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FERMAT NUMBERS IN NARAYANA'S COWS SEQUENCE

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Abstract

Narayana's cows sequence is a ternary recurrent sequence which satisfies the recurrence relation $N_{n+3} = N_{n+2} + N_n$ for $n \geq 0$ with initial terms $N_0 = 0$ and $N_1 = N_2 = 1$, where N_n is the *nth* Narayana number. In this study, we find all Narayana numbers that are also Fermat numbers.

1. Introduction

Narayana's cows sequence $\{N_n\}_{n\geq 0}$ originated from a herd of cows and calves problem, proposed by the Indian mathematician Narayana Pandit in his book *Ganita Kaumudi* [1]. It is the sequence A000930 in the OEIS satisfying the recurrence relation

$$N_{n+3} = N_{n+2} + N_n \tag{1}$$

for $n \geq 0$ with initial terms $N_0 = 0$ and $N_1 = N_2 = 1$. The first few terms of $\{N_n\}_{n\geq 0}$ are

$$0, 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, \cdots .$$

The characteristic equation of (1) is $f(x) = x^3 - x^2 - 1 = 0$, which has roots α (≈ 1.46557) and $\gamma = \bar{\beta}$ with $|\beta| = |\gamma| < 1$. Narayana's cows sequence has Binet's formula

$$N_n = a\alpha^n + b\beta^n + c\gamma^n \text{ for all } n \geq 0, \tag{2}$$

where

$$a = \frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)},\ b = \frac{\beta}{(\beta-\alpha)(\beta-\gamma)},\ c = \frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}.$$

The formula (2) can also be written in the form

$$N_n = C_\alpha \alpha^{n+2} + C_\beta \beta^{n+2} + C_\gamma \gamma^{n+2} \quad \text{for all} \quad n \geq 0, \tag{3}$$

where

$$C_x = \frac{1}{x^3+2}, \quad x \in \{\alpha, \beta, \gamma\}.$$

The coefficient C_α has the minimal polynomial $31x^3 - 31x^2 + 10x - 1$ over $\mathbb{Z}$ and all the zeros of this polynomial lie strictly inside the unit circle. Numerically, we can calculate

$$C_\alpha^{-1} \approx 5.1479 \quad \text{and} \quad C_\beta \approx 0.407506.$$

Using induction, one can prove that the nth Narayana number satisfies the following relation

$$\alpha^{n-2} \leq N_n \leq \alpha^{n-1} \text{ for all } n \geq 1, \tag{4}$$

(see [2]).

In the recent past, the study of Narayana's cows sequence has been a source of attraction for many authors. For instance, Bravo et al. [2] searched for the presence of repdigits in Narayana's cows sequence. They also obtained results on the existence of Mersenne prime numbers and numbers with distinct blocks of digits in this sequence. Ramírez and Sirvent [6] introduced the k-Narayana sequence and derived its combinatorial properties. They also established some relations between the k-Narayana sequence and convolved k-Narayana sequence.

Recently, Bravo and Herrera [3] showed the existence of Fermat numbers in k-Fibonacci and k-Lucas sequences. Later, Rihane et al. [7] searched for Fermat numbers in Padovan and Perrin Sequences. They proved that the only Fermat numbers in Padovan and Perrin sequences are $\{3, 5\}$ and {3,5,17} respectively. In this note, we search for Fermat numbers in Narayana's cows sequence. We use linear forms in logarithms of algebraic numbers and a version of the Baker-Davenport reduction method to prove the following result.

Theorem 1. *The only Fermat number in Narayana's cows sequence is* $N_5 = 3$.

2. Preliminary Results

We use Baker's theory of linear forms in logarithms of algebraic numbers for the proof of our result. Let η be an algebraic number with minimal primitive polynomial

$$f(X) = a_0(X - \eta^{(1)}) \ldots (X - \eta^{(k)}) \in \mathbb{Z}[X],$$

where $a_0 > 0$, and $\eta^{(i)}$'s are conjugates of η. Then,

$$h(\eta) = \frac{1}{k}\left(\log a_0 + \sum_{j=1}^{k} \max\{0, \log|\eta^{(j)}|\}\right)$$

is called the *logarithmic height* of η. In particular, if $\eta = a/b$ is a rational number with $\gcd(a,b) = 1$ and $b > 1$, then $h(\eta) = \log(\max\{|a|, b\})$. The following are some properties of logarithmic height function $h(\cdot)$ which will be used in this paper.

- $h(\eta + \gamma) \leq h(\eta) + h(\gamma) + \log 2$,
- $h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma)$,
- $h(\eta^k) = |k|h(\eta)$.

With the above notations, Matveev([5] or [4, Theorem 9.4]) proved the following result which is one of our main tools.

Theorem 2. *Let $\mathbb{L}$ be an algebraic number field of degree $d_{\mathbb{L}}$. Let $\eta_1, \eta_2, \ldots, \eta_l \in \mathbb{L}$ not 0 or 1 and let $b_1, b_2, \ldots, b_l$ be non zero integers. If $\Gamma = \prod_{i=1}^{l} \eta_i^{b_i} - 1$ is not zero, then*

$$\log|\Gamma| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 A_2 \ldots A_l,$$

where $D = \max\{|b_1|, |b_2|, \ldots, |b_l|\}$ and $A_1, A_2, \ldots, A_l$ are positive integers such that

$$A_j \geq \max\{d_{\mathbb{L}} h(\eta_j), |\log \eta_j|, 0.16\} \quad \textit{for } j = 1, \ldots, l.$$

After using Theorem 2, we get an upper bound on the variable n which is too large, thus we need to reduce that bound. To do so, we use the reduction method of Baker and Davenport due to de Weger [8].

Let $\vartheta_1, \vartheta_2, \beta \in \mathbb{R}$ be given and $x_1, x_2 \in \mathbb{Z}$ be unknowns. Suppose

$$\Lambda = \beta + x_1\vartheta_1 + x_2\vartheta_2. \tag{5}$$

Set $X = \max\{|x_1|, |x_2|\}$ and X_0, Y be positive. Assume that

$$X \leq X_0, \tag{6}$$

and

$$|\Lambda| < c \cdot \exp(-\delta \cdot Y), \tag{7}$$

where c, δ be positive constants. When $\beta \neq 0$ in (5), put $\vartheta = -\vartheta_1/\vartheta_2$ and $\psi = \beta/\vartheta_2$. Then we have

$$\frac{\Lambda}{\vartheta_2} = \psi - x_1\vartheta + x_2.$$

Let p/q be a convergent of ϑ with $q > X_0$. For a real number x, we let $\|x\| = \min\{|x - n|, n \in \mathbb{Z}\}$ be the distance from x to the nearest integer. We have the following result.

Lemma 1. *([8, Lemma 3.3]) Suppose that*

$$\|q\psi\| > \frac{2X_0}{q}.$$

Then, the solutions of (7) and (6) satisfy

$$Y < \frac{1}{\delta}\log\left(\frac{q^2 c}{|\vartheta_2|X_0}\right).$$

3. Proof of Theorem 1

Consider the Diophantine equation

$$N_n = 2^m + 1. \tag{8}$$

We search for the solutions to the above equation using *Mathematica* in the interval $n \in [0, 200]$ and find $N_4 = 2, N_5 = 3, N_8 = 9, N_{15} = 129$. One can check that the only Fermat number is $N_5 = 3$. We have to prove this is the only Fermat number in Narayana's cows sequence.

Now, we assume that $n > 200$. The following result shows that n is bigger than m which will be used later in our proof.

Lemma 2. *All solutions of (8) satisfy*

$$(n-2)\frac{\log\alpha}{\log 2} - 1 < m < (n-1)\frac{\log\alpha}{\log 2}.$$

Proof. By virtue of (4), we have

$$\alpha^{n-2} < N_n < 2^{m+1}.$$

This implies

$$(n-2)\log\alpha < (m+1)\log 2,$$

which leads to

$$(n-2)\frac{\log\alpha}{\log 2} - 1 < m.$$

Similarly, $2^m < N_n < \alpha^{n-1}$ gives

$$(n-1)\frac{\log\alpha}{\log 2} > m.$$

This ends the proof. □

Substituting Binet's formula (3) in (8), we get

$$C_\alpha \alpha^{n+2} + C_\beta \beta^{n+2} + C_\gamma \gamma^{n+2} = 2^m + 1.$$

That is

$$2^m - C_\alpha \alpha^{n+2} = C_\beta \beta^{n+2} + C_\gamma \gamma^{n+2} - 1.$$

Taking absolute values on both sides and dividing by $C_\alpha \alpha^{n+2}$, we get

$$\left| 2^m C_\alpha^{-1} \alpha^{-(n+2)} - 1 \right| < \frac{1.5}{C_\alpha \alpha^{n+2}} < 3.59 \alpha^{-n}. \tag{9}$$

For the left-hand side, we apply Theorem 2 with the following data. Set

$$\Gamma := 2^m C_\alpha^{-1} \alpha^{-(n+2)} - 1. \tag{10}$$

Now, we need to check $\Gamma \neq 0$. If $\Gamma = 0$, then

$$C_\alpha \alpha^{n+2} = 2^m. \tag{11}$$

Applying σ on both sides of (11), where σ is an automorphism of the Galois group of the splitting field $f(x)$ over $\mathbb{Q}$ defined by $\sigma(\alpha) = \beta$, we get

$$|C_\beta \beta^{n+2}| = 2^m.$$

Since $|C_\beta \beta^{n+2}| < |C_\beta| \approx 0.407506\ldots < 1$, whereas $2^m > 1$, the equality (11) is not possible. Therefore, $\Gamma \neq 0$. We take

$$\eta_1 = 2,\ \eta_2 = C_\alpha,\ \eta_3 = \alpha,\ b_1 = m,\ b_2 = -1,\ b_3 = -(n+2),\ l = 3,$$

where $\eta_1,\ \eta_2,\ \eta_3 \in \mathbb{Q}(\alpha)$ and $b_1,\ b_2,\ b_3 \in \mathbb{Z}$. The degree $d_{\mathbb{L}} = [\mathbb{Q}(\alpha) : \mathbb{Q}]$ is 3.

Since $m < n$, we take $D = \max\{1, m, n+2\} = n+2$. The logarithmic heights for η_1, η_2 and η_3 are calculated as follows:

$$h(\eta_1) = \log 2,\ h(\eta_2) = h(C_\alpha) = \frac{\log 31}{3} \text{ and } h(\eta_3) = h(\alpha) = \frac{\log \alpha}{3}.$$

Thus, we can take

$$A_1 = 2.08, \quad A_2 = 3.44 \quad \text{and} \quad A_3 = 0.39.$$

Applying Theorem 2, we have

$$\log |\Gamma| > -1.4 \cdot 30^6 3^{4.5} 3^2 (1 + \log 3)(1 + \log(n+2))(2.08)(3.44)(0.39).$$

Comparing the above inequality with (9) implies that

$$n \log \alpha < \log 3.59 + 7.54 \cdot 10^{12}(1 + \log(n+2)).$$

Thus, we get

$$n < 6.94 \cdot 10^{14}. \tag{12}$$

Our next aim is to reduce the bound on n. Set

$$\Lambda := m \log 2 - (n+2)\log\alpha - \log C_\alpha. \tag{13}$$

The inequality (9) can be written as

$$|e^\Lambda - 1| < 3.59\alpha^{-n}. \tag{14}$$

Observe that $\Lambda \neq 0$ as $|e^\Lambda - 1| = \Gamma \neq 0$. Since $n > 200$, the right side of (14) is smaller than $1/2$. The inequality $|e^x - 1| < y < 1/2$ for real values of x and y implies $|x| < 2y$. Thus, we get

$$|\Lambda| < 7.18 \cdot \alpha^{-n},$$

which implies that

$$|m\log 2 - (n+2)\log\alpha + \log(1/C_\alpha)| < 7.18\exp(-0.38Y),$$

where $Y := n < 6.94 \cdot 10^{14}$. We also have

$$\frac{\Lambda}{\log 2} = m - n\frac{\log\alpha}{\log 2} + \frac{\log(1/\alpha^2 C_\alpha)}{\log 2}.$$

Thus, we take

$$c = 7.18,\ \delta = 0.38,\ X_0 = 6.94 \cdot 10^{14},\ \psi = \left(\frac{\log(1/\alpha^2 C_\alpha)}{\log 2}\right)$$
$$\vartheta = \frac{\log\alpha}{\log 2},\ \vartheta_1 = -\log\alpha,\ \vartheta_2 = \log 2,\ \beta = \log(1/\alpha^2 C_\alpha).$$

We find that $\frac{p_{38}}{q_{38}} = \frac{453285617800432}{821969096806723}$ is the 38th convergent of ϑ such that $q_{38} > X_0$ and $q_{39} = 3140144568890233$ satisfies $\|q\psi\| > \frac{2X_0}{q}$. Applying Lemma 1, we get $n \leq 104$, which is a contradiction. Hence the theorem is proved.

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