots q_r)=1}}1$$

$$=\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\prod_{j=1}^{r}\left(1+\frac{1}{q_j^{sh}}\right)^{-1}\frac{\zeta(sh)}{\zeta(2sh)}\mathcal{H}_{q_1\cdots q_r}(s)\frac{x^s}{s}ds$$

$$+O\left(\frac{x^{\sigma_0+\varepsilon}}{T}\right).$$

To compute this integral we consider the rectangle of vertical sides $[\sigma_0-iT,\sigma_0+iT]$ and $[\sigma_1-iT,\sigma_1+iT]$ and horizontal sides $[\sigma_0\pm iT,\sigma_1\pm iT]$. The integral over the sides is equal to the residue from the pole at $s=\frac{1}{h}$, which can be computed as follows:

$$\prod_{j=1}^{r}\left(1+\frac{1}{q_j}\right)^{-1}\frac{h}{\zeta(2)}\mathcal{H}_{q_1\cdots q_r}\left(\frac{1}{h}\right)x^{\frac{1}{h}}\mathrm{Res}_{s=\frac{1}{h}}\zeta(sh)$$
$$=\prod_{j=1}^{r}\left(1+\frac{\frac{1}{q_j}-\frac{1}{q_j^{\frac{k}{h}}}}{1-\frac{1}{q_j^{\frac{1}{h}}}}\right)^{-1}\frac{1}{\zeta(2)}\prod_{p}\left(1+\frac{\frac{1}{p^{\frac{1}{h}+1}}-\frac{1}{p^{\frac{k}{h}}}}{\left(1-\frac{1}{p^{\frac{1}{h}}}\right)\left(1+\frac{1}{p}\right)}\right)x^{\frac{1}{h}}.$$

Since we are interested in the integral over the segment $[\sigma_0-iT,\sigma_0+iT]$, we proceed to bound the integral at the vertical segment $[\sigma_1-iT,\sigma_1+iT]$ and at the horizontal lines $[\sigma_0\pm iT,\sigma_1\pm iT]$. First we note that Inequality (33) gives a uniform bound for $\mathcal{H}_{q_1\cdots q_r}(s)$ which is independent of the choice of $q_1,\ldots,q_r$. Next notice that we have, over the same segments,

$$\left|1+\frac{1}{q^{sh}}\right|^{-1}\le\frac{1}{1-\frac{1}{q^{\mathrm{Re}(s)h}}}\le\frac{1}{1-\frac{1}{q^{\frac{h}{h+1}}}}\le\frac{1}{1-\frac{1}{q^{\frac{1}{2}}}},$$

and the above bound is less than or equal to 2 when $q\neq 2,3$, and for $q=2,3$ it is bounded by 4 and 3, respectively. Thus, we have the following bound over the vertical segment $[\sigma_1-iT,\sigma_1+iT]$ and at the horizontal lines $[\sigma_0\pm iT,\sigma_1\pm iT]$:

$$\left|\prod_{j=1}^{r}\left(1+\frac{1}{q_j^{sh}}\right)^{-1}\right|<12\cdot 2^r.$$

Since $\zeta(\sigma\pm iT)=O\left(T^{\frac{1}{2}}\right)$ uniformly for $\varepsilon\le\sigma\le 1$ as $T\to\infty$ (see for example, [6, Theorem 1.9]), the horizontal integrals on $[\sigma_0\pm iT,\sigma_1\pm iT]$ contribute $O\left(2^r\frac{x^{\sigma_0}T^{-\frac{1}{2}}}{\log x}\right)$.

The vertical line $[\sigma_1-iT,\sigma_1+iT]$ contributes to $O\left(2^r x^{\sigma_1}T^{\frac{1}{2}}\right)$.

Finally, taking $T=x^{\frac{1}{h(h+1)}}$ gives a final estimate of

$$\mathfrak{Q}_{k,h,q_1\cdots q_r}(x)=\prod_{j=1}^{r}\left(1+\frac{\frac{1}{q_j}-\frac{1}{q_j^{\frac{k}{h}}}}{1-\frac{1}{q_j^{\frac{1}{h}}}}\right)^{-1}\frac{1}{\zeta(2)}\prod_{p}\left(1+\frac{\frac{1}{p^{\frac{1}{h}+1}}-\frac{1}{p^{\frac{k}{h}}}}{\left(1-\frac{1}{p^{\frac{1}{h}}}\right)\left(1+\frac{1}{p}\right)}\right)x^{\frac{1}{h}}$$
$$+O\left(2^r x^{\frac{2h+1}{2h(h+1)}+\varepsilon}\right).$$

□

We remark that the main term in Lemma 5 reduces to the main term in Lemma 3 when $h=1$. However, the error term has size $O\left(2^r x^{\frac{3}{4}+\varepsilon}\right)$ and is worse. The reason for this is that we are we are only considering the pole at $s = \frac{1}{h}$ in Perron's formula. To eliminate the dependence on $h$ we would need to remove all the poles up to $\frac{1}{k}$.

Another interesting case is when $k \to \infty$ and $r = 0$. This counts the $h$-full numbers not exceeding $x$ and recovers the formula

$$\gamma_{0,h} x^{\frac{1}{h}} + O\left(x^{\frac{2h+1}{2h(h+1)}+\varepsilon}\right).$$

This is a much weaker version of the result of Ivić and Shiu [7], who estimate this number to be

$$\gamma_{0,h} x^{\frac{1}{h}} + \gamma_{1,h} x^{\frac{1}{h+1}} + \cdots + \gamma_{h-1,h} x^{\frac{1}{2h-1}} + \Delta_h(x),$$

where $\gamma_{0,h}, \gamma_{1,h} \ldots, \gamma_{h-1,h}$ are certain computable constants and $\Delta_h(x) \ll x^{\rho}$ for $\rho$ small.

*Proof of Theorem 3.* First, we proceed to prove Equations (14) and (17). Fix $0 < B \le x$ (to be determined later) and suppose that $U = U_k(n)$ is such that $U \le B$. We start by counting all the possible $L = L_k(n)$ satisfying $L \le x/U$. Since $L$ must be both $k$-free and $h$-full, Lemma 5 implies that the number of possible values of $L$ is given by

$$\mathfrak{Q}_{k,h,q_1\cdots q_r}\left(\frac{x}{U}\right) = \prod_{j=1}^{r}\left(1 + \frac{\frac{1}{q_j} - \frac{1}{q_j^{\frac{k}{h}}}}{1 - \frac{1}{q_j^{\frac{1}{h}}}}\right)^{-1} \prod_{p}\left(1 - \frac{1}{p}\right)\left(1 + \frac{\frac{1}{p} - \frac{1}{p^{\frac{k}{h}}}}{1 - \frac{1}{p^{\frac{1}{h}}}}\right)\frac{x^{\frac{1}{h}}}{U^{\frac{1}{h}}}$$
$$+ O\left(2^r \frac{x^{\frac{2h+1}{2h(h+1)}+\varepsilon}}{U^{\frac{2h+1}{2h(h+1)}+\varepsilon}}\right),$$

where $q_1, \ldots, q_r$ are the primes in the factorization of $U$.

To make the proof easier to follow, we define

$$f(k,h) := \prod_{p}\left(1 - \frac{1}{p}\right)\left(1 + \frac{\frac{1}{p} - \frac{1}{p^{\frac{k}{h}}}}{1 - \frac{1}{p^{\frac{1}{h}}}}\right).$$

Thus we have

$$\begin{aligned}\sum_{\substack{n\in\mathcal{N}_h\\ n\le x}}\Omega(U_k(n)) = &\sum_{\substack{n\in\mathcal{N}_h\\ n\le x\\ U_k(n)\le B}}\Omega(U_k(n)) + \sum_{\substack{n\in\mathcal{N}_h\\ n\le x\\ B<U_k(n)\le x}}\Omega(U_k(n))\\ =&f(k,h)x^{\frac{1}{h}}\sum_{\substack{U\in\mathcal{N}_k\\ U\le B}}\prod_{q|U}\left(1+\frac{\frac{1}{q}-\frac{1}{q^{\frac{k}{h}}}}{1-\frac{1}{q^{\frac{1}{h}}}}\right)^{-1}\frac{\Omega(U)}{U^{\frac{1}{h}}}\\ &+O\left(\sum_{\substack{U\in\mathcal{N}_k\\ U\le B}}2^{\omega(U)}\Omega(U)\frac{x^{\frac{2h+1}{2h(h+1)}+\varepsilon}}{U^{\frac{2h+1}{2h(h+1)}+\varepsilon}}\right)+\sum_{\substack{n\in\mathcal{N}_h\\ n\le x\\ B<U_k(n)\le x}}\Omega(U_k(n)).\end{aligned}$$

Notice that for $U\in\mathcal{N}_k$, we have $2^{\omega(U)}\le q_1\cdots q_r\le U^{\frac{1}{k}}$. Using this to bound the error term above gives

$$\begin{aligned}\sum_{\substack{n\in\mathcal{N}_h\\ n\le x}}\Omega(U_k(n)) =&f(k,h)x^{\frac{1}{h}}\sum_{U\in\mathcal{N}_k}\prod_{q|U}\left(1+\frac{\frac{1}{q}-\frac{1}{q^{\frac{k}{h}}}}{1-\frac{1}{q^{\frac{1}{h}}}}\right)^{-1}\frac{\Omega(U)}{U^{\frac{1}{h}}}\\ &+O\left(x^{\frac{2h+1}{2h(h+1)}+\varepsilon}\sum_{\substack{U\in\mathcal{N}_k\\ U\le B}}\Omega(U)U^{\frac{1}{k}-\frac{2h+1}{2h(h+1)}-\varepsilon}\right)\\ &+\sum_{\substack{n\in\mathcal{N}_h\\ n\le x\\ B<U_k(n)\le x}}\Omega(U_k(n))\\ &-f(k,h)x^{\frac{1}{h}}\sum_{\substack{U\in\mathcal{N}_k\\ B<U}}\prod_{q|U}\left(1+\frac{\frac{1}{q}-\frac{1}{q^{\frac{k}{h}}}}{1-\frac{1}{q^{\frac{1}{h}}}}\right)^{-1}\frac{\Omega(U)}{U^{\frac{1}{h}}}.\end{aligned}\tag{34}$$

We have the following estimate, analogous to Equation (28):

$$\sum_{\substack{n\in\mathcal{N}_h\\ n\le x\\ B<U_k(n)\le x}}\Omega(U_k(n))\le\sum_{\substack{U\in\mathcal{N}_k\\ B<U\le x}}\left\lfloor\frac{x}{U}\right\rfloor\Omega(U)\le\sum_{\substack{U\in\mathcal{N}_k\\ U\le x}}\frac{x}{U}\Omega(U).\tag{35}$$

Applying Lemma 4 to Equations (34) and (35), we have

$$\begin{aligned}\sum_{\substack{n\in\mathcal{N}_h\\ n\le x}}\Omega(U_k(n)) =& f(k,h)x^{\frac{1}{h}}\sum_{U\in\mathcal{N}_k}\prod_{q|U}\left(1+\frac{\frac{1}{q}-\frac{1}{q^{\frac{k}{h}}}}{1-\frac{1}{q^{\frac{1}{h}}}}\right)^{-1}\frac{\Omega(U)}{U^{\frac{1}{h}}}\\ &+O\left(x^{\frac{2h+1}{2h(h+1)}+\varepsilon}B^{\frac{2}{k}-\frac{2h+1}{2h(h+1)}-\varepsilon}\log\log B\right)\\ &+O\left(x^{\frac{1}{k}}\log\log x\right)+O\left(x^{\frac{1}{h}}B^{\frac{1}{k}-\frac{1}{h}}\log\log B\right).\end{aligned}$$

We choose $B=x^{\frac{k}{k+2h(h+1)}}$ and get

$$\begin{aligned}\sum_{\substack{n\in\mathcal{N}_h\\ n\le x}}\Omega(U_k(n)) =& f(k,h)x^{\frac{1}{h}}\sum_{U\in\mathcal{N}_k}\prod_{q|U}\left(1+\frac{\frac{1}{q}-\frac{1}{q^{\frac{k}{h}}}}{1-\frac{1}{q^{\frac{1}{h}}}}\right)^{-1}\frac{\Omega(U)}{U^{\frac{1}{h}}}\\ &+O\left(x^{\frac{1}{h}-\left(\frac{k}{h}-1\right)\frac{1}{k+2h(h+1)}+\varepsilon}\log\log x\right).\end{aligned}$$

We now proceed to find a closed expression for

$$f(k,h)\sum_{U\in\mathcal{N}_k}\prod_{q|U}\left(1+\frac{\frac{1}{q}-\frac{1}{q^{\frac{k}{h}}}}{1-\frac{1}{q^{\frac{1}{h}}}}\right)^{-1}\frac{\Omega(U)}{U^{\frac{1}{h}}}.$$

We consider a generating function given by

$$\begin{aligned}\mathcal{E}_{\Omega,k,h}(z) &= \sum_{n\in\mathcal{N}_k}\frac{z^{\Omega(n)}}{n^{\frac{1}{h}}}\prod_{q|n}\left(1+\frac{\frac{1}{q}-\frac{1}{q^{\frac{k}{h}}}}{1-\frac{1}{q^{\frac{1}{h}}}}\right)^{-1}\\ &=\prod_p\left(1+\left(1+\frac{\frac{1}{p}-\frac{1}{p^{\frac{k}{h}}}}{1-\frac{1}{p^{\frac{1}{h}}}}\right)^{-1}\frac{z^k}{p^{\frac{k}{h}}}\left(1+\frac{z}{p^{\frac{1}{h}}}+\frac{z^2}{p^{\frac{2}{h}}}+\cdots\right)\right)\\ &=\prod_p\left(1+\left(1+\frac{\frac{1}{p}-\frac{1}{p^{\frac{k}{h}}}}{1-\frac{1}{p^{\frac{1}{h}}}}\right)^{-1}\frac{\frac{z^k}{p^{\frac{k}{h}}}}{1-\frac{z}{p^{\frac{1}{h}}}}\right),\end{aligned}$$

which is absolutely convergent over compact sets.

We will recover our term of interest by computing $\mathcal{E}'_{\Omega,k,h}(1)$, which we find by considering the logarithmic derivative:

$$\frac{\mathcal{E}'_{\Omega,k,h}(z)}{\mathcal{E}_{\Omega,k,h}(z)}=\sum_p\frac{\frac{\frac{kz^{k-1}}{p^{\frac{k}{h}}}-\frac{(k-1)z^k}{p^{\frac{k+1}{h}}}}{\left(1-\frac{z}{p^{\frac{1}{h}}}\right)^2}}{\left(1+\frac{\frac{1}{p}-\frac{1}{p^{\frac{k}{h}}}}{1-\frac{1}{p^{\frac{1}{h}}}}\right)+\frac{\frac{z^k}{p^{\frac{k}{h}}}}{1-\frac{z}{p^{\frac{1}{h}}}}}.$$

Therefore,

$$\left.\frac{\mathcal{E}'_{\Omega,k,h}(z)}{\mathcal{E}_{\Omega,k,h}(z)}\right|_{z=1} = \sum_p \frac{\frac{k}{p^{\frac{k}{h}}} - \frac{k-1}{p^{\frac{k+1}{h}}}}{\left(1 - \frac{1}{p^{\frac{1}{h}}}\right)\left(1 + \frac{1}{p} - \frac{1}{p^{\frac{1}{h}}}\right)}.$$

By multiplying the above by $\mathcal{E}_{\Omega,k,h}(1)$ and by the coefficient $f(k,h)$, we get an expression for $E_{\Omega,k,h}$:

$$\begin{aligned}
f(k,h)\mathcal{E}'_{\Omega,k,h}(1) =& f(k,h) \sum_p \frac{\frac{k}{p^{\frac{k}{h}}} - \frac{k-1}{p^{\frac{k+1}{h}}}}{\left(1 - \frac{1}{p^{\frac{1}{h}}}\right)\left(1 + \frac{1}{p} - \frac{1}{p^{\frac{1}{h}}}\right)} \\
& \times \prod_p \left(1 + \left(1 + \frac{\frac{1}{p} - \frac{1}{p^{\frac{k}{h}}}}{1 - \frac{1}{p^{\frac{1}{h}}}}\right)^{-1} \frac{\frac{1}{p^{\frac{k}{h}}}}{1 - \frac{1}{p^{\frac{1}{h}}}}\right) \\
=& \sum_p \frac{\frac{k}{p^{\frac{k}{h}}} - \frac{k-1}{p^{\frac{k+1}{h}}}}{\left(1 - \frac{1}{p^{\frac{1}{h}}}\right)\left(1 + \frac{1}{p} - \frac{1}{p^{\frac{1}{h}}}\right)} \\
& \times \prod_p \left(1 + \frac{\frac{1}{p^{\frac{k}{h}}}}{1 - \frac{1}{p^{\frac{1}{h}}} + \frac{1}{p} - \frac{1}{p^{\frac{k}{h}}}}\right)\left(1 - \frac{1}{p}\right)\left(1 + \frac{\frac{1}{p} - \frac{1}{p^{\frac{k}{h}}}}{1 - \frac{1}{p^{\frac{1}{h}}}}\right) \\
=& \sum_p \frac{kp^{\frac{1}{h}} - k + 1}{p^{\frac{k-h-1}{h}}\left(p^{\frac{1}{h}} - 1\right)\left(p^{1+\frac{1}{h}} + p^{\frac{1}{h}} - p\right)} \prod_p \left(1 + \frac{p - p^{\frac{1}{h}}}{p^2\left(p^{\frac{1}{h}} - 1\right)}\right).
\end{aligned}$$

Equations (15) and (18) are proven analogously. Here instead we must consider

$$f(k,h) \sum_{U \in \mathcal{N}_k} \prod_{q|U} \left(1 + \frac{\frac{1}{q} - \frac{1}{q^{\frac{k}{h}}}}{1 - \frac{1}{q^{\frac{1}{h}}}}\right)^{-1} \frac{\omega(U)}{U^{\frac{1}{h}}}.$$

The corresponding generating function is given by

$$\begin{aligned}
\mathcal{E}_{\omega,k,h}(z) &= \sum_{n \in \mathcal{N}_k} \frac{z^{\omega(n)}}{n^{\frac{1}{h}}} \prod_{q|n} \left(1 + \frac{\frac{1}{q} - \frac{1}{q^{\frac{k}{h}}}}{1 - \frac{1}{q^{\frac{1}{h}}}}\right)^{-1} \\
&= \prod_p \left(1 + \left(1 + \frac{\frac{1}{p} - \frac{1}{p^{\frac{k}{h}}}}{1 - \frac{1}{p^{\frac{1}{h}}}}\right)^{-1} \frac{z}{p^{\frac{k}{h}}}\left(1 + \frac{1}{p^{\frac{1}{h}}} + \frac{1}{p^{\frac{2}{h}}} + \cdots\right)\right) \\
&= \prod_p \left(1 + \left(1 + \frac{\frac{1}{p} - \frac{1}{p^{\frac{k}{h}}}}{1 - \frac{1}{p^{\frac{1}{h}}}}\right)^{-1} \frac{\frac{z}{p^{\frac{k}{h}}}}{1 - \frac{1}{p^{\frac{1}{h}}}}\right),
\end{aligned}$$

which is absolutely convergent.

In order to find $\mathcal{E}'_{\omega,k,h}(1)$, we consider the logarithmic derivative:

$$\frac{\mathcal{E}'_{\omega,k,h}(z)}{\mathcal{E}_{\omega,k,h}(z)} = \sum_p \frac{\frac{1}{p^{\frac{k}{h}}}}{1 - \frac{1}{p^{\frac{1}{h}}} + \frac{1}{p} - \frac{1}{p^{\frac{k}{h}}} + \frac{z}{p^{\frac{k}{h}}}}.$$

Therefore,

$$\left.\frac{\mathcal{E}'_{\omega,k,h}(z)}{\mathcal{E}_{\omega,k,h}(z)}\right|_{z=1} = \sum_p \frac{\frac{1}{p^{\frac{k}{h}}}}{1 - \frac{1}{p^{\frac{1}{h}}} + \frac{1}{p}}.$$

By multiplying the above by $\mathcal{E}_{\omega,k,h}(1)$ and by the coefficient $f(k,h)$, we get an expression for $E_{\omega,k,h}$:

$$\begin{aligned}
f(k,h)\mathcal{E}'_{\omega,k,h}(1) =& f(k,h) \sum_p \frac{\frac{1}{p^{\frac{k}{h}}}}{1 - \frac{1}{p^{\frac{1}{h}}} + \frac{1}{p}} \prod_p \left(1 + \left(1 + \frac{\frac{1}{p} - \frac{1}{p^{\frac{k}{h}}}}{1 - \frac{1}{p^{\frac{1}{h}}}}\right)^{-1} \frac{\frac{1}{p^{\frac{k}{h}}}}{1 - \frac{1}{p^{\frac{1}{h}}}}\right) \\
=& \sum_p \frac{\frac{1}{p^{\frac{k}{h}}}}{1 - \frac{1}{p^{\frac{1}{h}}} + \frac{1}{p}} \\
& \times \prod_p \left(1 + \frac{\frac{1}{p^{\frac{k}{h}}}}{1 - \frac{1}{p^{\frac{1}{h}}} + \frac{1}{p} - \frac{1}{p^{\frac{k}{h}}}}\right) \left(1 - \frac{1}{p}\right) \left(1 + \frac{\frac{1}{p} - \frac{1}{p^{\frac{k}{h}}}}{1 - \frac{1}{p^{\frac{1}{h}}}}\right) \\
=& \sum_p \frac{1}{p^{\frac{k-h-1}{h}} \left(p^{1+\frac{1}{h}} + p^{\frac{1}{h}} - p\right)} \prod_p \left(1 + \frac{p - p^{\frac{1}{h}}}{p^2 \left(p^{\frac{1}{h}} - 1\right)}\right).
\end{aligned}$$

This concludes the proof of Theorem 3. □

*Proof of Corollary 3.* Recall that $n = L_k(n)U_k(n)$ and this implies

$$\Omega(n) = \Omega(L_k(n)) + \Omega(U_k(n)).$$

Combining Equations (14) and (26), we immediately obtain Equation (19). □

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