



**PARTIAL SUMS OF HORADAM SEQUENCES: SUM-FREE
REPRESENTATIONS VIA GENERATING FUNCTIONS**

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Abstract

Horadam sequences and their partial sums are computed via generating functions. The results are as simple as possible.

1. Introduction

Horadam sequences [1, 2] $W_n = W_n(a, b; p, q)$ are defined via

$$W_n = pW_{n-1} + qW_{n-2}, \quad n \geq 2, \quad W_0 = a, \quad W_1 = b.$$

These numbers are of course a generalization of Fibonacci numbers, Lucas numbers and many others. The characteristic equation $X^2 - pX - q = 0$ is essential, and the two roots are

$$\lambda = \frac{p + \sqrt{p^2 + 4q}}{2}, \quad \mu = \frac{p - \sqrt{p^2 + 4q}}{2}.$$

We define

$$F_n = \frac{\lambda^n - \mu^n}{\lambda - \mu} \quad \text{and} \quad L_n = \lambda^n + \mu^n,$$

as these sequences resemble Fibonacci and Lucas numbers, respectively, and each solution of the recursion may be expressed as a linear combination of these two sequences. We have $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, $L_1 = p$. For instance

$$W_n = \left(b - \frac{ap}{2}\right)F_n + \frac{a}{2}L_n.$$

The paper [1] concentrates on finding expressions for

$$\sum_{n \leq k \leq n+m} W_k = \sum_{0 \leq k \leq n+m} W_k - \sum_{0 \leq k \leq n-1} W_k.$$

In the rest of this short paper, we will find simple expressions for

$$S_n := \sum_{0 \leq k \leq n} W_k$$

using generating functions. The results do not contain summations, and can be expressed with the sequences F_n and L_n .

2. Generating Functions

Standard computations produce

$$W(z) = \sum_{k \geq 0} W_k z^k = \frac{a + z(b - pa)}{1 - pz - qz^2};$$

furthermore

$$F(z) = \sum_{k \geq 0} F_k z^k = \frac{z}{1 - pz - qz^2} \quad \text{and} \quad L(z) = \sum_{k \geq 0} L_k z^k = \frac{2 - pz}{1 - pz - qz^2}.$$

By general principles,

$$\begin{aligned} S(z) &= \sum_{k \geq 0} S_k z^k = \frac{1}{1 - z} \frac{a + z(b - pa)}{1 - pz - qz^2} \\ &= \frac{a - pa + b}{1 - p - q} \frac{1}{1 - z} + \frac{-b - qa + qz(pa - a - b)}{1 - p - q} \frac{1}{1 - pz - qz^2} \\ &= \frac{a - pa + b}{1 - p - q} \frac{1}{1 - z} \\ &\quad - \frac{2qa - pqa + 2qb + pb}{2(1 - p - q)} \frac{z}{1 - pz - qz^2} - \frac{qa + b}{2(1 - p - q)} \frac{2 - pz}{1 - pz - qz^2}. \end{aligned}$$

Reading off the coefficient of z^n on both sides leads to

$$S_n = \frac{a - pa + b}{1 - p - q} - \frac{2qa - pqa + 2qb + pb}{2(1 - p - q)} F_n - \frac{qa + b}{2(1 - p - q)} L_n.$$

This answers the finite sum problem addressed in [1] completely, since the answer is

$$S_{n+m} - S_{n-1} = -\frac{2qa - pqa + 2qb + pb}{2(1 - p - q)} (F_{n+m} - F_{n-1}) - \frac{qa + b}{2(1 - p - q)} (L_{n+m} - L_{n-1}).$$

References

- [1] C. Cooper, Finite sums of consecutive terms of a second order linear recurrence relation, *Integers*, **21** (2021), #A114.
- [2] A. F. Horadam, Basic properties of a certain generalized sequence of numbers, *Fibonacci Quart.*, **3** (1965), 161–176.