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IDEAL SOLUTIONS OF THE TARRY-ESCOTT PROBLEM OF DEGREE SEVEN

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Abstract

In this paper we obtain four new parametric ideal solutions of the Tarry-Escott problem of degree 7, that is, of the simultaneous diophantine equations, $\sum_{i=1}^{8} x_i^r = \sum_{i=1}^{8} y_i^r,\ r = 1, 2, \ldots, 7$. While all the known parametric solutions of the problem, with one exception, are given by polynomials of degrees greater than 4, the solutions obtained in this paper are given by quartic polynomials, and are thus simpler than almost all of the known solutions.

1. Introduction

The Tarry-Escott problem (written briefly as TEP) of degree k consists of finding two distinct multisets of integers $\{x_1, x_2, \ldots, x_s\}$ and $\{y_1, y_2, \ldots, y_s\}$ such that

$$\sum_{i=1}^{s} x_i^r = \sum_{i=1}^{s} y_i^r, \quad r = 1, 2, \ldots, k, \tag{1.1}$$

where k is a given positive integer. When $s = k + 1$, solutions of the diophantine system (1.1) are known as ideal solutions. Parametric ideal solutions of the TEP are known only when $k \leq 7$.

This paper is concerned with parametric ideal solutions of the Tarry-Escott problem of degree 7, that is, of the diophantine system,

$$\sum_{i=1}^{8} x_i^r = \sum_{i=1}^{8} y_i^r, \quad r = 1, 2, \ldots, 7. \tag{1.2}$$

Except for a single parametric ideal solution that is expressible by quadratic polynomials (see [1, p. 633]), all other known parametric ideal solutions of (1.2) are given by univariate polynomials of degrees greater than 4. Recently MacLeod [4, 5] has examined various methods of solving the diophantine system (1.2) and applied the theory of elliptic curves to obtain several parametric ideal solutions of degree 5.

An alternative method give by Choudhry [3, pp. 410–412] also yields a parametric ideal solution of degree 5. The complete ideal solution of the TEP of degree 7 is not yet known.

In this paper we give an elementary method of obtaining ideal solutions of the TEP of degree 7. We apply the method to obtain four parametric ideal solutions of degree 4. This is an improvement over the known results since, as mentioned above, all known parametric solutions, with one exception, are of degrees greater than 4. We also show how more parametric ideal solutions of degrees greater than 4 may be obtained.

2. Parametric Ideal Solutions of the TEP of Degree 7

To solve the diophantine system (1.2), we assume

$$x_{i+4} = -x_i, \;\; y_{i+4} = -y_i, \;\; i = 1, \ldots, 4, \tag{2.1}$$

which makes (1.2) identically satisfied for odd values of the exponent r, and the diophantine system reduces to

$$\sum_{i=1}^{4} x_i^r = \sum_{i=1}^{4} y_i^r, \quad r = 2, 4, 6. \tag{2.2}$$

In fact, all known solutions of the TEP of degree 7 are obtained by applying the simplifying conditions (2.1).

In view of the symmetry and homogeneity of each of the equations of the diophantine system (2.2) and the fact that each variable occurs only in even degrees in these equations, if $(X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4)$ is any rational solution of (2.2), any other solution obtained by permuting either the members of the set $\{X_i, i = 1, 2, 3, 4\}$ or of the set $\{Y_i, i = 1, 2, 3, 4\}$, or changing the signs of any of the X_i or Y_i, or interchanging the sets $\{X_i, i = 1, 2, 3, 4\}$ and $\{Y_i, i = 1, 2, 3, 4\}$, or scaling the solution by a nonzero factor, or performing several such operations any number of times in any order, will be considered equivalent to the solution $(X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4)$. Further, any rational solution yields, on appropriate scaling, a solution in integers. Therefore, it suffices to obtain rational solutions of the diophantine system (2.2).

We note that Choudhry [2, p. 119] has given the following solution of the diophantine system $\sum_{i=1}^4 x_i^r = \sum_{i=1}^4 y_i^r, \quad r = 1, 2, 4$:

$$\begin{aligned}
x_1 &= \alpha_1 m + (\alpha_1 + 2\alpha_3)(\alpha_1 + 2\alpha_2 + \alpha_3)n,\\
x_2 &= \alpha_2 m - (\alpha_1 - \alpha_3)(2\alpha_1 + \alpha_2 + 2\alpha_3)n,\\
x_3 &= \alpha_3 m - (2\alpha_1 + \alpha_3)(\alpha_1 + 2\alpha_2 + \alpha_3)n,\\
x_4 &= -(\alpha_1 + \alpha_2 + \alpha_3)m + (\alpha_1 - \alpha_3)(\alpha_1 - \alpha_2 + \alpha_3)n,\\
y_1 &= -\alpha_1 m + (\alpha_1 + 2\alpha_3)(\alpha_1 + 2\alpha_2 + \alpha_3)n,\\
y_2 &= -\alpha_2 m - (\alpha_1 - \alpha_3)(2\alpha_1 + \alpha_2 + 2\alpha_3)n,\\
y_3 &= -\alpha_3 m - (2\alpha_1 + \alpha_3)(\alpha_1 + 2\alpha_2 + \alpha_3)n,\\
y_4 &= (\alpha_1 + \alpha_2 + \alpha_3)m + (\alpha_1 - \alpha_3)(\alpha_1 - \alpha_2 + \alpha_3)n,
\end{aligned} \tag{2.3}$$

where $\alpha_1, \alpha_2, \alpha_3, m$ and n are arbitrary parameters. Accordingly, we choose x_i, y_i, $i = 1, \dots, 4$, as given by (2.3) when (2.2) is identically satisfied for $r = 2$ and $r = 4$, while for $r = 6$, we may write (2.2), after transposing all terms to one side, as follows:

$$\begin{aligned}
12mn(\alpha_1^2 - \alpha_3^2)(\alpha_1 + \alpha_2)(\alpha_1 + 2\alpha_2 + \alpha_3)(\alpha_2 + \alpha_3)\{m^2 - 9(\alpha_1 + \alpha_3)^2 n^2\}\\
\times \{(2\alpha_1^2 - \alpha_1\alpha_2 + 5\alpha_1\alpha_3 - \alpha_2^2 - \alpha_2\alpha_3 + 2\alpha_3^2)m^2 - (2\alpha_1^4 - \alpha_1^3\alpha_2\\
+ 9\alpha_1^3\alpha_3 - \alpha_1^2\alpha_2^2 + 37\alpha_1^2\alpha_2\alpha_3 + 14\alpha_1^2\alpha_3^2 + 38\alpha_1\alpha_2^2\alpha_3\\
+ 37\alpha_1\alpha_2\alpha_3^2 + 9\alpha_1\alpha_3^3 - \alpha_2^2\alpha_3^2 - \alpha_2\alpha_3^3 + 2\alpha_3^4)n^2\} = 0.
\end{aligned} \tag{2.4}$$

Except for the last factor, if we equate to zero any of the other factors on the left-hand side of (2.4), we get trivial solutions of the diophantine system (2.2). To obtain nontrivial solutions, we equate the last factor to 0 so that we get the equation,

$$\begin{aligned}
(2\alpha_1^2 - \alpha_1\alpha_2 + 5\alpha_1\alpha_3 - \alpha_2^2 - \alpha_2\alpha_3 + 2\alpha_3^2)m^2 - (2\alpha_1^4 - \alpha_1^3\alpha_2 + 9\alpha_1^3\alpha_3 - \alpha_1^2\alpha_2^2 + 37\alpha_1^2\alpha_2\alpha_3\\
+ 14\alpha_1^2\alpha_3^2 + 38\alpha_1\alpha_2^2\alpha_3 + 37\alpha_1\alpha_2\alpha_3^2 + 9\alpha_1\alpha_3^3 - \alpha_2^2\alpha_3^2 - \alpha_2\alpha_3^3 + 2\alpha_3^4)n^2\} = 0.
\end{aligned} \tag{2.5}$$

We now substitute

$$m = ny/(2\alpha_1^2 - \alpha_1\alpha_2 + 5\alpha_1\alpha_3 - \alpha_2^2 - \alpha_2\alpha_3 + 2\alpha_3^2), \tag{2.6}$$

to obtain,

$$y^2 = \phi(\alpha_1, \alpha_2, \alpha_3), \tag{2.7}$$

where

$$\begin{aligned}\phi(\alpha_1,\alpha_2,\alpha_3) &= (2\alpha_1^2-\alpha_1\alpha_2+5\alpha_1\alpha_3-\alpha_2^2-\alpha_2\alpha_3+2\alpha_3^2)(2\alpha_1^4-\alpha_1^3\alpha_2\\ &\quad+9\alpha_1^3\alpha_3-\alpha_1^2\alpha_2^2+37\alpha_1^2\alpha_2\alpha_3+14\alpha_1^2\alpha_3^2+38\alpha_1\alpha_2^2\alpha_3\\ &\quad+37\alpha_1\alpha_2\alpha_3^2+9\alpha_1\alpha_3^3-\alpha_2^2\alpha_3^2-\alpha_2\alpha_3^3+2\alpha_3^4),\\ &= (\alpha_1^2-38\alpha_1\alpha_3+\alpha_3^2)\alpha_2^4+2(\alpha_1+\alpha_3)(\alpha_1^2-38\alpha_1\alpha_3+\alpha_3^2)\alpha_2^3 \qquad (2.8)\\ &\quad-(3\alpha_1^4-26\alpha_1^3\alpha_3-98\alpha_1^2\alpha_3^2-26\alpha_1\alpha_3^3+3\alpha_3^4)\alpha_2^2\\ &\quad-2(2\alpha_1+\alpha_3)(\alpha_1+2\alpha_3)(\alpha_1+\alpha_3)(\alpha_1^2-18\alpha_1\alpha_3+\alpha_3^2)\alpha_2\\ &\quad+(\alpha_1+2\alpha_3)^2(2\alpha_1+\alpha_3)^2(\alpha_1+\alpha_3)^2.\end{aligned}$$

Now $\phi(\alpha_1,\alpha_2,\alpha_3)$ may be considered as a quartic function of α_2 in which the constant term is a perfect square, and the obvious method to find solutions of the diophantine equation (2.7) is to consider it as a quartic model of an elliptic curve over the function field $\mathbb{Q}(\alpha_1,\alpha_3)$. We can readily obtain rational points on this elliptic curve and thus obtain parametric ideal solutions of the TEP of degree 7 but these solutions are of degrees greater than 4.

We will now describe an elementary method of choosing $\alpha_1,\alpha_2,\alpha_3$ such that $\phi(\alpha_1,\alpha_2,\alpha_3)$ becomes a perfect square, and thus obtain parametric ideal solutions of degree 4 of the TEP of degree 7. We first assume $\alpha_2=f\alpha_1+g\alpha_3$, where f and g are arbitrary parameters, and consider $\phi(\alpha_i)$ as a homogeneous sextic polynomial in the variables α_1 and α_3. We choose one of the parameters (either f or g) such that the discriminant of $\phi(\alpha_i)$ with respect to α_1 vanishes, and now $\phi(\alpha_i)$ has a squared factor which may be factored out leaving us with a quartic polynomial in the variables α_1 and α_3. We now repeat the process with the quartic polynomial, that is, we choose the remaining parameter (either f or g) such that the discriminant of the quartic polynomial with respect to α_1 vanishes, and now we can again factor out a squared factor from the quartic polynomial leaving us with a quadratic polynomial in the variables α_1 and α_3. We can readily equate this quadratic polynomial in α_1 and α_3 to a perfect square, and thus obtain parametric ideal solutions of degree 4 of the TEP of degree 7.

We will now illustrate the method described above to obtain a specific parametric solution of the TEP of degree 7. On substituting $\alpha_2=f\alpha_1+g\alpha_3$ in $\phi(\alpha_1,\alpha_2,\alpha_3)$, and equating to zero the discriminant, with respect to α_1, of the resulting sextic polynomial, we get the following condition:

$$\begin{aligned}(f+2)^2(f-1)^2(f-g)^2(2f-g)^2(f-2g)^2(f+g+1)^2(f-2g-1)^2\\ \times(2f-g+1)^2(g+2)^2(g-1)^2(9f^2-22fg+9g^2-2f-2g+9)\\ \times(f^6+114f^5g+4335f^4g^2+55100f^3g^3+4335f^2g^4+114fg^5\\ +g^6+60f^5+4620f^4g+91320f^3g^2+91320f^2g^3+4620fg^4\\ +60g^5-20667f^4+48228f^3g+141678f^2g^2+48228fg^3\\ -20667g^4-35620f^3+72420f^2g+72420fg^2-35620g^3\\ -20667f^2+46254fg-20667g^2+60f+60g+1)=0. \quad (2.9)\end{aligned}$$

The left-hand side of Equation (2.9) has 12 factors of which 10 are linear in f and g. In the next two subsections we consider the solutions of the TEP of degree 7 obtained by equating to 0 the linear and the nonlinear factors, respectively.

2.1. Solutions Obtained from the Linear Factors on the Left-Hand Side of Equation (2.9)

We can readily obtain solutions of Equation (2.9) by choosing f or g such that one of the 10 linear factors on the left-hand side of Equation (2.9) becomes zero. As an example, we take $f=-2$, and now $\phi(\alpha_1,\alpha_2,\alpha_3)$ reduces to $\alpha_3^2\phi_1(\alpha_1,\alpha_3)$, where

$$\begin{aligned}\phi_1(\alpha_1,\alpha_3)=\{(3g+7)\alpha_1-(g+2)(g-1)\alpha_3\}\{(3g+87)\alpha_1^3-\\ (g^2+115g+64)\alpha_1^2\alpha_3+(19g+11)(2g+1)\alpha_1\alpha_3^2-(g+2)(g-1)\alpha_3^3\}. \quad (2.10)\end{aligned}$$

Now, on equating to zero the discriminant of $\phi_1(\alpha_1,\alpha_3)$ with respect to α_1, we get the following condition:

$$\begin{aligned}(g-1)^4(g+1)^2(g+2)^4(g+3)^2(g+4)^2(2g+3)^2\\ \times(g^4-226g^3-300g^2-130g-155)=0. \quad (2.11)\end{aligned}$$

This condition is again satisfied quite simply by choosing the value of g in several ways. We take $g=-1$ when $\phi_1(\alpha_1,\alpha_3)$ reduces to $4(2\alpha_1+\alpha_3)^2(21\alpha_1^2+2\alpha_1\alpha_3+\alpha_3^2)$. We have now reduced our problem to choosing α_1 and α_3 such that $21\alpha_1^2+2\alpha_1\alpha_3+\alpha_3^2$ becomes a perfect square. It is readily seen that the complete solution of this condition in rational numbers is given by

$$\alpha_1=k(2t+2),\quad \alpha_3=k(t^2-21), \quad (2.12)$$

where k and t are arbitrary rational parameters. Since we took $f=-2$ and $g=-1$, we now get $\alpha_2=-2\alpha_1-\alpha_3$ and, on using (2.12), this gives us

$$\alpha_2=k(-t^2-4t+17). \quad (2.13)$$

With the values of $\alpha_1, \alpha_2, \alpha_3$ given by (2.12) and (2.13), the value of $\phi(\alpha_1, \alpha_2, \alpha_3)$ is a perfect square, and hence Equation (2.4) can be solved for m and n, and finally, using the relations (2.3), we get a solution of the simultaneous diophantine equations (2.2). This solution may be written, after appropriate scaling, as follows:

$$\begin{aligned} x_1 &= t^4 + 6t^3 - 32t^2 - 158t + 279, & x_2 &= 4t^3 + 28t^2 + 4t - 420,\\ x_3 &= t^4 + 6t^3 - 4t^2 - 102t - 93, & x_4 &= t^4 - 50t^2 - 56t + 393,\\ y_1 &= t^4 + 8t^3 - 26t^2 - 112t + 321, & y_2 &= t^4 + 2t^3 - 16t^2 + 46t + 63,\\ y_3 &= 4t^3 - 4t^2 - 60t + 348, & y_4 &= t^4 + 2t^3 - 44t^2 - 10t + 435, \end{aligned} \tag{2.14}$$

where t is an arbitrary parameter. As a numerical example, on taking $t = 2$, we get the solution,

$$101^r + 268^r + 249^r + 97^r = 73^r + 123^r + 244^r + 271^r, \quad r = 2, 4, 6.$$

We will now find more solutions of the simultaneous diophantine equations (2.2) following the method illustrated above. We have already noted that the condition (2.9) can readily be satisfied in 10 different ways. We consider each possibility, obtain the quartic polynomial corresponding to the polynomial $\phi_1(\alpha_i)$ of the above illustrative example, and equate its discriminant to zero. Like the condition (2.11), each such condition can be readily satisfied in several ways. We exhaustively analyzed all the possibilities, and could obtain only four distinct nontrivial solutions of the simultaneous diophantine equations (2.2), one of which has already been given above.

The remaining three solutions of degree 4 of the simultaneous diophantine equations (2.2) may be written, in terms of an arbitrary parameter t, as follows:

$$\begin{aligned} x_1 &= 3t^4 + 40t^3 - 274t^2 + 48t + 1383, & x_2 &= t^4 - 88t^3 - 214t^2 + 736t - 675,\\ x_3 &= 7t^4 + 14t^3 - 152t^2 + 962t + 609, & x_4 &= 4t^4 - 70t^3 + 46t^2 + 254t - 1914,\\ y_1 &= 4t^4 - 6t^3 - 274t^2 - 642t + 1158, & y_2 &= 3t^4 - 92t^3 - 62t^2 + 676t + 1155,\\ y_3 &= 7t^4 + 12t^3 - 14t^2 - 1124t - 81, & y_4 &= t^4 + 66t^3 + 184t^2 + 238t - 1929, \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} x_1 &= 2t^4 + 42t^3 - 170t^2 + 942t - 48, & x_2 &= t^4 - 84t^3 - 242t^2 - 68t - 1911,\\ x_3 &= 5t^4 + 12t^3 - 10t^2 + 1724t + 1341, & x_4 &= 3t^4 - 58t^3 + 92t^2 - 310t - 1263,\\ y_1 &= 3t^4 - 16t^3 - 170t^2 - 1320t - 1569, & y_2 &= 2t^4 - 86t^3 - 106t^2 - 146t + 1872,\\ y_3 &= 5t^4 + 10t^3 + 164t^2 - 1562t - 921, & y_4 &= t^4 + 56t^3 + 266t^2 + 928t - 483, \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} x_1 &= 4t^4 + 18t^3 + 34t^2 + 526t + 378, & x_2 &= 3t^4 + 16t^3 - 86t^2 - 32t + 579,\\ x_3 &= t^4 - 12t^3 + 74t^2 + 1020t + 1317, & x_4 &= 2t^4 + 42t^3 + 110t^2 + 230t + 1056,\\ y_1 &= 4t^4 + 14t^3 - 122t^2 - 958t - 1338, & y_2 &= 2t^4 - 14t^3 - 242t^2 - 658t - 48,\\ y_3 &= 3t^4 + 40t^3 + 74t^2 - 696t - 861, & y_4 &= t^4 + 44t^3 + 202t^2 + 164t - 891. \end{aligned} \tag{2.17}$$

The four solutions (2.14), (2.15), (2.16), and (2.17) of the diophantine system (2.2) immediately yield four solutions of the TEP of degree 7 in terms of quartic polynomials.

We now give the relevant details of our exhaustive analysis to show that the above four solutions are essentially the only nontrivial solutions that can be obtained from the 10 linear factors on the left-hand side of Equation (2.9).

While obtaining the first solution (2.14), we had equated to 0 the first linear factor on the left-hand side of Equation (2.9) giving $f = -2$, and then we obtained Equation (2.11) which is satisfied by 6 rational values of g yielding 6 values for the ordered pair (f, g) to be considered for solving Equation (2.7). These set of 6 values of (f, g) are given below:

$$(-2, 1),\ (-2, -1),\ (-2, -2),\ (-2, -3),\ (-2, -4),\ (-2, -3/2). \tag{2.18}$$

Similarly, by equating to 0 each of the remaining 9 linear factors on the left-hand side of Equation (2.9), and proceeding as above, we get 9 more sets of values of (f, g) which may yield solutions of Equation (2.7). These 9 sets are given in the 9 rows below.

$$\begin{aligned}
&\{(1, 0),\ (1, 1),\ (1, 2),\ (1, 3),\ (1, 1/2),\ (1, -2)\},\\
&\{(0, 0),\ (1, 1),\ (-1, -1),\ (-2, -2),\ (-1/2, -1/2)\},\\
&\{(0, 0),\ (1, 1/2),\ (2, 1),\ (-2, -1),\ (-4, -2),\ (-2/3, -1/3)\},\\
&\{(0, 0),\ (2, 1),\ (-4, -2),\ (-2, -1),\ (1, 1/2),\ (-2/3, -1/3)\},\\
&\{(-2, 1),\ (1, -2),\ (-1/2, -1/2),\ (-1/3, -2/3),\ (-2/3, -1/3)\},\\
&\{(-3, -2),\ (-1, -1),\ (1, 0),\ (3, 1),\ (-2, -3/2),\ (-1/3, -2/3)\},\\
&\{(-2, -3),\ (-1, -1),\ (0, 1),\ (1, 3),\ (-3/2, -2),\ (-2/3, -1/3)\},\\
&\{(1, -2),\ (-1, -2),\ (-2, -2),\ (-3, -2),\ (-4, -2),\ (-3/2, -2)\},\\
&\{(0, 1),\ (1, 1),\ (2, 1),\ (-2, 1),\ (3, 1),\ (1/2, 1)\}.
\end{aligned} \tag{2.19}$$

We get a total of 25 distinct values for the order pair (f, g) from the sets (2.18) and (2.19).

We will now show that if f' and g' are two unequal rational numbers, we get essentially only one solution of the diophantine system (2.2) by taking $(f, g) = (f', g')$ or $(f, g) = (g', f')$ and following the method described above. On taking $(f, g) = (f', g')$, we get $\alpha_2 = f'\alpha_1 + g'\alpha_3$, and if $\phi(\alpha_1, \alpha_2, \alpha_3)$ becomes a perfect square when $(\alpha_1, \alpha_3) = (\alpha_1', \alpha_3')$, that is, $\phi(\alpha_1', \alpha_2', \alpha_3')$ is a perfect square where $\alpha_2' = f'\alpha_1' + g'\alpha_3'$, then Equation (2.5) has two rational solutions, say $(m, n) = (\pm m', n')$. Taking $(\alpha_1, \alpha_2, \alpha_3) = (\alpha_1', \alpha_2', \alpha_3')$ and $(m, n) = (+m', n')$, we get, on using (2.3), a solution of the diophantine system (2.2) which we write as $(X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4)$. We now note that the solution obtained by taking $(m, n) = (-m', n')$ is the equivalent solution $(Y_1, Y_2, Y_3, Y_4, X_1, X_2, X_3, X_4)$.

We now consider the solutions obtained by taking $(f, g) = (g', f')$. We note that on interchanging α_1 and α_3, both $\phi(\alpha_1, \alpha_2, \alpha_3)$ and Equation (2.5) remain unaltered, and moreover, the value of α_2 also remains the same since both f, g

and α_1, α_3 are interchanged. Thus, on taking $(\alpha_1, \alpha_2, \alpha_3) = (\alpha_3', \alpha_2', \alpha_1')$, Equation (2.5) has the same two solutions $(m, n) = (\pm m', n')$ as before. Now on taking $(\alpha_1, \alpha_2, \alpha_3) = (\alpha_3', \alpha_2', \alpha_1')$ and $(m, n) = (+m', n')$, we get the solution $(-Y_3, -Y_2, -Y_1, -Y_4, -X_3, -X_2, -X_1, -X_4)$ while on taking $(m, n) = (-m', n')$, we get the solution $(-X_3, -X_2, -X_1, -X_4, -Y_3, -Y_2, -Y_1, -Y_4)$. Both of these solutions are equivalent to the solution $(X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4)$ obtained by taking $(f, g) = (f', g')$ and $(m, n) = (m', n')$. This proves our assertion that both (f', g') and (g', f') yield equivalent solutions of our diophantine system.

Since taking (f, g) either as (f', g') or as (g', f') yields equivalent solutions, it suffices to consider only those values of the ordered pair (f, g) in which $f \leq g$. Thus the set of 25 distinct values of (f, g) gets reduced to the following set of 15 values:

$$\begin{aligned}&\{(-2,-1),(0,1),(-4,-2),(1,3),(-3,-2),(1,2),(-2,-3/2),(1/2,1),\\&\quad(0,0),(1,1),(-2,-2),(-2,1),(-1,-1),(-1/2,-1/2),(-2/3,-1/3)\}.\end{aligned} \tag{2.20}$$

We will now consider the solutions obtained from these 15 values of (f, g).

We have already considered the case $(f, g) = (-2, -1)$ and described our procedure which yielded the solution (2.14). We carried out a similar procedure for each value of (f, g) listed at (2.20), that is, we substituted $\alpha_2 = f\alpha_1 + g\alpha_3$, computed the value of $\phi(\alpha_1, \alpha_2, \alpha_3)$, then found all values of α_1, α_3 such that $\phi(\alpha_1, \alpha_2, \alpha_3)$ becomes a perfect square, and eventually obtained a solution of the diophantine system (2.2). The results of our investigations for the first 8 values of (f, g) listed at (2.20) are given in Table I where we have written the values of α_1, α_3 without the scaling factor k. It would be seen from Table I that these 8 values of (f, g) yield just the four solutions given by (2.14), (2.15), (2.16), and (2.17). The remaining 7 values of (f, g) listed at (2.20) yield only trivial solutions of the diophantine system (2.2).

2.2. Solutions Obtained from the Nonlinear Factors on the Left-Hand Side of Equation (2.9)

We have obtained four solutions of the TEP of degree 7 by equating to 0 the 10 linear factors on the left-hand side of Equation (2.9). We now consider the possibility of obtaining additional solutions by equating to 0 the remaining two nonlinear factors on the left-hand side of Equation (2.9). Equating to 0 the second-last factor, we get,

$$9f^2 - 22fg + 9g^2 - 2f - 2g + 9 = 0. \tag{2.21}$$

The complete rational solution of Equation (2.21) is readily obtained and is given by

$$\begin{aligned}f &= -2(t^2 - 22t + 141)/(t^2 - 22t + 81),\\ g &= -(7t^2 - 114t + 567)/(3(t^2 - 22t + 81)),\end{aligned} \tag{2.22}$$

where t is an arbitrary rational parameter.

(f,g)	$\phi(\alpha_1,\alpha_2,\alpha_3)$	Values of α_1,α_3	Solution generated
$(-2,-1)$	$4(2\alpha_1+\alpha_3)^2$ $\times(21\alpha_1^2+2\alpha_1\alpha_3+\alpha_3^2)$	$\alpha_1=2t+2,$ $\alpha_3=t^2-21$	(2.14)
$(0,1)$	$4\alpha_1^2(\alpha_1+2\alpha_3)^2$ $\times(\alpha_1^2+2\alpha_1\alpha_3+21\alpha_3^2)$	$\alpha_1=t^2-21,$ $\alpha_3=2t+2$	(2.14)
$(-4,-2)$	$25\alpha_1^2(2\alpha_1+\alpha_3)^2$ $\times(\alpha_1^2-46\alpha_1\alpha_3-15\alpha_3^2)$	$\alpha_1=t^2+15,$ $\alpha_3=2t-46$	(2.15)
$(1,3)$	$25\alpha_3^2(\alpha_1+2\alpha_3)^2$ $\times(-15\alpha_1^2-46\alpha_1\alpha_3+\alpha_3^2)$	$\alpha_1=2t-46,$ $\alpha_3=t^2+15$	(2.15)
$(-3,-2)$	$4\alpha_1^2(2\alpha_1+\alpha_3)^2$ $(\alpha_1^2-58\alpha_1\alpha_3-39\alpha_3^2)$	$\alpha_1=t^2+39,$ $\alpha_3=2t-58$	(2.16)
$(1,2)$	$-4\alpha_3^2(\alpha_1+2\alpha_3)^2$ $\times(39\alpha_1^2+58\alpha_1\alpha_3-\alpha_3^2)$	$\alpha_1=2t-58,$ $\alpha_3=t^2+39$	(2.16)
$(-2,-3/2)$	$25\alpha_3^2(2\alpha_1+\alpha_3)^2$ $\times(33\alpha_1^2+26\alpha_1\alpha_3+\alpha_3^2)/16$	$\alpha_1=2t+26$ $\alpha_3=t^2-33$	(2.17)
$(1/2,1)$	$25\alpha_1^2(\alpha_1+2\alpha_3)^2$ $\times(\alpha_1^2+26\alpha_1\alpha_3+33\alpha_3^2)/16$	$\alpha_1=t^2-33,$ $\alpha_3=2t+26$	(2.17)

Table 1: Nontrivial solutions obtained from linear factors of (2.9)

On substituting $\alpha_2 = f\alpha_1 + g\alpha_3$ in (2.8) where the values of f and g are given by (2.22), we get,

$$\begin{aligned}\phi(\alpha_1,\alpha_2,\alpha_3) &= 100\{(18t-198)\alpha_1+(t^2-81)\alpha_3\}^2\{324(t-11)^2\alpha_1^4-36(2t^2-45t+333)\\ &\times(t^2-22t+161)\alpha_1^3\alpha_3-(179t^4-6552t^3+101070t^2-745200t+2354103)\alpha_1^2\alpha_3^2\\ &-2(5t^2-72t+387)(11t^2-162t+891)\alpha_1\alpha_3^3+(t^2-81)^2\alpha_3^4\}/\{81(t^2-22t+81)^4\}.\end{aligned}$$

Now, three simple solutions of Equation (2.7) are given by $\alpha_3 = -\alpha_1$, $\alpha_3 = -2\alpha_1$ and $\alpha_3 = -\alpha_1/2$, but all of these lead to trivial solutions of the TEP of degree 7. The other solutions of Equation (2.7) are cumbersome and lead to nontrivial solutions of high degree.

Finally we note that it is difficult to find solutions of the equation obtained by equating to 0 the last factor on the left-hand side of Equation (2.9). We are thus able to obtain only four solutions of the TEP of degree 7 in terms of polynomials of degree 4. This, however, does not rule out the existence of more such solutions which may be obtained by some other method.

3. Concluding Remark

We note that our solutions, like all other known solutions of the TEP of degree 7, are symmetric solutions, that is, they satisfy the conditions (2.1). It would be of interest to find nonsymmetric solutions, that is, solutions that do not satisfy the simplifying conditions (2.1).

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