RELATIONS BETWEEN DERANGEMENT AND FACTORIAL NUMBERS

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Abstract

In this paper, we present several properties of derangement and factorial numbers. We give a relationship between these numbers using a lower triangular matrix. We obtain these numbers using the factorization of some lower triangular matrices. Also, we give several determinant and permanent representations of these numbers.

1. Introduction

The n-th factorial number, $n! = n(n-1)\cdots(2)(1)$, is the order of the symmetric group S_n, the number of permutations of n letters. These numbers are used in many different areas and can be obtained in different ways. For example

$$n! = (n-1)((n-1)! + (n-2)!),\ 0! = 1,\ 1! = 1.$$

A derangement is a permutation of the elements of a set, such that no element appears in its original position. The number of derangements of a set of size n, usually written as !n, is called the n-th derangement number. This subject was first studied by Pierre Raymond de Montmort in 1708 [8]. These numbers satisfy the following relations:

$$!n = \left\| \frac{n!}{e} \right\| = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} = \sum_{i=0}^{n} (-1)^{n-i} i! \binom{n}{i} = \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n-i)^i (n-i-1)^{n-i}.$$

Several recurrence formulas have been obtained for these numbers, some of which are:

$$!n = (n-1)[!(n-1)+!(n-2)],\ !0 = 1,\ !1 = 0,$$

and

$$!n = n(!(n-1)) + (-1)^n,\ !0 = 1.$$

These numbers have been studied in several papers during the last thirty years. Kittappa [15] defined an $n \times n$ band matrix H_n and showed that the determinant of H_n

equals the $(n+2)$-th derangement number. Janjic [11] gave several determinantal representations of derangement numbers. In [3] and [5], the authors gave several permanental representations of derangement and factorial numbers. Bona [4] obtained important properties of the Catalan, derangement, and factorial numbers. In [24], [25] and [26], the authors discussed several determinantal representations of derangement numbers. More examples can be found in [1, 10, 21, 22, 23]. Recently there has been an increase in the number of studies concerning matrices that contain special numbers and sequences. In [17], the authors examined the linear algebra of the k-Fibonacci matrix and symmetric k-Fibonacci matrix. More examples can be found in [2, 6, 7, 12, 13, 14, 16, 18, 19, 27, 28].

Lemma 1 ([6]). *Let A_n be an $n \times n$ lower Hessenberg matrix for all $n \geq 1$ and define* $\det(A_0) = 1$*. Then,* $\det(A_1) = a_{11}$ *and for $n \geq 2$*

$$\det(A_n) = a_{n,n}\det(A_{n-1}) + \sum_{r=1}^{n-1}[(-1)^{n-r}a_{n,r}(\prod_{j=r}^{n-1}a_{j,j+1})\det(A_{r-1})].$$

The Hessenberg matrices defined in this study are also band matrices ($a_{i,i\pm k} = a_{j,j\pm k}$, for $k \geq 0$). For this case, Lemma 1 has the corollary that

$$\det(A_n) = \sum_{k=1}^{n}(-1)^{k-1}a_{k,1}(a_{1,2})^{k-1}\det(A_{n-k}). \tag{1}$$

In this paper, we give the definitions of the n-th factorial and derangement matrices and obtain the inverse of these matrices via the determinants of some Hessenberg matrices. We also obtain a relationship between these matrices. Secondly, we give some factorizations of these matrices. In the last section, we obtain several determinant and permanent representations of the n-th factorial and derangement numbers.

2. The Factorial and Derangement Matrices

Definition 1. The n-th factorial matrix of order n will be denoted by $\mathcal{F}_n = [a_{i,j}]_{i,j=1,2,\ldots,n}$, and is defined by

$$a_{i,j} = \begin{cases} (i-j)!, & \text{if } i-j \geqslant 0; \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

where n is any positive integer.

For example,

$$\mathcal{F}_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 6 & 2 & 1 & 1 & 0 \\ 24 & 6 & 2 & 1 & 1 \end{bmatrix}$$

is a factorial matrix.

Definition 2. The n-th derangement matrix of order n, will be denoted by $\mathcal{D}_n = [b_{i,j}]_{i,j=1,2,\ldots,n}$, and is defined by

$$b_{i,j} = \begin{cases} 1, & i = j; \\ !(i-j+1), & \text{if } i-j \geqslant 1; \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

where n is any positive integer.

For example,

$$\mathcal{D}_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 9 & 2 & 1 & 1 & 0 \\ 44 & 9 & 2 & 1 & 1 \end{bmatrix}$$

is a derangement matrix.

Definition 3. The $n \times n$ lower Hessenberg matrix sequence $F_n^S = [a_{i,j}]_{i,j=1,2,\ldots,n}$ is defined by

$$a_{i,j} = \begin{cases} (i-j+1)!, & \text{if } i-j+1 \geq 0; \\ 0, & \text{otherwise,} \end{cases} \tag{4}$$

where n is any positive integer.

Definition 4. The $n \times n$ lower Hessenberg matrix sequence $D_n^S = [a_{i,j}]_{i,j=1,2,\ldots,n}$ is defined by

$$a_{i,j} = \begin{cases} 1, & \text{if } j = i+1; \\ !(i-j+2), & \text{if } i-j \geq 0; \\ 0, & \text{otherwise,} \end{cases} \tag{5}$$

where n is any positive integer.

Lemma 2. *Let* $d_0 = 1$, $d_n = \sum_{k=0}^{n-1}(-1)^{n-k+1}!(n-k+1)d_k$, *for* $n \geq 1$. *Then,* $\det(D_n^S) = d_n$ *for any integer* $n \geq 1$.

Proof. We proceed by induction on n. The result clearly holds for $n = 1$. Now suppose that the result is true for all positive integers less than or equal to n. We prove it for $(n+1)$. In fact, by using (1), we obtain that

$$\begin{aligned}\det(D_{n+1}^S) &= \sum_{k=1}^{n+1}\left[(-1)^{k-1}a_{k,1}(a_{1,2})^{k-1}\det(D_{n-k+1}^S)\right] \\ &= \sum_{k=1}^{n+1}\left[(-1)^{k-1}!(k-1+2)\det(D_{n-k+1}^S)\right].\end{aligned}$$

From the hypothesis of induction, we obtain

$$\det(D_{n+1}^S) = \sum_{k=1}^{n+1}\left[(-1)^{k-1}!(k+1)d_{n-k+1}\right].$$

Therefore, $\det(D_n^S) = d_n$ holds for all positive integers n. □

Lemma 3. *Let $f_0 = 1$, $f_n = \sum_{k=0}^{n-1}(-1)^{n-k+1}(n-k)!f_k$, for $n \geq 1$. Then, $\det(F_n^S) = f_n$ for any integer $n \geq 1$.*

Proof. The proof is similar to the proof of Lemma 2. □

Example 1. Using Lemma 2 and Lemma 3, we obtain d_3 and f_4 as follows:

$$\det\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 9 & 2 & 1 \end{bmatrix} = 6 = d_3,\ \det\begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 1 & 1 \\ 24 & 6 & 2 & 1 \end{bmatrix} = -13 = f_4.$$

With the help of the recurrence relations, we give the definitions of d_n and f_n. From these recurrence relations, we can obtain the equations

$$!n = d_{n-1} + \sum_{k=1}^{n-2}(-1)^{k-1}!(n-k)d_k \tag{6}$$

and

$$n! = \sum_{k=1}^{n}(-1)^{k-1}(n-k)!f_k. \tag{7}$$

Theorem 1. *Let n be any positive integer, $\mathcal{F}_n$ be the $n \times n$ factorial matrix in (2) and F_n^S be the $n \times n$ Hessenberg matrix in (4). The inverse of $\mathcal{F}_n$, denoted by $(\mathcal{F}_n)^{-1} = [c_{i,j}]$, is equal to*

$$[c_{i,j}] = \begin{cases} (-1)^{i-j}\det(F_{i-j}^S), & \text{if } i-j > 0; \\ 1, & \text{if } i-j = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. It suffices to prove that $\mathcal{F}_n(\mathcal{F}_n)^{-1} = I_n$. We take $\mathcal{F}_n(\mathcal{F}_n)^{-1} = [r_{i,j}]_{1\le i,j\le n}$. It is obvious that $r_{i,j} = \sum_{k=0}^{n} a_{i,k}c_{k,j} = 0$ for $i-j<0$ and $r_{i,j} = \sum_{k=0}^{n} a_{i,k}c_{k,j} = a_{i,i}c_{i,i} = 1$ for $i=j$. For $i>j\ge 1$, we have

$$\begin{aligned} r_{i,j} = \sum_{k=0}^{n} a_{i,k}c_{k,j} &= \sum_{k=j}^{i} a_{i,k}c_{k,j} \\ &= (i-j)! - (i-j-1)!f_1 + \cdots + 1(-1)^{i-j+1}f_{i-j}. \end{aligned}$$

Since $(i-j+1)! = \sum_{s=1}^{i-j+1}(-1)^{s+1}f_s(i-j+1-s)!$, we obtain $r_{i,j} = \sum_{k=0}^{n} a_{i,k}c_{k,j} = 0$ for $i>j\ge 1$ which, implies that $\mathcal{F}_n(\mathcal{F}_n)^{-1} = I_n$. □

Theorem 2. *Let n be any positive integer, $\mathcal{D}_n$ be the $n\times n$ derangement matrix in (3) and D_n^S be the $n\times n$ Hessenberg matrix in (5). The inverse of $\mathcal{D}_n$, denoted by $(\mathcal{D}_n)^{-1} = [c_{i,j}]$, is equal to*

$$[c_{i,j}] = \begin{cases} (-1)^{i-j}\det(D_{i-j}^S), & \text{if } i-j>0; \\ 1, & \text{if } i-j=0; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The proof is similar to the proof of Theorem 1. □

Now, we show the relation between the factorial and derangement matrices. The $n\times n$ lower triangular matrix $\mathcal{L}_n = [t_{i,j}]$, $(1\le i,j\le n)$ is defined by

$$t_{i,j} = \begin{cases} \sum_{k=0}^{i-j}(-1)^k(i-j-k)!d_k, & \text{if } i>j; \\ 1, & \text{if } i=j; \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3. *Let n be any positive integer, $\mathcal{F}_n$ and $\mathcal{D}_n$ be any $n\times n$ lower triangular factorial and derangement matrices. Then,*

$$\mathcal{L}_n\mathcal{D}_n = \mathcal{F}_n.$$

Proof. It suffices to prove that $\mathcal{F}_n(\mathcal{D}_n)^{-1} = \mathcal{L}_n$. It is obvious that $t_{i,j} = 0$ for $i-j<0$ and $t_{i,j}=1$ for $i-j=0$. For $i>j\ge 1$, we have

$$\begin{aligned} \sum_{k=0}^{n} a_{i,k}c_{k,j} &= \sum_{k=0}^{n}(i-k)!(-1)^{k-j}d_{k-j} \\ &= \sum_{k=j}^{i}(i-k)!(-1)^{k-j}d_{k-j} = \sum_{k=0}^{i-j}(i-j-k)!(-1)^k d_k = t_{i,j} \end{aligned}$$

which implies that $\mathcal{F}_n(\mathcal{D}_n)^{-1} = \mathcal{L}_n$, as desired. □

Example 2. Using Theorem 3, we obtain the relation between the factorial and derangement matrices for $n = 5$ as

$$\begin{aligned}\mathcal{L}_5\mathcal{D}_5 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 0 & 1 & 0 \\ -17 & -3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 9 & 2 & 1 & 1 & 0 \\ 44 & 9 & 2 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 6 & 2 & 1 & 1 & 0 \\ 24 & 6 & 2 & 1 & 1 \end{bmatrix} = \mathcal{F}_5.\end{aligned}$$

In the last part of this section we consider companion matrices. We define the companion matrices $Q_{n,F}$ and $Q_{n,D}$ by

$$Q_{n,F} = \begin{bmatrix} 0 & 1 & & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 1 & 0 \\ 0 & 0 & & 0 & 0 & 1 \\ (-1)^{n+1}f_n & (-1)^n f_{n-1} & \cdots & f_3 & -f_2 & f_1 \end{bmatrix}_{n\times n} \tag{8}$$

and

$$Q_{n,D} = \begin{bmatrix} 0 & 1 & & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 1 & 0 \\ 0 & 0 & & 0 & 0 & 1 \\ (-1)^{n+1}d_n & (-1)^n d_{n-1} & \cdots & d_3 & -d_2 & d_1 \end{bmatrix}_{n\times n}. \tag{9}$$

Theorem 4. *Let $Q_{n,F}$ be the $n \times n$ matrix in (8), m and n be the positive integers such that $m \leq n$. The last column of the matrix $(Q_{n,F})^m$ is equal to*

$$\begin{bmatrix} (m-n+1)! \\ \vdots \\ (m-1)! \\ m! \end{bmatrix},$$

where we assume $k!$ to be 0 when $k < 0$.

Proof. The proof is done by mathematical induction on m. The result clearly holds for $m = 1$. Now suppose that the result is true for all positive integers less than or equal to m. We prove it for $(m+1)$ using Equation (7). The last column of matrix $(Q_{n,F})^{m+1}$ is

$$\begin{bmatrix} 0 & 1 & & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 1 & 0 \\ 0 & 0 & & 0 & 0 & 1 \\ (-1)^{n+1}f_n & (-1)^n f_{n-1} & \cdots & f_3 & -f_2 & f_1 \end{bmatrix} \begin{bmatrix} (m-n+1)! \\ \vdots \\ (m-1)! \\ m! \end{bmatrix}$$

$$= \begin{bmatrix} (m-n+2)! \\ \vdots \\ m! \\ (-1)^{n+1}f_n(m-n+1)! + \cdots + f_1 m! \end{bmatrix} = \begin{bmatrix} (m-n+2)! \\ \vdots \\ m! \\ (m+1)! \end{bmatrix}.$$

So by the principle of mathematical induction, the statement is true for all positive integers. □

Theorem 5. *Let $Q_{n,D}$ be the $n \times n$ matrix in (9), and let m and n be the positive integers such that $m \leq n$. The last column of the matrix $(Q_{n,D})^m$ is equal to $(a_1\ a_2\ \ldots\ a_n)^t$, where*

$$a_k = \begin{cases} !(m-n+k+1), & \textit{if } m-n+k+1 \geq 2; \\ 1, & \textit{if } m-n+k+1 = 1; \\ 0, & \textit{if } m-n+k+1 \leq 0. \end{cases}$$

Proof. The proof is done by mathematical induction on m. When $m = 1$,

$$a_k = \begin{cases} !2 = 1 = d_1, & \text{if } 2-n+k = 2; \\ 1, & \text{if } 2-n+k = 1; \\ 0, & \text{if } 2-n+k \leq 0, \end{cases}$$

so the given statement is true in this case. Now suppose that the result is true for all positive integers less than or equal to m. We prove it for $(m+1)$ using Equation

(6). The last column of matrix $(Q_{n,D})^{m+1}$ is

$$\begin{bmatrix} 0 & 1 & & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 1 & 0 \\ 0 & 0 & & 0 & 0 & 1 \\ (-1)^{n+1}d_n & (-1)^n d_{n-1} & \cdots & d_3 & -d_2 & d_1 \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix}$$

$$= \begin{bmatrix} a_2 \\ \vdots \\ a_n \\ (-1)^{n+1}d_n a_1 + (-1)^n d_{n-1} a_2 + \cdots + d_1 a_n \end{bmatrix} = \begin{bmatrix} a_2 \\ \vdots \\ a_n \\ a_{n+1} \end{bmatrix}.$$

So by the principle of mathematical induction, the statement is true for all positive integers. □

2.1. Factorizations of the Factorial and Derangement Matrices

Lee et al. [16, 17] and Sahin [27] obtained the factorization of some matrices. In this study, we consider the factorization of $\mathcal{F}_n$ and $\mathcal{D}_n$. Let I_n be the identity matrix of order n. We define the matrices $\overline{\mathcal{F}}_n$, $\overline{\mathcal{D}}_n$, C_n^F, C_n^D as

$$\overline{\mathcal{F}}_n = [1] \oplus (\mathcal{F}_{n-1}),\ \overline{\mathcal{D}}_n = [1] \oplus (\mathcal{D}_{n-1}),$$

$$C_n^F = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ f_1 & & & \\ \vdots & & I_{n+2} & \\ (-1)^{n+1} f_{n+2} & & & \end{bmatrix} \text{ and } C_n^D = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ d_1 & & & \\ \vdots & & I_{n+2} & \\ (-1)^{n+1} d_{n+2} & & & \end{bmatrix}. \tag{10}$$

Lemma 4. *Let k be any integer such that $k \geq 3$. Then, $(\overline{\mathcal{F}}_k)(C_{k-3}^F) = \mathcal{F}_k$.*

Proof. We take $(\overline{\mathcal{F}}_k) = [\overline{a}_{i,j}]$, $(C_{k-3}^F) = [c_{i,j}]$ and $\mathcal{F}_k = [a_{i,j}]$ and obtain $\sum_{s=1}^{k} \overline{a}_{i,s} c_{s,j}$ for $i, j = 1, 2, \ldots, k$. It is obvious from the matrix product and the definition of I_{n+2} that $a_{11} = 1$, and $\sum_{s=1}^{k} \overline{a}_{i,s} c_{s,j} = a_{i,j}$ for $i = 1, 2, \ldots, k$ and $j = 2, \ldots, k$. For $j = 1$,

$$a_{i,1} = \sum_{s=1}^{k} \overline{a}_{i,s} c_{s,1} = (i-2)! f_1 - \cdots (-1)^{k-1} 1.f_{i-1}.$$

We obtain $a_{i,1} = (i-2)! f_1 - \cdots + (-1)^{k-1}(0!) f_{(i-1)} = (i-1)!$ from Equation (7). □

If we change the roles of f_n and d_n, then we have the following identity.

Lemma 5. *Let k be any integer such that $k \geq 3$. Then,* $(\overline{\mathcal{D}}_k)(C^D_{k-3}) = \mathcal{D}_k$.

Corollary 1. *Let n be any integer such that $n \geq 3$. Then,* $\mathcal{F}_n = (I_{n-2} \oplus (C^F_{-1})) \cdots (I_1 \oplus (C^F_{n-4}))(C^F_{n-3})$ *and* $\mathcal{D}_n = (I_{n-2} \oplus (C^D_{-1})) \cdots (I_1 \oplus (C^D_{n-4}))(C^D_{n-3})$.

Proof. This is obvious from Lemma 4 and the matrix product. □

Example 3. Using Corollary 1, we give a factorization of the matrix $\mathcal{F}_5$ as

$(I_3 \oplus C^F_{-1})(I_2 \oplus C^F_0)(I_1 \oplus C^F_1)C^F_2$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 13 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 6 & 2 & 1 & 1 & 0 \\ 24 & 6 & 2 & 1 & 1 \end{bmatrix} = \mathcal{F}_5.$$

Lemma 6. *Let C^F_n and C^D_n be the $(n+3) \times (n+3)$ Hessenberg matrices in (10). Then,*

$$(C^F_n)^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -f_1 & & & \\ \vdots & & I_{n+2} & \\ (-1)^{n+2} f_{n+2} & & & \end{bmatrix}$$

and

$$(C^D_n)^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -d_1 & & & \\ \vdots & & I_{n+2} & \\ (-1)^{n+2} d_{n+2} & & & \end{bmatrix}.$$

Proof. The proof is obvious from the definition of the matrix product. □

Corollary 2. *Let n be any integer such that $n \geq 3$. Then,* $(\mathcal{F}_n)^{-1} = (C^F_{n-3})^{-1}(I_1 \oplus (C^F_{n-4})^{-1}) \cdots (I_{n-2} \oplus (C^F_{-1})^{-1})$ *and* $(\mathcal{D}_n)^{-1} = (C^D_{n-3})^{-1}(I_1 \oplus (C^D_{n-4})^{-1}) \cdots (I_{n-2} \oplus (C^D_{-1})^{-1})$.

Proof. The proof is obvious from the previous lemma and the equation $(I_k \oplus (C_{n-k-3}))^{-1} = I_k \oplus (C_{n-k-3})^{-1}$. □

3. Determinantal and Permanental Representation of Factorial and Derangement Numbers

Theorem 6. *Let n be any integer such that $n \geq 1$, and let ${}_{-}A_n^F = [a_{i,j}]_{i,j=1,2,\ldots,n}$ be an $n \times n$ Hessenberg matrix defined as*

$$a_{i,j} = \begin{cases} -1, & \text{if } i-j=-1; \\ (-1)^{i-j} f_{i-j+1}, & \text{if } i \geq j; \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

Then,

$$\det({}_{-}A_n^F) = n!.$$

Proof. We proceed by induction on n. The result clearly holds for $n = 1$. Now suppose that the result is true for all positive integers less than n. We prove it for n. In fact, by using (1) we obtain that

$$\begin{aligned} \det({}_{-}A_n^F) &= \sum_{k=1}^{n} \left[(-1)^{k-1} a_{k,1} (a_{1,2})^{k-1} \det({}_{-}A_{n-k}^F)\right] \\ &= \sum_{k=1}^{n} \left[(-1)^{k-1}(-1)^{k-1} f_k (-1)^{k-1} \det({}_{-}A_{n-k}^F)\right] \\ &= \sum_{k=1}^{n} \left[(-1)^{k-1} f_k (n-k)!\right] \\ &= n!. \end{aligned}$$

□

Theorem 7. *Let n be any integer such that $n \geq 1$, and let ${}_{-}A_n^D = [a_{i,j}]_{i,j=1,2,\ldots,n}$ be an $n \times n$ Hessenberg matrix defined as*

$$a_{i,j} = \begin{cases} -1, & \text{if } i-j=-1; \\ (-1)^{i-j} d_{i-j+1}, & \text{if } i \geq j; \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

Then,

$$\det({}_{-}A_n^D) = !(n+1).$$

Proof. The proof is similar to the proof of Theorem 6. □

Corollary 3. *Let m and n be positive integers such that $m \leq n$, and let e_n be the n-th row of the identity matrix I_n. Then,*

$$e_n (Q_{n,F})^m e_n^T = (m)! \text{ and } e_n (Q_{n,D})^n e_n^T = !(n+1).$$

Proof. The proof is obvious from the definition of matrix product. □

Theorem 8. *Let n be any integer such that $n \geq 1$, and let ${}_{+}B_n^F = [b_{s,t}]_{s,t=1,2,\ldots,n}$ be an $n \times n$ Hessenberg matrix defined as*

$$b_{s,t} = \begin{cases} -i, & \text{if } s-t=-1; \\ (i)^{s-t} f_{s-t+1}, & \text{if } s \geq t; \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\det({}_{+}B_n^F) = n!$.

Proof. We proceed by induction on n. The result clearly holds for $n = 1$. Now suppose that the result is true for all positive integers less than n. We prove it for n. In fact, by using (1) we obtain that

$$\begin{aligned} \det({}_{+}B_n^F) &= \sum_{k=1}^{n} \left[(-1)^{k-1} b_{k,1} (b_{1,2})^{k-1} \det({}_{+}B_{n-k}^F)\right] \\ &= \sum_{k=1}^{n} \left[(-1)^{k-1} (i)^{k-1} f_k (-i)^{k-1} \det({}_{+}B_{n-k}^F)\right] \\ &= \sum_{k=1}^{n} \left[(-1)^{k-1} f_k (n-k)!\right] \\ &= n!. \end{aligned}$$

□

Example 4. By using Theorem 8, we obtain 5! as

$$\det \begin{bmatrix} 1 & -i & 0 & 0 & 0 \\ -i & 1 & -i & 0 & 0 \\ -3 & -i & 1 & -i & 0 \\ 13i & -3 & -i & 1 & -i \\ 71 & 13i & -3 & -i & 1 \end{bmatrix} = 120.$$

Theorem 9. *Let n be any integer such that $n \geq 1$, and let ${}_{+}B_n^D = [b_{s,t}]_{s,t=1,2,\ldots,n}$ be an $n \times n$ Hessenberg matrix defined as*

$$b_{s,t} = \begin{cases} -i, & \text{if } s-t=-1; \\ (i)^{s-t} d_{s-t+1}, & \text{if } s \geq t; \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\det({}_{+}B_n^D) = !(n+1).$$

Proof. The proof is similar to the proof of Theorem 6. □

There exists a relation between the determinant and the permanent of a Hessenberg matrix (cf. [9, 12]). The following corollaries are obvious from this relation.

Corollary 4. *Let n be any integer such that $n \geq 1$, and let ${}_{+}A_n^F = [u_{s,t}]_{s,t=1,2,\ldots,n}$ and ${}_{-}B_n^F = [v_{s,t}]_{s,t=1,2,\ldots,n}$ be the $n \times n$ Hessenberg matrices defined as*

$$u_{s,t} = \begin{cases} 1, & \text{if } s-t=-1; \\ (-1)^{s-t} f_{s-t+1}, & \text{if } s \geq t; \\ 0, & \text{otherwise;} \end{cases} \quad \text{and } v_{s,t} = \begin{cases} i, & \text{if } s-t=-1; \\ (i)^{s-t} f_{s-t+1}, & \text{if } s \geq t; \\ 0, & \text{otherwise,} \end{cases}$$

where $i = \sqrt{-1}$. Then, $per({}_{+}A_n^F) = per({}_{-}B_n^F) = n!$.

Corollary 5. *Let n be any integer such that $n \geq 1$, and let ${}_{+}A_n^D = [u_{s,t}]_{s,t=1,2,\ldots,n}$ and ${}_{-}B_n^D = [v_{s,t}]_{s,t=1,2,\ldots,n}$ be the $n \times n$ Hessenberg matrices defined as*

$$u_{s,t} = \begin{cases} 1, & \text{if } s-t=-1; \\ (-1)^{s-t} d_{s-t+1}, & \text{if } s \geq t; \\ 0, & \text{otherwise;} \end{cases} \quad \text{and } v_{s,t} = \begin{cases} i, & \text{if } s-t=-1; \\ (i)^{s-t} d_{s-t+1}, & \text{if } s \geq t; \\ 0, & \text{otherwise,} \end{cases}$$

where $i = \sqrt{-1}$. Then, $per({}_{+}A_n^D) = per({}_{-}B_n^D) = !(n+1)$.

Corollary 6. *Let n be any integer such that $n \geq 1$,*

$$\widehat{{}_{-}A_n^F} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ & & & \vdots \\ & -({}_{-}A_{n-1}^F) & & 0 \\ & & & 1 \end{bmatrix} \text{ and } \widehat{{}_{-}A_n^D} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ & & & \vdots \\ & -({}_{-}A_{n-1}^D) & & 0 \\ & & & 1 \end{bmatrix}.$$

Then,

$$(\widehat{{}_{-}A_n^F})^{-1} = \mathcal{F}_{(n+1)} \text{ and } (\widehat{{}_{-}A_n^D})^{-1} = \mathcal{D}_{(n+1)}$$

Proof. The proof is obvious from Theorem 1 and Theorem 2. □

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