Abstract
In this paper, we establish some combinatorial properties of a generalization of the hyper-Fibonacci numbers and then we apply these properties to extend the Cassini identity.

1. Introduction
Given two integers $\alpha$ and $\beta$, the Lucas sequence of the first kind $(u_n)_{n \geq 0}$ is defined by

$$\begin{cases} u_0 = 0, & u_1 = 1, \\ u_{n+2} = \alpha u_{n+1} + \beta u_n & (n \geq 0). \end{cases}$$

For $\alpha = \beta = 1$ we obtain the well known Fibonacci sequence $(F_n)_{n \geq 0}$. When $\beta \neq 0$ and $\alpha^2 + 4\beta \neq 0$, the Lucas sequence for negative subscripts is defined by $u_{-n} = (-\beta)^{-n}u_n$ for $n \geq 0$. If $\beta \neq 0$ and $\alpha^2 + 4\beta = 0$ then $\alpha = 2s$ and $\beta = -s^2$, where $s$ is the double root of $x^2 - \alpha x - \beta$. Thus, $u_n = ns^{n-1}$ and $u_{-n} = -ns^{-n-1}$ for $n \geq 0$.

If $(b_n)_{n \geq 0}$ and $(c_n)_{n \geq 0}$ are two sequences satisfying the recurrence relation $a_{n+2} = \alpha a_{n+1} + \beta a_n$, then we have the identity [10]

$$b_n c_{n-1} - b_{n-1} c_n = (-\beta)^{n-1}(b_1 c_0 - b_0 c_1).$$  

If we take $b_n = F_{n+2}$ and $c_n = F_{n+1}$, then the identity (1) reduces to

$$F_n F_{n+2} - F_{n+1}^2 = (-1)^{n+1},$$  

(2)
which is called the Cassini identity \[4, 7, 11\]. The identity (2) can also be written in matrix form as

$$\det\left(\begin{array}{cc} F_n & F_{n+1} \\ F_{n+1} & F_{n+2} \end{array}\right) = (-1)^{n+1}. \quad (3)$$

In \[5, 9\] we find a generalization of the Cassini identity for the \(p\)-Fibonacci numbers. Martinjak and Urbiha \[6\] extended the Cassini identity (3) to the hyper-Fibonacci numbers defined by

$$F_n^{(r+1)} = \sum_{k=0}^{n} F_k^{(r)}, \quad F_n^{(0)} = F_n, \quad F_0^{(r)} = 0, \quad F_1^{(r)} = 1, \quad (4)$$

where \(r\) is a nonnegative integer. The number \(F_n^{(r)}\) is called the \(n\)th hyper-Fibonacci number of the \(r\)th generation. Hyper-Fibonacci numbers were introduced by Dil and Mezö \[3\], they satisfy many interesting number-theoretical and combinatorial properties, e.g. \[2\]. Martinjak and Urbiha \[6\] defined the matrix

$$A_{r,n} = \begin{pmatrix} F_n^{(r)} & F_{n+1}^{(r)} & \cdots & F_{n+r+1}^{(r)} \\ F_n^{(r)} & F_{n+1}^{(r)} & \cdots & F_{n+r+2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ F_n^{(r)} & F_{n+r+1}^{(r)} & \cdots & F_{n+2r+2}^{(r)} \end{pmatrix}$$

and proved that

$$\det(A_{r,n}) = (-1)^{n+[r(r+3)/2]}. \quad (5)$$

If we take \(b_n = u_{n+2}\) and \(c_n = u_{n+1}\), then the identity (1) reduces to

$$u_n u_{n+2} - u_{n+1}^2 = (-1)^{n+1} \beta^n. \quad (6)$$

In Section 2 we define generalized hyper-Fibonacci numbers associated with the sequence \((u_n)_{n \geq 0}\) and we give some properties. In Section 3 we extend the identity (6) to these generalized hyper-Fibonacci numbers.

Throughout this paper we denote by \(C_k^n\) the binomial coefficient which is defined for a nonnegative integer \(n\) and an integer \(k\) by

$$C_k^n = \begin{cases} \frac{n!}{k!(n-k)!} & \text{if } 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}$$

and for a negative integer \(n\) and an integer \(k\) by

$$C_k^n = \begin{cases} (-1)^k C_{n+k}^{n-k-1} & \text{if } k \geq 0 \\ (-1)^{n-k} C_{-k}^{n-k} & \text{if } k \leq n \\ 0 & \text{otherwise}. \end{cases}$$
2. Generalized Hyper-Fibonacci Numbers

Classically, the hyper-sequence of the $r$th generation of a sequence $(T_n)_n$ is defined by $T_r^{(r+1)} = \sum_{k=0}^{n} T_r^{(k)}$. In this paper we define the generalized hyper-Fibonacci numbers associated with the Lucas sequence $(u_n)_n \geq 0$ by

$$u_n^{(r+1)} = \sum_{k=0}^{n} \alpha^{n-k} u_k^{(r)}, \quad u_n^{(0)} = u_n, \quad u_0^{(r)} = 0, \quad u_1^{(r)} = 1,$$

where $r$ is a nonnegative integer. The coefficients $\alpha^{n-k}$ are especially added in the definition of $(u_n^{(r)})_{n \geq 0}$ to get the following starting lemma. In the rest of this paper we assume that $\beta \neq 0$.

Lemma 1. If $n \geq 0$ is an integer, then

$$u_n^{(1)} = \frac{1}{\beta} u_{n+2} - \frac{\alpha^{n+1}}{\beta}.$$  \hspace{1cm} (8)

Proof. We prove the lemma by induction on $n$. For $n = 0$, the identity (8) is trivially checked. Now assume that (8) is true for an integer $n \geq 0$. Then

$$u_n^{(1)} = \sum_{k=0}^{n+1} \alpha^{n+1-k} u_k = \alpha \sum_{k=0}^{n} \alpha^{n-k} u_k + u_{n+1} = \alpha u_n^{(1)} + u_{n+1} = \alpha \left( \frac{1}{\beta} u_{n+2} - \frac{\alpha^{n+1}}{\beta} \right) + u_{n+1} = \frac{1}{\beta} u_{n+3} - \frac{\alpha^{n+2}}{\beta}.$$  \hspace{1cm}

We conclude that (8) is true for all $n \geq 0$. \hfill \Box

Proposition 1. Let $r \geq 0$ be an integer. Then

$$u_n^{(r+1)} = \frac{1}{\beta} u_n^{(r+2)} - C_n^{r+1} \frac{\alpha^{n+1}}{\beta}, \quad n \geq 0.$$  \hspace{1cm} (9)

Proof. We prove the proposition by induction on $r$. It follows from Lemma 1 that
We deduce that (9) is true for all $r \geq 0$. Now assume that (9) is true for an integer $r \geq 0$. Then
\[
\begin{align*}
 u_{n+2}(r+2) &= \sum_{k=0}^{n} \alpha^{n-k} u_{k}^{(r+1)} \\
 &= \sum_{k=0}^{n} \alpha^{n-k} \left( \frac{1}{\beta} u_{k+2}^{(r)} - C_{k+r+1}^{r} \frac{\alpha^{k+1}}{\beta} \right) \\
 &= \frac{1}{\beta} \sum_{k=0}^{n} \alpha^{n-k} u_{k+2}^{(r)} - \frac{1}{\beta} \sum_{k=0}^{n} C_{k+r+1}^{r} \alpha^{n+1} \\
 &= \frac{1}{\beta} \sum_{k=2}^{n+2} \alpha^{n+2-k} u_{k}^{(r)} - \frac{1}{\beta} \sum_{l=2}^{n+2} C_{l+r-1}^{r} \alpha^{n+1} \\
 &= \frac{1}{\beta} \left[ \sum_{l=0}^{n+2} \alpha^{n+2-l} u_{l}^{(r)} - \alpha^{n+1} \right] - \frac{1}{\beta} \left[ \sum_{l=1}^{n+2} C_{l+r-1}^{r} \alpha^{n+1} \right] \\
 &= \frac{1}{\beta} u_{n+2}^{(r+1)} - \frac{1}{\beta} \sum_{l=1}^{n+2} C_{l+r-1}^{r} \alpha^{n+1} \\
 &= \frac{1}{\beta} \left[ \frac{1}{\alpha} u_{n+1}^{(r+1)} + \beta u_{n}^{(r+1)} - \alpha^{n+1} \right] - \frac{1}{\beta} \sum_{l=1}^{n+2} C_{l+r-1}^{r} \alpha^{n+1}.
\end{align*}
\]

We deduce that (9) is true for all $r \geq 0$.

We get the following as a simple and immediate consequence, it allows us to define the generalized hyper-Fibonacci numbers of negative subscripts.

**Corollary 1.** Let $r \geq 0$ and $n \geq 0$ be integers. Then
\[
 u_{n+2}(r+1) = \alpha u_{n+1}^{(r+1)} + \beta u_{n}^{(r+1)} + C_{n+r+1}^{r} \alpha^{n+1}.
\]

**Proof.** According to definition (7), we have
\[
 u_{n+1}^{(r)} = u_{n+1}^{(r+1)} - \alpha u_{n}^{(r+1)}.
\]

From (9) and (10) we get
\[
 u_{n}^{(r+1)} = \frac{1}{\beta} u_{n+2}^{(r+1)} - \frac{\alpha}{\beta} u_{n+1}^{(r+1)} - C_{n+r+1}^{r} \frac{\alpha^{n+1}}{\beta}.
\]

We deduce that
\[
 u_{n+2}^{(r+1)} = \alpha u_{n+1}^{(r+1)} + \beta u_{n}^{(r+1)} + C_{n+r+1}^{r} \alpha^{n+1}.
\]

**Remark 1.** For $r \geq 1$ and $\alpha \neq 0$, the generalized hyper-Fibonacci numbers for negative subscripts are defined by
\[
 u_{-n}^{(r)} = \frac{1}{\beta} u_{-n+2}^{(r)} - \frac{\alpha}{\beta} u_{-n+1}^{(r)} - C_{-n+r}^{r-1} \frac{\alpha^{-n+1}}{\beta}, \quad n \geq 0.
\]
Theorem 1. Assume that $\alpha \neq 0$ and let $r \geq 0$ be an integer. Then

$$u_{n}^{(r)} = 0 \text{ for } 0 \leq n \leq r \quad \text{and} \quad u_{r-1}^{(r)} = \frac{1}{\beta} \left( \frac{-1}{\alpha} \right)^{r} \neq 0.$$  

If $\alpha = 0$, we have from (7) that $u_{n}^{(r)} = u_n$ for $n \geq 0$. Thus,

$$u_{n}^{(r)} = u_{-n}, \quad n \geq 0.$$  

The following theorem is the key behind all the results in Section 3.

**Theorem 1.** Assume that $\alpha \neq 0$ and let $r \geq 0$ be an integer. Then

$$u_{n+r+2}^{(r)} = \sum_{k=0}^{r+1} (-1)^{r+1-k} \alpha^{r-k} \left( \alpha^{2} C_{r+1}^{k-1} - \beta C_{r}^{k} \right) u_{n+k}^{(r)}, \quad n \geq -r, \quad (11)$$

where the coefficients $p_k = (-1)^{r+1-k} \alpha^{r-k} \left( \alpha^{2} C_{r+1}^{k-1} - \beta C_{r}^{k} \right)$ are unique.

**Proof.** Let us prove the theorem by induction on $r$. For $r = 0$ we get $u_{n+2} = \beta u_n + \alpha u_{n+1}$ which is true by definition of the sequence $(u_n)_{n \geq 0}$. Now assume that it is true for an integer $r \geq 0$. Since $n + 1 \geq n + r$, we deduce from the induction hypothesis that

$$u_{n+r+3}^{(r)} = \sum_{k=0}^{r+1} (-1)^{r+1-k} \alpha^{r-k} \left( \alpha^{2} C_{r+1}^{k-1} - \beta C_{r}^{k} \right) u_{n+k+1}^{(r)}.$$  

(12)

Since $n + r + 3 \geq n + r + 2 \geq 0$, we have from (7) that

$$u_{n+r+3}^{(r+1)} = u_{n+2}^{(r+1)} = u_{n+2}^{(r+1)} - \alpha u_{n+r+2}^{(r)}.$$  

(13)

On the other hand, for $k = 0, 1, \ldots, r + 1$, we have

$$u_{n+k+1}^{(r)} = u_{n+k+1}^{(r+1)} - \alpha u_{n+k}^{(r+1)}.$$  

(14)

Indeed,

- if $n + k + 1 \leq 0$ we obtain $0 = 0 - 0$,
- if $n + k + 1 > 0$ and $n + k \leq 0$, then $n + k = 0$ and we obtain $1 = 1 - 0$,
- if $n + k > 0$, then $n + k + 1 > 0$ and we obtain $u_{n+k+1}^{(r)} = u_{n+k+1}^{(r+1)} - \alpha u_{n+k}^{(r+1)}$.

We get from (12), (13) and (14) that

$$u_{n+r+3}^{(r+1)} - \alpha u_{n+r+2}^{(r+1)} = \sum_{k=0}^{r+1} (-1)^{r+1-k} \alpha^{r-k} \left( \alpha^{2} C_{r+1}^{k-1} - \beta C_{r}^{k} \right) \left( u_{n+k+1}^{(r+1)} - \alpha u_{n+k}^{(r+1)} \right).$$
We deduce that

\[ u^{(r+1)}_{n+r+3} = au^{(r+1)}_{n+r+2} + \sum_{k=0}^{r+1} (-1)^{r+1-k} \alpha^{r-k} \left( \alpha^2 C_{r+1} - \beta C_r \right) u^{(r+1)}_{n+k+1} \]

\[ + \sum_{k=0}^{r+1} (-1)^{-k} \alpha^{r+1-k} \left( \alpha^2 C_{r+1} - \beta C_r \right) u^{(r+1)}_{n+k} \]

\[ = (\alpha r + 2\alpha) u^{(r+1)}_{n+r+2} + \sum_{k=0}^{r} (-1)^{r+1-k} \alpha^{r-k} \left( \alpha^2 C_{r+1} - \beta C_r \right) u^{(r+1)}_{n+k+1} \]

\[ + (-\alpha)^{r+1} \beta u^{(r+1)}_{n+r+2} + \sum_{k=0}^{r+1} (-1)^{-k} \alpha^{r+1-k} \left( \alpha^2 C_{r+1} - \beta C_r \right) u^{(r+1)}_{n+k} \]

\[ = (\alpha r + 2\alpha) u^{(r+1)}_{n+r+2} + \sum_{k=0}^{r} (-1)^{r+1} \alpha^{r+1-k} \left( \alpha^2 C_{r+1} - \beta C_r \right) u^{(r+1)}_{n+k+1} \]

\[ + (-\alpha)^{r+1} \beta u^{(r+1)}_{n+r+2} + \sum_{k=0}^{r+1} (-1)^{-k} \alpha^{r+1-k} \left( \alpha^2 C_{r+1} - \beta C_r \right) u^{(r+1)}_{n+k} \]

\[ = \sum_{k=0}^{r+2} (-1)^{-k} \alpha^{r+1-k} \left( \alpha^2 C_{r+1} + \alpha^2 C_{r+1} - \beta C_r \right) u^{(r+1)}_{n+k} \]

\[ = \sum_{k=0}^{r+2} (-1)^{-k} \alpha^{r+1-k} \left( \alpha^2 C_{r+1} - \beta C_r \right) u^{(r+1)}_{n+k} . \]

We conclude that (11) is true for all \( r \geq 0. \)

Let us show the uniqueness of the coefficients \( p_k. \) Assume that for \( r \geq 0 \) we have

\[ u^{(r)}_{n+r+2} = \sum_{k=0}^{r+1} p_k u^{(r)}_{n+k} = \sum_{k=0}^{r+1} p_k' u^{(r)}_{n+k}, \quad p_k, p_k' \in \mathbb{Z}, n + r \geq 0. \]

Therefore,

\[ \sum_{k=0}^{r+1} (p_k - p_k') u^{(r)}_{n+k} = 0. \] (15)

For \( n = -r \) we get from (15) that \( p_{r+1} = p'_{r+1} \). Thus, (15) becomes

\[ \sum_{k=0}^{r} (p_k - p_k') u^{(r)}_{n+k} = 0. \]

For \( n = -r + 1, \) we obtain \( p_r = p'_{r}. \) Proceeding in the same way, we get

\[ p_k = p'_k, \quad 0 \leq k \leq r + 1. \]

\[ \square \]

**Corollary 2.** Let \( r \geq 0 \) be an integer and \( \left( F^{(r)}_n \right) \) be the hyper-Fibonacci numbers defined by (4). Then

\[ F^{(r)}_{n+r+2} = \sum_{k=0}^{r+1} (-1)^{-k} \left( C_r - C_{r+1} \right) F^{(r)}_{n+k}, \quad n \geq -r. \]
We conclude that (16) is true for all Proposition 2.

Let \( r \) be an integer. Then
\[
\begin{align*}
    u_n^{(r+1)} &= \frac{1}{\beta} u_n^{(r)} - C_{n+r+1}^r \frac{\alpha^{n+1}}{\beta} \\
    &= \frac{1}{\beta + 1} u_n^{(r)} - \frac{1}{\beta + 1} \sum_{k=1}^{n} C_{n+2r-2k}^{k-1} \frac{\alpha^{n+2r-2k+1}}{\beta^{r+2-k}} - C_{n+r+1}^r \frac{\alpha^{n+1}}{\beta} \\
    &= \frac{1}{\beta+r+1} u_n^{(r)} - \sum_{k=1}^{r+1} C_{n+2r+2}^{k-1} \frac{\alpha^{n+2(r+1)-2k+1}}{\beta^{r+2-k}}.
\end{align*}
\]
We conclude that (16) is true for all \( r \geq 1 \).

Remark 2. The coefficients \( p_n = (-1)^{-k} (C_r^k - C_{r+1}^k) \) given explicitly in Corollary 2 are the same as those given recursively in [6].

Proof. We prove the proposition by induction on \( r \). It follows from Lemma 1 that (16) is true for \( r = 1 \). Now assume that (16) is true for an integer \( r \geq 1 \). Then from (9) we get
\[
\begin{align*}
    u_n^{(r+1)} &= \frac{1}{\beta} u_n^{(r)} - C_{n+r+1}^r \frac{\alpha^{n+1}}{\beta} \\
    &= \frac{1}{\beta + 1} u_n^{(r)} - \frac{1}{\beta + 1} \sum_{k=1}^{n} C_{n+2r-2k}^{k-1} \frac{\alpha^{n+2r-2k+1}}{\beta^{r+2-k}} - C_{n+r+1}^r \frac{\alpha^{n+1}}{\beta} \\
    &= \frac{1}{\beta+r+1} u_n^{(r)} - \sum_{k=1}^{r+1} C_{n+2r+2}^{k-1} \frac{\alpha^{n+2(r+1)-2k+1}}{\beta^{r+2-k}}.
\end{align*}
\]

We conclude that (16) is true for all \( r \geq 1 \).
The next corollary gives an extension of the Binet formula to any positive generation of the generalized hyper-Fibonacci numbers.

**Corollary 4.** Let \( a = \left( \alpha + \sqrt{\alpha^2 + 4\beta} \right) / 2 \) and \( b = \left( \alpha - \sqrt{\alpha^2 + 4\beta} \right) / 2 \) with \( \alpha^2 + 4\beta \neq 0 \). Then for all \( r \geq 1 \) we have

\[
\frac{u_n^{(r)}}{\beta^r} = \frac{a^{n+2r} - b^{n+2r}}{a - b} - \sum_{k=1}^{r} C_{n+2r-k}^{k-1} \frac{\alpha^{n+2r-2k+1}}{\beta^{r+1-k}} , \quad n \geq 0.
\]

**Proof.** It is well known [8] that

\[ u_n = \frac{a^n - b^n}{a - b} , \quad n \geq 0. \]

Thus, the corollary follows from Proposition 2. \( \square \)

### 3. Cassini Identity for the Generalized Hyper-Fibonacci Numbers

Assume that \( \alpha \neq 0 \). We deduce from Theorem 1 that the generalized hyper-Fibonacci numbers \( \left( u_n^{(r)} \right)_n \) can be defined by the vector recurrence relation

\[
\begin{pmatrix}
\begin{array}{c}
\vdots \\
u_{n+1}^{(r)} \\
u_n^{(r)} \\
u_{n+r+2}^{(r)} \\
u_{n+1}^{(r)} \\
u_n^{(r)} \\
u_{n+r+1}^{(r)}
\end{array}
\end{pmatrix} = \begin{pmatrix} U_{r+2} \\ \vdots \end{pmatrix},
\]

where \( n + r \geq 0 \). The matrix \( U_{r+2} \) is defined by

\[
U_{r+2} = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 \\
p_0 & p_1 & p_2 & \cdots & p_r & p_{r+1}
\end{pmatrix}.
\]

The coefficients \( p_k \), for \( k = 0, 1, \ldots, r + 1 \), are those described in Theorem 1. For \( \alpha, n, r \in \mathbb{Z} \) such that \( n \geq 0 \) and \( r \geq 0 \), let us define another matrix by

\[
B_{r,n} = \begin{pmatrix}
\begin{array}{cccc}
\vdots \\
u_n^{(r)} & u_{n+1}^{(r)} & \cdots & u_{n+r+1}^{(r)} \\
\vdots \\
u_n^{(r)} & u_{n+1}^{(r)} & \cdots & u_{n+r+2}^{(r)} \\
\vdots \\
u_{n+r+1}^{(r)} & u_{n+r+2}^{(r)} & \cdots & u_{n+2r+2}^{(r)}
\end{array}
\end{pmatrix}.
\]
Lemma 2. Let $\alpha, n, r \in \mathbb{Z}$ such that $\alpha \neq 0, n \geq 0$ and $r \geq 0$. Then

$$B_{r,n} = U_{r+2}^n B_{r,0}.$$ 

Proof. From the relation (19) we can write

$$B_{r,n} = U_{r+2} B_{r,n-1} = U_{r+2}^2 B_{r,n-2} = \ldots = U_{r+2}^n B_{r,0}.$$

Lemma 3. Assume that $\alpha \neq 0$ and let $r \geq 0$ be an integer. Then

$$\det(U_{r+2}) = -\alpha^r \beta.$$ 

Proof. It is clear that

$$\det(U_{r+2}) = (-1)^{r+3} p_0 = (-1)^{r+3} (-1)^{r+1} \alpha^r (-\beta) = -\alpha^r \beta.$$

We are now ready to extend the Cassini identity (5).

Theorem 2. Let $\alpha, n, r \in \mathbb{Z}$ such that $n \geq 0$ and $r \geq 0$. Then

$$\det(B_{r,n}) = (-1)^{n+\lceil (r+3)/2 \rceil} \alpha^{r+2} \beta^{n+r}.$$ 

Proof. Assume that $\alpha \neq 0$. We deduce from (19) that the multiplication by $U_{r+2}^{-1}$ decreases by 1 the subscript of each component, i.e.,

$$U_{r+2}^{-1} B_{r,0} = \begin{pmatrix}
  u_{-1}^{(r)} & u_0^{(r)} & \cdots & u_r^{(r)} \\
  u_{-1}^{(r)} & u_0^{(r)} & \cdots & u_r^{(r)} \\
  \vdots & \vdots & \ddots & \vdots \\
  u_{-1}^{(r)} & u_0^{(r)} & \cdots & u_r^{(r)}
\end{pmatrix}.$$

Thus,

$$U_{r+2}^{-r} B_{r,0} = \begin{pmatrix}
  u_{-r}^{(r)} & u_{-r+1}^{(r)} & \cdots & u_1^{(r)} \\
  u_{-r}^{(r)} & u_{-r+1}^{(r)} & \cdots & u_1^{(r)} \\
  \vdots & \vdots & \ddots & \vdots \\
  u_{-r}^{(r)} & u_{-r+1}^{(r)} & \cdots & u_1^{(r)}
\end{pmatrix}.$$ 

Since $u_{-n}^{(r)} = 0$ for $0 \leq n \leq r$ and $u_1^{(r)} = 1$, then

$$U_{r+2}^{-r} B_{r,0} = \begin{pmatrix}
  0 & 0 & \cdots & 0 & 1 \\
  0 & 0 & \cdots & 1 & u_2^{(r)} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 1 & \cdots & u_r^{(r)} & u_{r+1}^{(r)} \\
  1 & u_2^{(r)} & \cdots & u_{r+1}^{(r)} & u_{r+2}^{(r)}
\end{pmatrix}.$$
Thus,
\[
\det \left( U_{r+2}^{-r} B_{r,0} \right) = (-1)^{\lfloor (r+2)/2 \rfloor}.
\]
From Lemma 3 we get
\[
\det(B_{r,0}) = (-1)^{\lfloor (r+3)/2 \rfloor} \alpha r^2 \beta^r.
\]
Finally, we deduce from Lemmas 2 and 3 that
\[
\det(B_{r,n}) = (-1)^{n + \lfloor (r+3)/2 \rfloor} \alpha^{rn+r^2} \beta^{n+r}.
\]
If \( \alpha = 0 \), we know that \( u_n^{(r)} = u_n \) for \( n \geq 0 \) and it is easy to see that
\[
\begin{aligned}
\begin{cases}
  u_{2n} = 0 \\
  u_{2n+1} = \beta^n
\end{cases}, & n \geq 0.
\end{aligned}
\]
Thus, \( \det(B_{0,n}) \) is given by (6). For \( r \geq 1 \), it is clear that \( \det(B_{r,n}) = 0 \).

References


