THE CONTINUED FRACTION MAP COMPOSED WITH FRACTIONAL LINEAR MAPS

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Abstract
The continued fraction expansion is related to the map $Tx = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$. We determine piecewise fractional linear maps S_N, defined on the partition $0 < \frac{1}{N} < \cdots < \frac{N-1}{N} < 1$, such that the composed map $S_N \circ T$ has a dual map. This helps to calculate the density of the invariant measure. We further investigate sequences of maps S_a, $a = \lfloor \frac{1}{x} \rfloor$, such that $S_a \circ T$ has a dual map.

1. Introduction

Maps of the unit interval into itself are widely used as models within ergodic theory. An important question is the existence of an invariant measure μ which is absolutely continuous with respect to Lebesgue measure. Furthermore, if for a measurable subset E we have

$$\mu(E) = \int_E h(x)dx$$

then one wants to know the density h. Two fractional linear maps on a subset of the unit interval are well known. Let $N \geq 2$ be a natural number, then the N-adic expansion is generated by the map

$$T : [0,1[\to [0,1[$$

$$T_N x = Nx - j,\ j = j(x) = \lfloor Nx \rfloor.$$

This map has the N inverse branches

$$\beta_N(j)x = \frac{j+x}{N},\ 0 \leq j < N.$$

Its invariant measure is Lebesgue measure.

The continued fraction expansion is related to the map

$$T :]0,1] \to [0,1[$$

$$Tx = \frac{1}{x} - a, a = a(x) = \lfloor \frac{1}{x} \rfloor, 0 < x \leq 1.$$

One can define $T0 = 0$ or just that the algorithm stops for $x = 0$. This map has infinitely many inverse branches

$$\beta(a)x = \frac{1}{a + x},\ a \geq 1.$$

For continued fractions the invariant meausure has the density $h(x) = \frac{1}{1+x}$. The composed map

$$(T_N \circ T)x = T_N(Tx) = \frac{N}{x} - (aN + j),\ a = \lfloor \frac{1}{x} \rfloor,\ j = \lfloor \frac{N}{x} - aN \rfloor$$

has the inverse branches defined by the matrices

$$\beta_N(a, j) = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} N & 0 \\ j & 1 \end{pmatrix} = \begin{pmatrix} aN + j & 1 \\ N & 0 \end{pmatrix}.$$

The associated maps are given as

$$\beta_N(a, j)x = \frac{N}{aN + j + x}.$$

Note that we use the same notation for a matrix $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and its associated fractional linear map $\alpha(x) = \frac{c+dx}{a+bx}$.

The density of its absolutely continuous measure can be calculated by the use of the *dual algorithm* which can be described as follows (see [5, 4, 3]). Let T be a piecewise fractional map with inverse branches $\beta(k)x = \frac{c_k+d_kx}{a_k+b_kx}$ then the dual map has the inverse branches $\beta(k)^{\#}y = \frac{b_k+d_ky}{a_k+c_ky}$. However, in many cases it is not easy to find an interval $I^{\#}$ with a suitable partition such that $T^{\#}$ is a piecewise fractional linear map from $I^{\#}$ into itself. If a map $M(x) = \frac{B+Dx}{A+Bx}$ exists such that $M(Tx) = T^{\#}(Mx)$ then $T^{\#}$ is defined on $I^{\#} = M[0, 1]$. Then we speak of a *natural dual*. If $T^{\#}$ exists on an interval $I^{\#}$ but no connecting map M can be found we call the situation an *exceptional dual*. In both cases

$$h(x) = \int_{I^{\#}} \frac{dy}{(1 + xy)^2}$$

is the density of an absolutely continuous invariant measure.

We apply this idea to the map $T_N \circ T$. The matrix

$$M = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}$$

satisfies the equations

$$M\beta_N(a, j) = \beta_N(a, j)^{\#}M.$$

Furthermore $\alpha^{\#} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ denotes the transposed matrix. Therefore $M(x) = \frac{x}{N}$ defines a *natural dual* on $[0, \frac{1}{N}]$ with the map

$$(T_N \circ T)^{\#} = M(T_N \circ T)M^{-1}.$$

Then the required density is given by

$$h(x) = \int_0^{\frac{1}{N}} \frac{dy}{(1+xy)^2} = \frac{1}{N+x}.$$

The algorithm $(T_N \circ T)x = \frac{N}{x} - \lfloor \frac{N}{x} \rfloor$ has been considered in [2]. Dajani et al. ([1]) also considered the map

$$Sx = (T_a \circ T)x = \frac{a}{x} - (a^2 + j),\ a = \lfloor \frac{1}{x} \rfloor,\ j = \lfloor \frac{a}{x} - a^2 \rfloor.$$

However, the density of the absolutely continuous invariant measure could not be determined.

In this paper we consider two problems related to continued fractions with variable numerators: (i) we determine piecewise fractional linear maps S_N, defined on the partition $0 < \frac{1}{N} < \cdots < \frac{N-1}{N} < 1$ such that the composed map $S_N \circ T$ has a dual which allows to construct the invariant density; (ii) we determine sequences of maps S_a, $a = \lfloor \frac{1}{x} \rfloor$ such that $S_a \circ T$ has a natural dual.

This paper extends the investigations started in [3].

2. Composed Maps with a Fixed Piecewise Increasing Fractional Linear Map

Let S_N be a piecewise fractional linear map which is defined on $[0, 1]$ with the partition $0 < \frac{1}{N} < \cdots < \frac{N-1}{N} < 1$. We assume that the map S_N is increasing on each partition interval. Then the N inverse branches can be described by the matrices

$$\begin{pmatrix} kN & N\lambda_k - N \\ k^2 - k & k\lambda_k \end{pmatrix},\ k + \lambda_k > 1,\ 1 \le k \le N,$$

where λ_1, ..., λ_N are given parameters. To avoid attractive fixed points $\xi = 0$ and $\xi = 1$ we further require $0 < \lambda_1 \le N$ and $2 \le N + \lambda_N$. Keep in mind that S_N depends on the choice of the parameters λ_1, ..., λ_N.

Theorem 1. *The map $S_N \circ T$ has a natural dual if and only if the parameters are related by the equations*

$$\lambda_k = k\lambda_1 - k + 1,\ 1 \le k \le N.$$

Proof. We calculate the inverse branches of the map $S_N \circ T$ as

$$V(a,k) = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} kN & N\lambda_k - N \\ k^2 - k & k\lambda_k \end{pmatrix}$$
$$= \begin{pmatrix} akN + k^2 - k & aN\lambda_k - aN + k\lambda_k \\ kN & N\lambda_k - N \end{pmatrix}.$$

We now find a matrix $M = \begin{pmatrix} A & B \\ B & D \end{pmatrix}$ such that $MV(a,k) = V(a,k)^{\#}M$. Then the equation

$$\begin{pmatrix} A & B \\ B & D \end{pmatrix} \begin{pmatrix} akN + k^2 - k & aN\lambda_k - aN + k\lambda_k \\ kN & N\lambda_k - N \end{pmatrix}$$
$$= \begin{pmatrix} akN + k^2 - k & kN \\ aN\lambda_k - aN + k\lambda_k & N\lambda_k - N \end{pmatrix} \begin{pmatrix} A & B \\ B & D \end{pmatrix}$$

leads to

$$A(aN\lambda_k - aN + k\lambda_k) + B(N\lambda_k - N) = B(akN + k^2 - k) + DkN.$$

Since this equation should be valid for all $a \geq 1$ we first obtain $A(\lambda_k - 1) = Bk$. Then setting $k = 1$ yields $B = A(\lambda_1 - 1)$ from which we see $\lambda_k - 1 = k(\lambda_1 - 1)$. Hence $\lambda_k = k\lambda_1 - k + 1$. One then verifies $m(\lambda_k - 1) = (\lambda_m - 1)k$ for any $m = 1, 2, ..., N$. Hence, we can define $A = 1$ and $B = \lambda_1 - 1$.

There is a second equation which must be fulfilled, namely

$$Ak\lambda_k + B(N\lambda_k - N) = B(k^2 - k) + DkN.$$

We insert $\lambda_k = k\lambda_1 - k + 1$ and divide by k. Then

$$A(k\lambda_1 - k + 1) + B(N\lambda_1 - N - k + 1) = DN.$$

This should be true for all $k = 1, ..., N$. Hence $A(\lambda_1 - 1) = B$ which is correct. There remains

$$DN = A + BN(\lambda_1 - 1) + B = N(\lambda_1 - 1)^2 + \lambda_1.$$

If we set $A = 1$ and $B = \lambda_1 - 1$ we obtain

$$D = \frac{\lambda_1}{N} + (\lambda_1 - 1)^2.$$

□

Remark 1. (1) The map S_N has a natural dual. The equations

$$\begin{pmatrix} A' & B' \\ B' & D' \end{pmatrix}\begin{pmatrix} kN & N\lambda_k - N \\ k^2-k & k\lambda_k \end{pmatrix} = \begin{pmatrix} kN & k^2-k \\ N\lambda_k - N & k\lambda_k \end{pmatrix}\begin{pmatrix} A' & B' \\ B' & D' \end{pmatrix}$$

have the solutions

$$A' = N - \lambda_1,\ B' = N\lambda_1 - N, D' = N(\lambda_1 - 1)^2.$$

Clearly, the relation $\lambda_k = k\lambda_1 - k + 1$ is essential.
(2) Our first theorem repeats the exercise stated at the beginning of Section 2 of [3]. If one observes the change in notation we have the connection

$$\lambda_k = \frac{N + \varepsilon k - \varepsilon}{N - \varepsilon}.$$

.

Remark 2. It is easy to see that the map $T \circ S_N$ has no natural dual. However, since the map $T \circ S_N$ is isomorphic to the map $S_N \circ T$ its invariant measure can be calculated. If h is the density of the invariant measure of $S_N \circ T$ and g is the density of the invariant measure of $T \circ S_N$ then

$$g(x) = \sum_{b=1}^{\infty} h(\frac{1}{b+x})\frac{1}{(b+x)^2}.$$

Example 1. Take $N = 2$ and $\lambda_1 = \lambda_2 = 1$. Then we obtain $h(x) = \frac{1}{2+x}$ and $g(x) = \sum_{b=1}^{\infty}\left(\frac{1}{b+x} - \frac{2}{2b+1+2x}\right)$.

We will show that exceptional duals for $S_N \circ T$ can also be found.

Theorem 2. *Define* $\eta = \frac{\lambda_N - 1}{N}$. *If the sequence* $\lambda_1, ..., \lambda_N$ *satisfies the conditions*

$$V(1,1)^{\#}\eta = \frac{\lambda_N + N - 1}{N}$$

$$V(1,k)^{\#}\eta = \frac{\lambda_{k-1} - 1}{k-1},\ 2 \le k \le N$$

then the map $S_N \circ T$ *has an exceptional dual.*

Proof. Note that the partition for $S_N \circ T$ is arranged as follows. The interval $B = [0,1]$ first is divided into the intervals $B(a) =]\frac{1}{a+1}, \frac{1}{a}]$, $a \ge 1$, and then the interval $B(a)$ is divided into N intervals. The exceptional dual on the interval $B^{\#} = [\eta, \theta]$ starts with a partition into N intervals $B^{\#}(k)$, $1 \le k \le N$, which will be further divided into the intervals $B^{\#}(a,k)$, $a \ge 1$. Using the transposed matrices of $V(a,k)$ we find

$$V(a,k)^{\#}y = \frac{aN(\lambda_k - 1) + k\lambda_k + N(\lambda_k - 1)y}{akN + k^2 - k + kNy}.$$

Then

$$\lim_{a\to\infty} V(a,k)^{\#}y = \frac{\lambda_k - 1}{k}.$$

Therefore we find $B^{\#}(1) =]\lambda_1-1, \theta]$, $B^{\#}(2) =]\frac{\lambda_2-1}{2}, \lambda_1-1]$, ..., $B^{\#}(N) =]\frac{\lambda_N-1}{N}, \frac{\lambda_{N-1}-1}{N-1}]$. This implies

$$\eta = \frac{\lambda_N - 1}{N}.$$

The equations

$$V(a,k)^{\#}\theta = V(a+1,k)^{\#}\eta$$

show that $\theta = 1 + \eta$. The conditions

$$V(1,1)^{\#}\eta = \frac{\lambda_N + N - 1}{N}$$

$$V(1,k)^{\#}\eta = \frac{\lambda_{k-1} - 1}{k-1},\ 2 \le k \le N$$

are just the necessary conditions of the theorem. □

Example 2. (a) Let $N = 2$. We find

$$\lim_{a\to\infty} V(a,1)^{\#}y = \lambda_1 - 1$$

and

$$\lim_{a\to\infty} V(a,2)^{\#}y = \frac{\lambda_2 - 1}{2} = \eta.$$

Calculations show that the further conditions reduce to the equation

$$2\lambda_1\lambda_2 + 4\lambda_1 = \lambda_2^2 + 4\lambda_2 + 3.$$

If we take $\lambda_1 = 1$ then $\lambda_2 = -1 + \sqrt{2}$. The invariant measure for $S_2 \circ T$ has the density

$$h(x) = \frac{\sqrt{2}}{2 + \sqrt{2}x} - \frac{-2 + \sqrt{2}}{2 + (-2 + \sqrt{2})x}.$$

For the map S_2 it is

$$g(x) = \frac{1}{1 + (-2 + \sqrt{2})x}.$$

(b) If $\lambda_N = 1$ then $S_N \circ T$ and S_N have the same invariant measure with density

$$h(x) = g(x) = \frac{1}{1+x}.$$

The condition $\lambda_N = 1$ implies $\eta = 0$ and $\theta = 1$. Therefore $h(x) = \frac{1}{1+x}$. The matrix $M' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ satisfies the condition $M'(S_N x) = S_N^{\#}(M'x)$ if

$$\lambda_k = \frac{kN + N - k^2 + k}{N + k}.$$

This condition conforms with

$$V(1,k)^{\#}(0) = \frac{\lambda_{k-1} - 1}{k-1},\ 2 \le k \le N.$$

3. Composed Maps with a Fixed Piecewise Decreasing Fractional Linear Map

Again, let S_N be a piecewise fractional linear map which is defined on the partition $0 < \frac{1}{N} < \cdots < \frac{N-1}{N} < 1$. But now we assume that the map S_N is decreasing on each interval of the partition. The N matrices related to the inverse branches are given as

$$\begin{pmatrix} N - N\lambda_k - kN & -N + N\lambda_k \\ -k\lambda_k - k^2 + k & k\lambda_k \end{pmatrix},\ 1 < k + \lambda_k,\ 1 \le k \le N.$$

Then

$$V(a,k) = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} N - N\lambda_k - kN & -N + N\lambda_k \\ -k\lambda_k - k^2 + k & k\lambda_k \end{pmatrix}$$
$$= \begin{pmatrix} aN(1 - \lambda_k - k) - k\lambda_k - k^2 + k & aN(\lambda_k - 1) + k\lambda_k \\ N(1 - \lambda_k - k) & N(\lambda_k - 1) \end{pmatrix}.$$

Theorem 3. *The map $S_N \circ T$ has a natural dual if and only if the parameters are related by the equations*

$$\lambda_k = k\lambda_1 - k + 1,\ 1 \le k \le N.$$

Proof. The same method as in the proof of Theorem 1 leads to the equations

$$A(-1 + \lambda_k) = (1 - \lambda_k - k)B.$$

$$Ak\lambda_k + B(-N + N\lambda_k) = B(-k\lambda_k - k^2 + k) + D(N - \lambda_k - kN).$$

Starting with $A = \lambda_1$ and $B = 1 - \lambda_1$ we find $\lambda_k = k\lambda_1 - k + 1$. If we substitue this in the second set of equations we eventually find

$$D = \frac{(1 - \lambda_1)^2}{\lambda_1} - \frac{1}{N}.$$

□

Remark 3. Again, the map S_N has a natural dual. In a similar way we find for $\lambda \ne 1$

$$A' = \frac{N\lambda_1^2 + \lambda_1}{-N\lambda_1 + N}, B' = \lambda_1, D' = 1 - \lambda_1.$$

If $\lambda_1 = 1$ then $B' = D' = 0$ and one can take $A' = 1$. Since $\lambda_k = 1$ for all k the map is piecewise linear and Lebesgue measure is invariant.

In this case we also find exceptional duals.

Theorem 4. *Let* $\eta = \frac{2-2\lambda_N-N}{\lambda_N+N-1}$. *If the parameters* $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_N$ *satisfy the conditions*

$$V(1,1)^{\#}\eta = \eta$$

$$V(1,k)^{\#}\eta = \frac{1-\lambda_{k-1}}{\lambda_{k-1}+k-2},\ 2 \le k \le N,$$

then the map $S_N \circ T$ *has an exceptional dual.*

Proof. The partition of $B^{\#} = [\eta, \theta]$ is similar as in Theorem 2, but here $B^{\#}(1,1)$ is the leftmost interval. Hence η satisfies the relation

$$V(1,1)^{\#}\eta = \eta.$$

Furthermore

$$\lim_{a\to\infty} V(a,k)^{\#}y = \frac{1-\lambda_k}{\lambda_k - 1 + k}.$$

We get $B^{\#}(1) = [\eta, \frac{1-\lambda_1}{\lambda_1}[$, $B^{\#}(2) = [\frac{1-\lambda_1}{\lambda_1}, \frac{1-\lambda_2}{\lambda_2+1}[$, ..., $B^{\#}(N) = [\frac{1-\lambda_{N-1}}{\lambda_{N-1}+N-2}, \frac{1-\lambda_N}{\lambda_N+N-1}]$, and therefore $\theta = \frac{1-\lambda_N}{\lambda_N+N-1}$. As before, the equations $V(a,k)^{\#}\theta = V(a+1,k)^{\#}\eta$ imply $\eta = 1-\theta$. Next, the conditions on the parameters come from

$$V(1,1)^{\#}\eta = \eta$$

$$V(1,k)^{\#}\eta = \frac{1-\lambda_{k-1}}{\lambda_{k-1}+k-2},\ 2 \le k \le N.$$

□

Example 3. (a)$N = 2$
We find

$$\lim_{a\to\infty} V(a,1)^{\#}y = \frac{1-\lambda_1}{\lambda_1}$$

and

$$\lim_{a\to\infty} V(a,2)^{\#}y = \frac{1-\lambda_2}{1+\lambda_2} = \theta.$$

Calculations show that the further conditions reduce to the equation

$$\lambda_1\lambda_2 - 3\lambda_1 + 2\lambda_2 + 2 = 0.$$

If we take $\lambda_1 = 2$, then $\lambda_2 = 1$, $\theta = 0$, and $\eta = -1$. Then the density of the invariant measure of the map $S_2 \circ T$ is $h(x) = \frac{1}{1-x}$, but S_2 has the density $g(x) = \frac{1}{1+x} + \frac{2}{3-2x}$.
(b) If $\lambda_N = \frac{2-N}{2}$ then again $S_N \circ T$ and S_N have the same invariant measure with density

$$h(x) = g(x) = \frac{1}{1+x}.$$

The condition $\lambda_N = \frac{2-N}{2}$ implies $\eta = 0$ and $\theta = 1$. Therefore $h(x) = \frac{1}{1+x}$. The matrix $M' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ satisfies the condition $M'(S_N x) = S_N^{\#}(M'x)$ if the matrices of S_N are symmetric. This amounts to the condition

$$\lambda_k = \frac{N - k^2 + k}{N + k}$$

which condition conforms with

$$V(1,k)^{\#}(0) = \frac{1 - \lambda_{k-1}}{\lambda_{k-1} + k - 2},\ 2 \leq k \leq N.$$

Remark 4. If the monotony behavior of S_N changes, then generally, the map $S_N \circ T$ does not admit a natural dual. This will be illustrated with $N = 2$.
(a) If S_2 has the two inverse branches

$$V(\lambda_1)x = \frac{\lambda_1 x}{2 + (2\lambda_1 - 2)x},\ V(\lambda_2)x = \frac{\lambda_2 + 1 - \lambda_2 x}{\lambda_2 + 1 + (1 - \lambda_2)x}$$

then the inverse branches of $S_2 \circ T$ are given by the matrices

$$\begin{pmatrix} 2a & (2\lambda_1 - 2)a + \lambda_1 \\ 2 & 2\lambda_1 - 2 \end{pmatrix},\ \begin{pmatrix} a(\lambda_2 + 1) + \lambda_2 + 1 & a(-\lambda_2 + 1) - \lambda_2 \\ \lambda_2 + 1 & -\lambda_2 + 1 \end{pmatrix}$$

for $1 \leq a$.
The search for a suitable matrix $M = \begin{pmatrix} A & B \\ B & D \end{pmatrix}$ leads to the equations

$$A(2\lambda_1 - 2)a + A\lambda_1 + B(2\lambda_1 - 2) = 2aB + 2D$$

$$A(-\lambda_2 + 1)a - A\lambda_2 + B(-\lambda_2 + 1) = a(\lambda_2 + 1)B + (\lambda_2 + 1)B + (\lambda_2 + 1)D.$$

Therefore we obtain

$$A(\lambda_1 - 1) = B,\ A(-\lambda_2 + 1) = (\lambda_2 + 1)B.$$

Hence $\lambda_2 = \frac{2-\lambda_1}{\lambda_1}$. We further have the equations

$$A\lambda_1 + (2\lambda_1 - 2)B = 2D$$

$$-A\lambda_2 - 2\lambda_2 B = D(\lambda_2 + 1).$$

We substitute $\lambda_2 = \frac{2-\lambda_1}{\lambda_1}$ and obtain

$$A\lambda_1 + (2\lambda_1 - 2)B = 2D$$

$$A(\lambda_1 - 2) + (2\lambda_1 - 4)B = 2D.$$

Hence $A+B=0$ and we may put $A=1$ and $B=-1$. Using $A(\lambda_1-1)=B$ this leads to $\lambda_1=0$, a contradiction.
(b) If S_2 has the inverse branches

$$V(\lambda_1)x=\frac{\lambda_1-\lambda_1 x}{2\lambda_1+(-2\lambda_1+2)x},\ V(\lambda_2)x=\frac{1+\lambda_2 x}{2+(-1+\lambda_2)x}$$

then we find a contradiction in a similar way.

In some cases an exceptional dual can be constructed. Take the increasing branch with $\lambda_1=1$ and the decreasing branch with $\lambda_2=0$. Then the inverse branches of $T\circ S_2$ are given by

$$V(a,0)x=\frac{2}{2a+x},\ V(a,1)=\frac{1+x}{a+1+ax}.$$

The dual algorithm has the inverse branches

$$V(a,0)^{\#}y=\frac{1}{2a+2y},\ V(a,1)^{\#}y=\frac{a+y}{a+1+y}.$$

This defines a suitable algorithm on the interval $[0,1]$. Hence $h(x)=\frac{1}{1+x}$ is the density of the invariant measure for $T\circ S_2$.

4. Composed Maps with Variable Fractional Linear Maps

If we replace S_N by S_a, $a=a(x)=\lfloor\frac{1}{x}\rfloor$ then the next result comes as a surprise.

Theorem 5. *Let $M(x)=\frac{B+Dx}{A+Bx}$ be a map such that M has no pole ξ with $0<\xi<1$. Then there is a collection of piecewise increasing fractional linear maps S_a, $a=a(x)=\lfloor\frac{1}{x}\rfloor$, such that $S_a\circ T$ has a natural dual via the given map M.*

Proof. Let S_a be piecewise increasing fractional linear map. The inverse branches of $S_a\circ T$ are given by the matrix products

$$\begin{pmatrix} a & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} ka & a\lambda_{ak}-a\\ k^2-k & k\lambda_{ak}\end{pmatrix},\ 1\le k\le a.$$

The search for a suitable matrix $\begin{pmatrix} A & B\\ B & D\end{pmatrix}$ leads to the equations

$$A(a^2\lambda_{ak}-a^2+k\lambda_{ak})+B(a\lambda_{ak}-a)=B(a^2k+k^2-k)+akD.$$

Then we find

$$\lambda_{ak}=\frac{Aa^2+B(a^2k+a+k^2-k)+akD}{Aa^2+Ak+aB}.$$

Unfortunately, the case $\lambda_{ak}=1$ does not appear as a possible solution. □

Example 4. (a) If $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $h(x) = \frac{1}{1+x}$. We get

$$\lambda_{ak} = \frac{a^2 + ak}{a^2 + k}.$$

The increasing map S_a has a natural dual too. One finds $M' = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$.

(b) If $M = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$, then $h(x) = \frac{1}{x}$. Here

$$\lambda_{ak} = \frac{a^2k + k^2 - k + a - ak}{a}.$$

The increasing map S_a has no natural dual if $a \geq 3$. If $M' = \begin{pmatrix} A' & B' \\ B' & D' \end{pmatrix}$ then the equations

$$A'(a^3 + ak - a - a^2) + B'(a^2k + k^2 - k + a - ak) = B'a^2 + D'(ak - a)$$

have no solution $M' \neq 0$.

For $a = 2$ we find $M' = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$. This is not surprising because any piecewise fractional linear map with two branches has a natural dual (see [6]).

Remark 5. Clearly Theorem 5 is also true if we suppose that every S_a is piecewise decreasing. A similar calculation shows

$$\lambda_{ak} = \frac{Aa^2 + B(a^2 + a - a^2k - k^2 + k) + D(a - ak)}{A(a^2 + k) + B(a^2 + a + k) + Da}.$$

Example 5. (a) If $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $h(x) = \frac{1}{1+x}$. We get

$$\lambda_{ak} = \frac{a^2 + a - ak}{a^2 + a + k}.$$

Again the map S_a has a natural dual with the matrix $M' = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$.

(b) If $M = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$, then $h(x) = \frac{1}{x}$. Here

$$\lambda_{ak} = \frac{a^2 - a^2k - k^2 + k + ak}{a^2 + k}.$$

Then the decreasing mapp S_a has no natural dual if $a \geq 3$. For $a = 2$ we find $\lambda_{21} = \frac{2}{5}$, $\lambda_{22} = -\frac{1}{3}$, and $M' = \begin{pmatrix} 5 & 2 \\ 2 & 9 \end{pmatrix}$.

Remark 6. Theorem 5 and Remark 5 generalize Theorem 1 of [3]. This Theorem 1 corresponds to the matrix $M = \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix}$. Part (a) is covered by

$$\lambda_{ak} = \frac{a^2 + ak\rho}{a^2 + k}$$

and Part (b) by

$$\lambda_{ak} = \frac{a^2 + a\rho - ak\rho}{a^2 + k + a\rho}.$$

Unfortunately, in the formulation of Theorem 1 the maps R_ρ and Q_ρ were mixed up with their inverses. It should be read $P_\rho x = \frac{-j+ax}{\eta-\varepsilon x}$ and $Q_\rho x = \frac{-j-1+ax}{\lambda-\kappa x}$ on $\frac{j}{a} \le x < \frac{j+1}{a}$.

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