GENERALIZED HYPER-FIBONACCI NUMBERS AND APPLICATIONS

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Abstract

In this paper, we establish some combinatorial properties of a generalization of the hyper-Fibonacci numbers and then we apply these properties to extend the Cassini identity.

1. Introduction

Given two integers α and β, the Lucas sequence of the first kind $(u_n)_{n\geq 0}$ is defined by

$$\begin{cases} u_0 = 0, \ u_1 = 1, \\ u_{n+2} = \alpha u_{n+1} + \beta u_n \qquad (n \geq 0). \end{cases}$$

For $\alpha = \beta = 1$ we obtain the well known Fibonacci sequence $(F_n)_{n\geq 0}$. When $\beta \neq 0$ and $\alpha^2 + 4\beta \neq 0$, the Lucas sequence for negative subscripts is defined by $u_{-n} = -(-\beta)^{-n} u_n$ for $n \geq 0$. If $\beta \neq 0$ and $\alpha^2 + 4\beta = 0$ then $\alpha = 2s$ and $\beta = -s^2$, where s is the double root of $x^2 - \alpha x - \beta$. Thus, $u_n = ns^{n-1}$ and $u_{-n} = -ns^{-n-1}$ for $n \geq 0$.

If $(b_n)_{n\geq 0}$ and $(c_n)_{n\geq 0}$ are two sequences satisfying the recurrence relation $a_{n+2} = \alpha a_{n+1} + \beta a_n$, then we have the identity [10]

$$b_n c_{n-1} - b_{n-1} c_n = (-\beta)^{n-1}(b_1 c_0 - b_0 c_1). \tag{1}$$

If we take $b_n = F_{n+2}$ and $c_n = F_{n+1}$, then the identity (1) reduces to

$$F_n F_{n+2} - F_{n+1}^2 = (-1)^{n+1}, \tag{2}$$

which is called the Cassini identity [4, 7, 11]. The identity (2) can also be written in matrix form as

$$\det\begin{pmatrix} F_n & F_{n+1} \\ F_{n+1} & F_{n+2} \end{pmatrix} = (-1)^{n+1}. \tag{3}$$

In [5, 9] we find a generalization of the Cassini identity for the p-Fibonacci numbers. Martinjak and Urbiha [6] extended the Cassini identity (3) to the hyper-Fibonacci numbers defined by

$$F_n^{(r+1)} = \sum_{k=0}^{n} F_k^{(r)}, \quad F_n^{(0)} = F_n, \quad F_0^{(r)} = 0, \quad F_1^{(r)} = 1, \tag{4}$$

where r is a nonnegative integer. The number $F_n^{(r)}$ is called the nth hyper-Fibonacci number of the rth generation. Hyper-Fibonacci numbers were introduced by Dil and Mező [3], they satisfy many interesting number-theoretical and combinatorial properties, e.g. [2]. Martinjak and Urbiha [6] defined the matrix

$$A_{r,n} = \begin{pmatrix} F_n^{(r)} & F_{n+1}^{(r)} & \cdots & F_{n+r+1}^{(r)} \\ F_{n+1}^{(r)} & F_{n+2}^{(r)} & \cdots & F_{n+r+2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n+r+1}^{(r)} & F_{n+r+2}^{(r)} & \cdots & F_{n+2r+2}^{(r)} \end{pmatrix}$$

and proved that

$$\det(A_{r,n}) = (-1)^{n+\lfloor (r+3)/2 \rfloor}. \tag{5}$$

If we take $b_n = u_{n+2}$ and $c_n = u_{n+1}$, then the identity (1) reduces to

$$u_n u_{n+2} - u_{n+1}^2 = (-1)^{n+1}\beta^n. \tag{6}$$

In Section 2 we define generalized hyper-Fibonacci numbers associated with the sequence $(u_n)_{n\geq 0}$ and we give some properties. In Section 3 we extend the identity (6) to these generalized hyper-Fibonacci numbers.

Throughout this paper we denote by C_n^k the binomial coefficient which is defined for a nonnegative integer n and an integer k by

$$C_n^k = \begin{cases} \dfrac{n!}{k!(n-k)!} & \text{if } \ 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}$$

and for a negative integer n and an integer k by

$$C_n^k = \begin{cases} (-1)^k C_{-n+k-1}^{k} & \text{if } \ k \geq 0 \\ (-1)^{n-k} C_{-k-1}^{n-k} & \text{if } \ k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

2. Generalized Hyper-Fibonacci Numbers

Classically, the hyper-sequence of the rth generation of a sequence $(T_n)_n$ is defined by $T_n^{(r+1)} = \sum_{k=0}^{n} T_k^{(r)}$. In this paper we define the generalized hyper-Fibonacci numbers associated with the Lucas sequence $(u_n)_{n\geq 0}$ by

$$u_n^{(r+1)} = \sum_{k=0}^{n} \alpha^{n-k} u_k^{(r)}, \quad u_n^{(0)} = u_n, \quad u_0^{(r)} = 0, \quad u_1^{(r)} = 1, \tag{7}$$

where r is a nonnegative integer. The coefficients α^{n-k} are especially added in the definition of $\left(u_n^{(r)}\right)_{n\geq 0}$ to get the following starting lemma. In the rest of this paper we assume that $\beta \neq 0$.

Lemma 1. *If $n \geq 0$ is an integer, then*

$$u_n^{(1)} = \frac{1}{\beta} u_{n+2} - \frac{\alpha^{n+1}}{\beta}. \tag{8}$$

Proof. We prove the lemma by induction on n. For $n = 0$, the identity (8) is trivially checked. Now assume that (8) is true for an integer $n \geq 0$. Then

$$\begin{aligned} u_{n+1}^{(1)} &= \sum_{k=0}^{n+1} \alpha^{n+1-k} u_k \\ &= \alpha \sum_{k=0}^{n} \alpha^{n-k} u_k + u_{n+1} \\ &= \alpha u_n^{(1)} + u_{n+1} \\ &= \alpha \left(\frac{1}{\beta} u_{n+2} - \frac{\alpha^{n+1}}{\beta} \right) + u_{n+1} \\ &= \frac{1}{\beta} u_{n+3} - \frac{\alpha^{n+2}}{\beta}. \end{aligned}$$

We conclude that (8) is true for all $n \geq 0$. □

Proposition 1. *Let $r \geq 0$ be an integer. Then*

$$u_n^{(r+1)} = \frac{1}{\beta} u_{n+2}^{(r)} - C_{n+r+1}^{r} \frac{\alpha^{n+1}}{\beta}, \quad n \geq 0. \tag{9}$$

Proof. We prove the proposition by induction on r. It follows from Lemma 1 that

(9) is true for $r = 0$. Now assume that (9) is true for an integer $r \geq 0$. Then

$$\begin{aligned}
u_n^{(r+2)} &= \sum_{k=0}^{n} \alpha^{n-k} u_k^{(r+1)} \\
&= \sum_{k=0}^{n} \alpha^{n-k} \left(\frac{1}{\beta} u_{k+2}^{(r)} - C_{k+r+1}^{r} \frac{\alpha^{k+1}}{\beta} \right) \\
&= \frac{1}{\beta} \sum_{k=0}^{n} \alpha^{n-k} u_{k+2}^{(r)} - \frac{1}{\beta} \sum_{k=0}^{n} C_{k+r+1}^{r} \alpha^{n+1} \\
&= \frac{1}{\beta} \sum_{l=2}^{n+2} \alpha^{n+2-l} u_l^{(r)} - \frac{1}{\beta} \sum_{l=2}^{n+2} C_{l+r-1}^{r} \alpha^{n+1} \\
&= \frac{1}{\beta} \left[\sum_{l=0}^{n+2} \alpha^{n+2-l} u_l^{(r)} - \alpha^{n+1} \right] - \frac{1}{\beta} \left[\sum_{l=1}^{n+2} C_{l+r-1}^{r} \alpha^{n+1} - \alpha^{n+1} \right] \\
&= \frac{1}{\beta} u_{n+2}^{(r+1)} - \frac{1}{\beta} \sum_{l=1}^{n+2} C_{l+r-1}^{r} \alpha^{n+1} \\
&= \frac{1}{\beta} u_{n+2}^{(r+1)} - C_{n+r+2}^{r+1} \frac{\alpha^{n+1}}{\beta}.
\end{aligned}$$

We deduce that (9) is true for all $r \geq 0$. □

We get the following as a simple and immediate consequence, it allows us to define the generalized hyper-Fibonacci numbers of negative subscripts.

Corollary 1. *Let $r \geq 0$ and $n \geq 0$ be integers. Then*

$$u_{n+2}^{(r+1)} = \alpha u_{n+1}^{(r+1)} + \beta u_n^{(r+1)} + C_{n+r+1}^{r} \alpha^{n+1}.$$

Proof. According to definition (7), we have

$$u_{n+1}^{(r)} = u_{n+1}^{(r+1)} - \alpha u_n^{(r+1)}. \tag{10}$$

From (9) and (10) we get

$$u_n^{(r+1)} = \frac{1}{\beta} u_{n+2}^{(r+1)} - \frac{\alpha}{\beta} u_{n+1}^{(r+1)} - C_{n+r+1}^{r} \frac{\alpha^{n+1}}{\beta}.$$

We deduce that

$$u_{n+2}^{(r+1)} = \alpha u_{n+1}^{(r+1)} + \beta u_n^{(r+1)} + C_{n+r+1}^{r} \alpha^{n+1}.$$

□

Remark 1. For $r \geq 1$ and $\alpha \neq 0$, the generalized hyper-Fibonacci numbers for negative subscripts are defined by

$$u_{-n}^{(r)} = \frac{1}{\beta} u_{-n+2}^{(r)} - \frac{\alpha}{\beta} u_{-n+1}^{(r)} - C_{-n+r}^{r-1} \frac{\alpha^{-n+1}}{\beta}, \quad n \geq 0.$$

It is easy to see that

$$u_{-n}^{(r)} = 0 \quad \text{for } 0 \le n \le r \quad \text{and} \quad u_{-r-1}^{(r)} = \frac{1}{\beta}\left(\frac{-1}{\alpha}\right)^r \ne 0.$$

If $\alpha = 0$, we have from (7) that $u_n^{(r)} = u_n$ for $n \ge 0$. Thus,

$$u_{-n}^{(r)} = u_{-n}, \quad n \ge 0.$$

The following theorem is the key behind all the results in Section 3.

Theorem 1. *Assume that $\alpha \ne 0$ and let $r \ge 0$ be an integer. Then*

$$u_{n+r+2}^{(r)} = \sum_{k=0}^{r+1}(-1)^{r+1-k}\alpha^{r-k}\left(\alpha^2 C_{r+1}^{k-1} - \beta C_r^k\right)u_{n+k}^{(r)}, \qquad n \ge -r, \tag{11}$$

where the coefficients $p_k = (-1)^{r+1-k}\alpha^{r-k}\left(\alpha^2 C_{r+1}^{k-1} - \beta C_r^k\right)$ are unique.

Proof. Let us prove the theorem by induction on r. For $r = 0$ we get $u_{n+2} = \beta u_n + \alpha u_{n+1}$ which is true by definition of the sequence $(u_n)_{n\ge 0}$. Now assume that it is true for an integer $r \ge 0$. Since $n+1 \ge n \ge -r$, we deduce from the induction hypothesis that

$$u_{n+r+3}^{(r)} = \sum_{k=0}^{r+1}(-1)^{r+1-k}\alpha^{r-k}\left(\alpha^2 C_{r+1}^{k-1} - \beta C_r^k\right)u_{n+k+1}^{(r)}. \tag{12}$$

Since $n+r+3 \ge n+r+2 \ge 0$, we have from (7) that

$$u_{n+r+3}^{(r)} = u_{n+r+3}^{(r+1)} - \alpha u_{n+r+2}^{(r+1)}. \tag{13}$$

On the other hand, for $k = 0, 1\ldots, r+1$, we have

$$u_{n+k+1}^{(r)} = u_{n+k+1}^{(r+1)} - \alpha u_{n+k}^{(r+1)}. \tag{14}$$

Indeed,

- if $n+k+1 \le 0$ we obtain $0 = 0 - 0$,
- if $n+k+1 > 0$ and $n+k \le 0$, then $n+k = 0$ and we obtain $1 = 1 - 0$,
- if $n+k > 0$, then $n+k+1 > 0$ and we obtain $u_{n+k+1}^{(r)} = u_{n+k+1}^{(r+1)} - \alpha u_{n+k}^{(r+1)}$.

We get from (12), (13) and (14) that

$$u_{n+r+3}^{(r+1)} - \alpha u_{n+r+2}^{(r+1)} = \sum_{k=0}^{r+1}(-1)^{r+1-k}\alpha^{r-k}\left(\alpha^2 C_{r+1}^{k-1} - \beta C_r^k\right)\left(u_{n+k+1}^{(r+1)} - \alpha u_{n+k}^{(r+1)}\right).$$

We deduce that

$$\begin{aligned} u_{n+r+3}^{(r+1)} &= \alpha u_{n+r+2}^{(r+1)} + \sum_{k=0}^{r+1} (-1)^{r+1-k} \alpha^{r-k} \left(\alpha^2 C_{r+1}^{k-1} - \beta C_r^k\right) u_{n+k+1}^{(r+1)} \\ &\quad + \sum_{k=0}^{r+1} (-1)^{r-k} \alpha^{r+1-k} \left(\alpha^2 C_{r+1}^{k-1} - \beta C_r^k\right) u_{n+k}^{(r+1)} \\ &= (\alpha r + 2\alpha) u_{n+r+2}^{(r+1)} + \sum_{k=0}^{r} (-1)^{r+1-k} \alpha^{r-k} \left(\alpha^2 C_{r+1}^{k-1} - \beta C_r^k\right) u_{n+k+1}^{(r+1)} \\ &\quad + (-\alpha)^{r+1} \beta u_n^{(r+1)} + \sum_{k=1}^{r+1} (-1)^{r-k} \alpha^{r+1-k} \left(\alpha^2 C_{r+1}^{k-1} - \beta C_r^k\right) u_{n+k}^{(r+1)} \\ &= (\alpha r + 2\alpha) u_{n+r+2}^{(r+1)} + \sum_{l=1}^{r+1} (-1)^{r-l} \alpha^{r+1-l} \left(\alpha^2 C_{r+1}^{l-2} - \beta C_r^{l-1}\right) u_{n+l}^{(r+1)} \\ &\quad + (-\alpha)^{r+1} \beta u_n^{(r+1)} + \sum_{k=1}^{r+1} (-1)^{r-k} \alpha^{r+1-k} \left(\alpha^2 C_{r+1}^{k-1} - \beta C_r^k\right) u_{n+k}^{(r+1)} \\ &= \sum_{k=0}^{r+2} (-1)^{r-k} \alpha^{r+1-k} \left(\alpha^2 C_{r+1}^{k-2} + \alpha^2 C_{r+1}^{k-1} - \beta C_r^{k-1} - \beta C_r^k\right) u_{n+k}^{(r+1)} \\ &= \sum_{k=0}^{r+2} (-1)^{r-k} \alpha^{r+1-k} \left(\alpha^2 C_{r+2}^{k-1} - \beta C_{r+1}^k\right) u_{n+k}^{(r+1)}. \end{aligned}$$

We conclude that (11) is true for all $r \geq 0$.

Let us show the uniqueness of the coefficients p_k. Assume that for $r \geq 0$ we have

$$u_{n+r+2}^{(r)} = \sum_{k=0}^{r+1} p_k u_{n+k}^{(r)} = \sum_{k=0}^{r+1} p'_k u_{n+k}^{(r)}, \quad p_k, p'_k \in \mathbb{Z}, n + r \geq 0.$$

Therefore,

$$\sum_{k=0}^{r+1} \left(p_k - p'_k\right) u_{n+k}^{(r)} = 0. \tag{15}$$

For $n = -r$ we get from (15) that $p_{r+1} = p'_{r+1}$. Thus, (15) becomes

$$\sum_{k=0}^{r} \left(p_k - p'_k\right) u_{n+k}^{(r)} = 0.$$

For $n = -r + 1$, we obtain $p_r = p'_r$. Proceeding in the same way, we get

$$p_k = p'_k, \qquad 0 \leq k \leq r + 1.$$

□

Corollary 2. *Let $r \geq 0$ be an integer and $\left(F_n^{(r)}\right)_n$ be the hyper-Fibonacci numbers defined by (4). Then*

$$F_{n+r+2}^{(r)} = \sum_{k=0}^{r+1} (-1)^{r-k} \left(C_r^k - C_{r+1}^{k-1}\right) F_{n+k}^{(r)}, \qquad n \geq -r.$$

Proof. The corollary follows from the identity (11) with $\alpha = \beta = 1$. □

Remark 2. The coefficients $p_k = (-1)^{r-k}\left(C_r^k - C_{r+1}^{k-1}\right)$ given explicitly in Corollary 2 are the same as those given recursively in [6].

Let us spread Lemma 1 to the rth generation.

Proposition 2. *Let $r > 0$ be an integer. Then*

$$u_n^{(r)} = \frac{1}{\beta^r} u_{n+2r} - \sum_{k=1}^{r} C_{n+2r-k}^{k-1} \frac{\alpha^{n+2r-2k+1}}{\beta^{r+1-k}}, \quad n \geq 0. \tag{16}$$

Proof. We prove the proposition by induction on r. It follows from Lemma 1 that (16) is true for $r = 1$. Now assume that (16) is true for an integer $r \geq 1$. Then from (9) we get

$$\begin{aligned} u_n^{(r+1)} &= \frac{1}{\beta} u_{n+2}^{(r)} - C_{n+r+1}^{r} \frac{\alpha^{n+1}}{\beta} \\ &= \frac{1}{\beta^{r+1}} u_{n+2+2r} - \sum_{k=1}^{r} C_{n+2+2r-k}^{k-1} \frac{\alpha^{n+2+2r-2k+1}}{\beta^{r+2-k}} - C_{n+r+1}^{r} \frac{\alpha^{n+1}}{\beta} \\ &= \frac{1}{\beta^{r+1}} u_{n+2(r+1)} - \sum_{k=1}^{r+1} C_{n+2(r+1)-k}^{k-1} \frac{\alpha^{n+2(r+1)-2k+1}}{\beta^{r+2-k}}. \end{aligned}$$

We conclude that (16) is true for all $r \geq 1$. □

Recall the formula [1]

$$u_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} C_{n-k-1}^{k} \alpha^{n-2k-1} \beta^k, \quad n \geq 1, \tag{17}$$

that we are now able to generalize to any generation r.

Corollary 3. *Let $n > 0$ and $r \geq 0$ be integers. Then*

$$u_n^{(r)} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} C_{n+r-k-1}^{k+r} \alpha^{n-2k-1} \beta^k. \tag{18}$$

Proof. If $r = 0$, then (18) is simply (17). Thus, assume that $r \geq 1$. From Proposition 2 and the identity (17) we get

$$\begin{aligned} u_n^{(r)} &= \frac{1}{\beta} u_{n+2r} - \sum_{k=1}^{r} C_{n+2r-k}^{k-1} \frac{\alpha^{n+2r-2k+1}}{\beta^{r+1-k}} \\ &= \sum_{k=0}^{\lfloor \frac{n+2r-1}{2} \rfloor} C_{n+2r-k-1}^{k} \frac{\alpha^{n+2r-2k-1}}{\beta^{r-k}} - \sum_{k=0}^{r-1} C_{n+2r-k-1}^{k} \frac{\alpha^{n+2r-2k-1}}{\beta^{r-k}} \\ &= \sum_{k=r}^{\lfloor \frac{n+2r-1}{2} \rfloor} C_{n+2r-k-1}^{k} \frac{\alpha^{n+2r-2k-1}}{\beta^{r-k}} \\ &= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} C_{n+r-k-1}^{k+r} \alpha^{n-2k-1} \beta^k. \end{aligned}$$

□

The next corollary gives an extension of the Binet formula to any positive generation of the generalized hyper-Fibonacci numbers.

Corollary 4. *Let* $a = \left(\alpha + \sqrt{\alpha^2 + 4\beta}\right)/2$ *and* $b = \left(\alpha - \sqrt{\alpha^2 + 4\beta}\right)/2$ *with* $\alpha^2 + 4\beta \neq 0$. *Then for all* $r \geq 1$ *we have*

$$u_n^{(r)} = \frac{1}{\beta^r}\frac{a^{n+2r} - b^{n+2r}}{a-b} - \sum_{k=1}^{r} C_{n+2r-k}^{k-1}\frac{\alpha^{n+2r-2k+1}}{\beta^{r+1-k}}, \quad n \geq 0.$$

Proof. It is well known [8] that

$$u_n = \frac{a^n - b^n}{a-b}, \qquad n \geq 0.$$

Thus, the corollary follows from Proposition 2. □

3. Cassini Identity for the Generalized Hyper-Fibonacci Numbers

Assume that $\alpha \neq 0$. We deduce from Theorem 1 that the generalized hyper-Fibonacci numbers $\left(u_n^{(r)}\right)_n$ can be defined by the vector recurrence relation

$$\begin{pmatrix} u_{n+1}^{(r)} \\ u_{n+2}^{(r)} \\ \vdots \\ u_{n+r+2}^{(r)} \end{pmatrix} = U_{r+2}\begin{pmatrix} u_{n}^{(r)} \\ u_{n+1}^{(r)} \\ \vdots \\ u_{n+r+1}^{(r)} \end{pmatrix}, \tag{19}$$

where $n + r \geq 0$. The matrix U_{r+2} is defined by

$$U_{r+2} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ p_0 & p_1 & p_2 & \cdots & p_r & p_{r+1} \end{pmatrix}.$$

The coefficients p_k, for $k = 0, 1, \ldots, r+1$, are those described in Theorem 1. For $\alpha, n, r \in \mathbb{Z}$ such that $n \geq 0$ and $r \geq 0$, let us define another matrix by

$$B_{r,n} = \begin{pmatrix} u_n^{(r)} & u_{n+1}^{(r)} & \cdots & u_{n+r+1}^{(r)} \\ u_{n+1}^{(r)} & u_{n+2}^{(r)} & \cdots & u_{n+r+2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n+r+1}^{(r)} & u_{n+r+2}^{(r)} & \cdots & u_{n+2r+2}^{(r)} \end{pmatrix}.$$

Lemma 2. *Let $\alpha, n, r \in \mathbb{Z}$ such that $\alpha \neq 0, n \geq 0$ and $r \geq 0$. Then*

$$B_{r,n} = U_{r+2}^n B_{r,0}.$$

Proof. From the relation (19) we can write $B_{r,n} = U_{r+2} B_{r,n-1}$. It follows that

$$B_{r,n} = U_{r+2} B_{r,n-1} = U_{r+2}^2 B_{r,n-2} = \ldots = U_{r+2}^n B_{r,0}.$$

□

Lemma 3. *Assume that $\alpha \neq 0$ and let $r \geq 0$ be an integer. Then*

$$\det(U_{r+2}) = -\alpha^r \beta.$$

Proof. It is clear that

$$\det(U_{r+2}) = (-1)^{r+3} p_0 = (-1)^{r+3}(-1)^{r+1}\alpha^r(-\beta) = -\alpha^r \beta.$$

□

We are now ready to extend the Cassini identity (5).

Theorem 2. *Let $\alpha, n, r \in \mathbb{Z}$ such that $n \geq 0$ and $r \geq 0$. Then*

$$\det(B_{r,n}) = (-1)^{n+\lfloor (r+3)/2 \rfloor} \alpha^{rn+r^2} \beta^{n+r}.$$

Proof. Assume that $\alpha \neq 0$. We deduce from (19) that the multiplication by U_{r+2}^{-1} decreases by 1 the subscript of each component, i.e.,

$$U_{r+2}^{-1} B_{r,0} = \begin{pmatrix} u_{-1}^{(r)} & u_0^{(r)} & \cdots & u_r^{(r)} \\ u_0^{(r)} & u_1^{(r)} & \cdots & u_{r+1}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ u_r^{(r)} & u_{r+1}^{(r)} & \cdots & u_{2r+1}^{(r)} \end{pmatrix}.$$

Thus,

$$U_{r+2}^{-r} B_{r,0} = \begin{pmatrix} u_{-r}^{(r)} & u_{1-r}^{(r)} & \cdots & u_1^{(r)} \\ u_{1-r}^{(r)} & u_{2-r}^{(r)} & \cdots & u_2^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{(r)} & u_2^{(r)} & \cdots & u_{r+2}^{(r)} \end{pmatrix}.$$

Since $u_{-n}^{(r)} = 0$ for $0 \leq n \leq r$ and $u_1^{(r)} = 1$, then

$$U_{r+2}^{-r} B_{r,0} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & u_2^{(r)} \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & u_r^{(r)} & u_{r+1}^{(r)} \\ 1 & u_2^{(r)} & \cdots & u_{r+1}^{(r)} & u_{r+2}^{(r)} \end{pmatrix}.$$

Thus,

$$\det\left(U_{r+2}^{-r}B_{r,0}\right) = (-1)^{\lfloor (r+2)/2 \rfloor}.$$

From Lemma 3 we get

$$\det(B_{r,0}) = (-1)^{\lfloor (r+3)/2 \rfloor}\alpha^{r^2}\beta^{r}.$$

Finally, we deduce from Lemmas 2 and 3 that

$$\det\left(B_{r,n}\right) = (-1)^{n+\lfloor (r+3)/2 \rfloor}\alpha^{rn+r^2}\beta^{n+r}.$$

If $\alpha = 0$, we know that $u_n^{(r)} = u_n$ for $n \geq 0$ and it is easy to see that

$$\begin{cases} u_{2n} = 0 \\ u_{2n+1} = \beta^n \end{cases}, \qquad n \geq 0.$$

Thus, $\det\left(B_{0,n}\right)$ is given by (6). For $r \geq 1$, it is clear that $\det\left(B_{r,n}\right) = 0$. □

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