TRACE AND DISCRIMINANT CRITERIA FOR A MATRIX TO BE A SUM OF SIXTH AND EIGHTH POWERS OF MATRICES

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Abstract

In this paper, we consider the Waring's problem for matrices. One version of the problem involves writing an $n \times n$ matrix over a commutative ring R with unity as a sum of k-th powers of matrices over R, where $n, k \geq 2$ are integers. In this paper, we will find criteria for a matrix (as well as every matrix) over a commutative ring R with unity to be a sum of sixth and eighth powers of matrices. Further, for orders $\mathcal{O}$ in algebraic number fields, following Katre and Khule, we derive a discriminant criteria for every matrix $n \times n$ matrix over $\mathcal{O}$ to be a sum of sixth (and eighth) powers of matrices.

1. Introduction

Let R be a commutative ring with unity and let $n, k \geq 2$ be integers. Let $M_n(R)$ denote the commutative ring with unity of all $n \times n$ matrices over R. Given $A \in M_n(R)$, we say that A is a sum of k-th powers in $M_n(R)$ if there exist matrices $A_1, A_2, \ldots, A_r \in M_n(R)$ such that $A = \sum_{i=1}^{r} A_i^k$. All the matrices in $M_n(R)$ need not have this property. So one needs to determine:

1. necessary and sufficient conditions for a matrix to be a sum of k-th powers of matrices and
2. the least value of r, the number of matrices required to express A as a sum of k-th powers of matrices.

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This is known as the classical Waring's problem for matrices after the British mathematician Edward Waring who was the first to describe such a problem for integers. This problem was studied by L. Carlitz and M. Newman over the integers (see [1] and [8]) and further by M. Griffin and M. Krusemeyer over fields (see [3]). L. N. Vaserstein as well as D. R. Richman studied this problem over commutative rings (see [9] and [10]). The majority of the above mathematicians had given the criteria (and the number of matrices required) for a matrix of size two, to be a sum of squares of matrices. M. Griffin and M. Krusemeyer had given the criteria for matrices of sizes two, three and four to be a sum of squares (see [3], Propositions 3.3, 3.4 and 3.5). It was L. N. Vaserstein who gave a trace criterion for a matrix of size $n \geq 2$ to be a sum of squares of matrices. We recall his theorem below.

Theorem 1.1 ([10], Theorem 1). *Let R be a ring with unity. Then for any integer $n \geq 2$, a matrix $A \in M_n(R)$ is a sum of squares if and only if Tr A is a sum of squares in the ring $R/2R$, where Tr A denotes the trace of the matrix A.*

It was D. R. Richman who gave the trace criterion for an $n \times n$ matrix to be a sum of k-th powers of matrices, provided $n \geq k \geq 2$, in the following theorem.

Theorem 1.2 ([9], Proposition 4.2). *Let R be a commutative ring with unity. If $n \geq k \geq 2$ are integers, then the following statements are equivalent.*

(1) $A \in M_n(R)$ is a sum of k-th powers in $M_n(R)$.

(2) $A \in M_n(R)$ is a sum seven k-th powers in $M_n(R)$.

(3) $A \in M_n(R)$ and for every prime power p^e dividing k, there exist elements $x_0, x_1, \ldots, x_e$ (depending on p) in R such that

$$Tr\ A = x_0^{p^e} + p x_1^{p^{e-1}} + \cdots + p^e x_e.$$

Moreover if $k = p$ is a prime, in statement (2) "seven" is replaced by "five" and statement (3) simplifies to Tr $A = x_0^p + p x_1$, for some $x_0, x_1 \in R$.

We will record here another nice result from Richman's paper, especially for trace zero matrices.

Theorem 1.3 ([9], Proposition 5.5). *Let $n, k \geq 2$ be integers. Let $A \in M_n(R)$ be such that Tr $A \equiv 0 (\text{mod } k!R)$. Then, A is a sum of k-th powers of matrices in $M_n(R)$. In particular, if Tr A is zero then A is a sum of k-th powers in $M_n(R)$.*

In fact, refined results were obtained by S. A. Katre and S. A. Khule for orders in algebraic number fields. Let K be an algebraic number field of degree n. A subring of K with unity which is also a $\mathbb{Z}$-module of K of rank n is called an order of K.

Let $\mathcal{O}$ be an order of K. The following theorem gives the discriminant criterion by the above authors for a matrix to be a sum of k-th powers of matrices over $\mathcal{O}$.

Theorem 1.4 ([5], Theorem 1). *Let $\mathcal{O}$ be an order of a number field K and let $n \geq k \geq 2$ be integers. Then every matrix $A \in M_n(\mathcal{O})$ is a sum of k-th powers in $M_n(\mathcal{O})$ if and only if $(k,\ disc\ \mathcal{O}) = 1$.*

Coming back to the trace condition in Theorem 1.2, the restriction $n \geq k$ was removed via the generalized trace criteria given by S. A. Katre and the second author in the form of the following theorem.

Theorem 1.5 ([4], Theorem 3.1). *Let R be a commutative ring with unity and let $n, k \geq 2$ be integers. Let $A \in M_n(R)$. Then the following statements are equivalent.*

(1) A is a sum of k-th powers of matrices in $M_n(R)$.

(2) Tr A is a sum of traces of k-th powers of matrices in $M_n(R)$.

(3) Tr A is in the subgroup of R generated by the traces of k-th powers of matrices in $M_n(R)$.

(4) Tr A is in the subgroup of R generated by the traces of k-th powers of matrices in $M_n(R) \pmod{k!R}$.

(5) Tr A is a sum of traces of k-th powers of matrices in $M_n(R) \pmod{k!R}$.

In the same paper, the proof of the following theorem was given, which we shall use here also.

Theorem 1.6 ([4], Theorem 3.5). *Let R be a commutative ring with unity and let $n \geq m \geq 1$ and $k \geq 2$ be integers. If every matrix in $M_m(R)$ is a sum of k-th powers of matrices in $M_m(R)$, then every matrix in $M_n(R)$ is also a sum of k-th powers of matrices in $M_n(R)$.*

The motivation for this paper comes from the following theorems due to S. A. Katre and the second author (see [2], [4]).

Theorem 1.7. *Let R be a commutative ring with unity and let $n \geq 2$ be an integer. Let $k = 3, 4, 5, 7$. Then the following statements are equivalent.*

(1) Every matrix in $M_n(R)$ is a sum of k-th powers of matrices in $M_n(R)$.

(2) For every prime p dividing k, every element of R is a p-th power $\pmod{pR}$. Further, if $\mathcal{O}$ is an order in an algebraic number field, these conditions are equivalent to $(p,\ \mathrm{disc}(\mathcal{O})) = 1$.

In this paper we shall derive similar results for $n \geq 2$ and $k = 6, 8$. All the proofs are based on a detailed analysis of traces of sixth and eighth power of matrices. The results obtained here are stated below.

Theorem 1.8. *Let R be a commutative ring with unity and let $n \geq 2$ be an integer.*

(a) The following statements are equivalent.

  (1) Every matrix in $M_n(R)$ is a sum of sixth powers of matrices in $M_n(R)$.

  (2) Every element of R is a square $(\mathrm{mod}\ 2R)$ and also a cube $(\mathrm{mod}\ 3R)$. Further, if $\mathcal{O}$ is an order in an algebraic number field, these conditions are equivalent to $(6,\ \mathrm{disc}(\mathcal{O})) = 1$.

(b) The following statements are equivalent.

  (1) Every matrix in $M_n(R)$ is a sum of eighth powers of matrices in $M_n(R)$.

  (2) Every element of R is a square $(\mathrm{mod}\ 2R)$. Further, if $\mathcal{O}$ is an order in an algebraic number field, the conditions are equivalent to $(2,\ \mathrm{disc}(\mathcal{O})) = 1$.

We would like to mention here that recently S. A. Katre, Wadikar K. G. and Deepa Krishnamurthi (see [6] and [7]) have obtained various results for matrices over non-commutative rings too as sums of powers of matrices.

2. Criteria for Matrices as Sums of Sixth Powers

We prove in this section, trace related results regarding sixth powers of matrices. This is the first composite, non-prime power under consideration. We have two subsections: the first one having results on 2×2 matrices and the second one on $n \times n$ matrices, where $n \geq 3$.

2.1. Two-by-two Matrices

Lemma 2.1. *Let $A \in M_2(R)$ be such that $A = B^6$, for some $B \in M_2(R)$. Then there exist $t, \delta \in R$ such that Tr $A = t^6 - 6t^4\delta + 9t^2\delta^2 - 2\delta^3$.*

Proof. For any $B' \in M_2(R)$, we have the equation $B'^2 = (\mathrm{Tr}\ B')B' - (\det\ B')I_2$. Hence, if $B' = B$ where $t = \mathrm{Tr}\ B$ and $\delta = \det\ B$, we get $B^2 = tB - \delta I_2$. This gives that Tr $B^2 = t^2 - 2\delta$. It now follows that $B^3 = tB^2 - \delta B$ and hence Tr $B^3 = t(t^2 - 2\delta) - \delta(t) = t^3 - 3t\delta$.

Thus for $A = B^6$, choosing $B' = B^3$ above gives $A = (B^3)^2 = (\mathrm{Tr}\ B^3)B^3 - (\det B^3)I_2$. Hence from the above calculations, we get that Tr $A = (t^3 - 3t\delta)^2 - 2\delta^3 = t^6 - 6t^4\delta + 9t^2\delta^2 - 2\delta^3$. □

Motivated by the above lemma, given a commutative ring R with unity we shall consider the set of traces of sixth powers of matrices modulo $6R$ and in fact prove that it is a subgroup of R.

Lemma 2.2. *Let R be a commutative ring with unity. Define the set*

$$S = \left\{x_0^6 - 2x_1^3 + 3x_2^2 \pmod{6R} : \ x_0, x_1, x_2 \in R\right\}.$$

Then S is a subgroup of R.

Proof. It is enough to prove S is closed under addition and every element of S has an additive inverse.

In order to prove that S is closed under addition, we note the identities:

(i) $(x+y)^3 = x^3 + 3x^2y + 3yx^2 + y^3$. Hence $-2(x+y)^3 \equiv -2x^3 - 2y^3 \pmod{6R}$.

(ii) $(z+w)^2 = z^2 + 2zw + w^2$. Hence $3(z+w)^2 \equiv 3z^2 + 3w^2 \pmod{6R}$.

(iii) Using the above two identities, we see that $(s+t)^6 = s^6 + 6s^5t + 15s^4t^2 + 20s^3t^3 + 15s^2t^4 + 6st^5 + t^6 \equiv s^6 + t^6 + 3(st^2 + s^2t)^2 + 2s^3t^3 \pmod{6R}$.

We now prove that S is closed under addition as follows: let $\alpha, \beta \in S$. Hence, there exist $x_0, x_1, x_2, y_0, y_1, y_2 \in R$ such that $\alpha = x_0^6 - 2x_1^3 + 3x_2^2 \pmod{6R}$ and $\beta = y_0^6 - 2y_1^3 + 3y_2^2 \pmod{6R}$. Hence using the above relations we get that

$$\alpha + \beta \pmod{6R} \equiv (x_0^6 - 2x_1^3 + 3x_2^2) + (y_0^6 - 2y_1^3 + 3y_2^2) \pmod{6R}$$

$$\equiv (x_0+y_0)^6 - 2(x_0y_0)^3 + 3(x_0y_0^2 + x_0^2y_0)^2 - 2(x_1+y_1)^3 + 3(x_2+y_2)^2 \pmod{6R}$$

$$\equiv (x_0+y_0)^6 - 2(x_0y_0 + x_1 + y_1)^3 + 3(x_0^2y_0 + x_0y_0^2 + x_2 + y_2)^2 \pmod{6R}.$$

Choose $z_0 = x_0 + y_0, z_1 = x_0y_0 + x_1 + y_1, z_2 = x_0^2y_0 + x_0y_0^2 + x_2 + y_2$, to get that $\alpha + \beta = z_0^6 - 2z_1^3 + 3z_2^2 \pmod{6R}$. This completes the proof that S is closed under addition. Note now that for any $a \in S$, we have that $a, 2a, 3a, \ldots, 6a \in S$. In fact, $6a = 0$ in S and thus $0 = a + 5a = 5a + a$ in S gives that $-a = 5a$. As $5a$ is in S, this proves that every element of S has an additive inverse in S. Therefore S is a subgroup of R. $\square$

Theorem 2.3. *Let R be a commutative ring with unity. Then $A \in M_2(R)$ is a sum of sixth powers of matrices in $M_2(R)$ if and only if Tr $A \pmod{6R} \in S$.*

Proof. Let $A = \sum_{i=1}^k A_i^6$, where $A, A_i \in M_2(R)$ for $1 \le i \le k$. By Lemma 2.1, there exist $x_{i0}, x_{i1}, x_{i2} \in R$ such that $\mathrm{Tr}\ A_i^6 \equiv x_{i0}^6 - 2x_{i1}^3 + 3x_{i2}^2 \pmod{6R}$. Hence, $\mathrm{Tr}\ A \equiv \sum_{i=1}^k \mathrm{Tr}\ A_i^6 \pmod{6R} \equiv \sum_{i=1}^k (x_{i0}^6 - 2x_{i1}^3 + 3x_{i2}^2) \pmod{6R}$. Since each term of this summation is in S and S is additively closed, from Lemma 2.2 we get that $\mathrm{Tr}\ A \pmod{6R} \in S$.

Conversely assume $A \in M_2(R)$ with Tr $A (\text{mod } 6R) \in S$. Therefore,

$$\begin{aligned}
\text{Tr } A &= x^6 - 2y^3 + 3z^2 + 6b' \text{ for some } b' \in R \\
&= x^6 - 2y^3 + (2z^3 + 3z^2 - 6z + 2) - 2z^3 - 2 + 6z + 6b' \\
&= \text{Tr}\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}^6 + \text{Tr}\begin{pmatrix} 0 & y \\ -1 & 0 \end{pmatrix}^6 + \text{Tr}\begin{pmatrix} 1 & -z+1 \\ -1 & 0 \end{pmatrix}^6 \\
&\quad + \text{Tr}\begin{pmatrix} 0 & z \\ -1 & 0 \end{pmatrix}^6 + \text{Tr}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^6 + 6(z + b').
\end{aligned}$$

Let $A_1 = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & y \\ -1 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & -z+1 \\ -1 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 0 & z \\ -1 & 0 \end{pmatrix}$ and $A_5 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The above equations give $\text{Tr}(A - \sum_{i=1}^{5} A_i^6) = 6(z + b')$. Suppose we now prove that every element of the type $6b, b \in R$, is a sum of traces of sixth powers of matrices, i.e., suppose we prove that $6b = \sum_{j=1}^{s} \text{Tr } B_j^6$, with $B_j \in M_2(R); s \geq 1$. Then we get that $\text{Tr}(A - \sum_{i=1}^{5} A_i^6) = \text{Tr}(\sum_{j=1}^{s} B_j^6)$, i.e., $\text{Tr}(A - \sum_{i=1}^{5} A_i^6 - \sum_{j=1}^{s} B_j^6) = 0$.

By Theorem 1.3 , we conclude that $(A - \sum_{i=1}^{5} A_i^6 - \sum_{j=1}^{s} B_j^6)$ is a sum of sixth powers in $M_2(R)$ and consequently A itself is also a sum of sixth powers in $M_2(R)$. Hence in order to prove Tr A is a sum of traces of sixth powers of matrices, the argument above shows that now it is enough to prove that every element of the type $6b$, $b \in R$, is a sum of traces of sixth powers of matrices in $M_2(R)$. So consider,

$$\begin{aligned}
6b &= (2b^3 + 9b^2 + 6b + 1) - 2b^3 - 9b^2 - 1 \\
&= (2b^3 + 9b^2 + 6b + 1) - 2b^3 + (-12b^2 + 3b^2) - 1 \\
&= \text{Tr}\begin{pmatrix} 1 & -b \\ -1 & 0 \end{pmatrix}^6 + \text{Tr}\begin{pmatrix} 0 & b \\ -1 & 0 \end{pmatrix}^6 + 2(2b^6 + 3b^4 - 6b^2 + 2) \\
&\quad + (2b^6 - 6b^4 + 3b^2 + 2) - 6b^6 - 6 - 1 \\
&= \text{Tr}\begin{pmatrix} 1 & -b \\ -1 & 0 \end{pmatrix}^6 + \text{Tr}\begin{pmatrix} 0 & b \\ -1 & 0 \end{pmatrix}^6 + 2\text{ Tr}\begin{pmatrix} -1 & b^2-1 \\ 1 & 0 \end{pmatrix}^6 + \\
&\quad \text{Tr}\begin{pmatrix} -b & b^2-1 \\ -1 & 0 \end{pmatrix}^6 + 3\text{ Tr}\begin{pmatrix} 0 & b^2 \\ -1 & 0 \end{pmatrix}^6 - 8 + 1 \\
&= \text{Tr}\begin{pmatrix} 1 & -b \\ -1 & 0 \end{pmatrix}^6 + \text{Tr}\begin{pmatrix} 0 & b \\ -1 & 0 \end{pmatrix}^6 + 2\text{ Tr}\begin{pmatrix} -1 & b^2-1 \\ 1 & 0 \end{pmatrix}^6 + \\
&\quad \text{Tr}\begin{pmatrix} -b & b^2-1 \\ -1 & 0 \end{pmatrix}^6 + 3\text{ Tr}\begin{pmatrix} 0 & b^2 \\ -1 & 0 \end{pmatrix}^6 + 4\text{ Tr}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^6 + \\
&\quad \text{Tr}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}^6.
\end{aligned}$$

Applying the above with b replaced by $z+b'$, we get that $6(z+b')$ is a sum of traces of sixth powers in $M_2(R)$. This completes the proof that if $A \in M_2(R)$ with Tr $A \pmod{6R} \in S$, then A is a sum of sixth powers in $M_2(R)$. □

Having obtained conditions for a single 2×2 matrix over R to be a sum of sixth powers of matrices, we now find an equivalent condition for every 2×2 matrix over R to be a sum of sixth powers of matrices.

Corollary 2.4. *Let R be a commutative ring with unity. The following statements are equivalent.*

(1) Every matrix in $M_2(R)$ is a sum of sixth powers of matrices in $M_2(R)$.

(2) Every element of R is a square $\pmod{2R}$ *and a cube* $\pmod{3R}$.

Proof. We first prove (1) implies (2). Given $\alpha \in R$, consider $A = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \in M_2(R)$. By (1), $A = \sum_{i=1}^{s} B_i^6$, with $B_i \in M_2(R)$. Thus, $A = \sum_{i=1}^{s} (B_i^3)^2 = \sum_{i=1}^{s} (B_i^2)^3$, which shows that A is a sum of squares as well as a sum of cubes in $M_2(R)$. We first prove that every element of R is a square $\pmod{2R}$. By Theorem 1.1 and use of the identity $(z+w)^2 \equiv z^2 + w^2 \pmod{2R}$, it follows that $\alpha = \text{Tr } A$ is a sum of squares and hence a square $\pmod{2R}$. Since $\alpha \in R$ was arbitrary, we get that every element of R is a square $\pmod{2R}$. From Theorem 1.7 applied with $n = 2$ and $k = 3$, it follows that every element of R is a cube $\pmod{3R}$. This completes the proof of (1) implies (2).

We now prove (2) implies (1). Let $A \in M_2(R)$. Since every element of R is a square modulo $2R$, there exist $x_0, x_1 \in R$ such that Tr $A = x_0^2 - 2x_1$. Also since every element of R is a cube modulo $3R$, there exist a_0, a_1, b_0, b_1 in R such that $x_0 = a_0^3 - 3a_1$ and $x_1 = b_0^3 - 3b_1$. Therefore Tr $A = (a_0^3 - 3a_1)^2 - 2(b_0^3 - 3b_1) = a_0^6 - 2b_0^3 + 3a_1^2 + 6(b_1 - a_0^3 a_1 + a_1^2)$. Hence, Tr $A \pmod{6R} \in S$ and by Theorem 2.3, we conclude that $A \in M_2(R)$ is a sum of sixth powers of matrices in $M_2(R)$. □

2.2. $n \times n$ Matrices, $n \geq 3$.

We begin this section by stating a partial implication in Theorem 2.3 which holds for all $n \geq 2$. This fact uses the same idea as the one in the proof of Theorem 1.6.

Corollary 2.5. *Let $A \in M_n(R)$, $n \geq 3$. If Tr $A \pmod{6R} \in S$, then A is a sum of sixth powers of matrices in $M_n(R)$.*

Proof. Let $A \in M_n(R)$ such that Tr $A \pmod{6R} \in S$. Then there exist $x, y, z, b \in R$ such that Tr $A = x^6 - 2y^3 + 3z^2 + 6b$. By Theorem 2.3, there exist $A_i \in M_2(R)$ such that $\sum_{i=1}^{r} \text{Tr } A_i^6 = x^6 - 2y^3 + 3x^2 + 6b = \text{Tr } A$. For each i, define $B_i =$

$\begin{pmatrix} A_i & 0 \\ 0 & 0_{n-2} \end{pmatrix} \in M_n(R)$. Then Tr $B_i^6 = =$ Tr A_i^6. Hence, $\mathrm{Tr}(A - \sum_{i=1}^r B_i^6) = 0$. By Theorem 1.3, we conclude that A itself is a sum of sixth powers in $M_n(R)$. □

We wish to continue our investigation for $n \geq 3$. Note that Corollary 2.4 above was motivated by the following result.

Lemma 2.6 ([5], Lemma 2). *Let R be a commutative ring with unity. Let $n \geq k \geq 2$. Then the following statements are equivalent.*

(1) Every matrix in $M_n(R)$ is a sum of k-th powers in $M_n(R)$.

(2) Every matrix in $M_n(R)$ is a sum of seven k-th powers in $M_n(R)$.

(3) For every prime p dividing k, every element of R is a p-th power $\pmod{pR}$.

From Lemma 2.6, it follows that for all $n \geq 6$, every matrix in $M_n(R)$ is a sum of sixth powers in $M_n(R)$ if and only if every element of R is a square $\pmod{2R}$ and a cube $\pmod{3R}$. Hence, in view of Corollary 2.4, it now remains to prove the equivalence of conditions (1) and (3) of Lemma 2.6 above for sixth powers only when $n = 3, 4$ and 5.

Corollary 2.7. *Let R be a commutative ring with unity and $n = 3, 4, 5$. The following statements are equivalent.*

(1) Every matrix in $M_n(R)$ is a sum of sixth powers of matrices in $M_n(R)$.

(2) Every element of R is a square $\pmod{2R}$ *and a cube* $\pmod{3R}$.

Proof. We first prove (1) implies (2). For this note that if every $A \in M_n(R)$ is a sum of sixth powers of matrices in $M_n(R)$, then it is a sum of squares of matrices in $M_n(R)$ and also a sum of cubes of matrices in $M_n(R)$. Since $n = 3, 4, 5$ are all greater than or equal to $k = 2, 3$ we can now apply Lemma 2.6 in each of the cases $n = 3, 4, 5$ above and $k = 2, 3$ (for each value of n) to deduce that every element of R is a square $\pmod{2R}$ and a cube $\pmod{3R}$.

We now prove (2) implies (1). The fact that every element of R is a square $\pmod{2R}$ and a cube $\pmod{3R}$ implies exactly as in Corollary 2.4 (see the part (2) implies (1)) that the trace $\pmod{6R}$ of every matrix in $M_n(R)$ lies in S. Hence, by Corollary 2.5, we get that $A \in M_n(R)$ is a sum of sixth powers in $M_n(R)$. □

The results proved in this section establish the following theorem:

Theorem 2.8. *Let R be a commutative ring with unity and let $n \geq 2$ be an integer. Then the following statements are equivalent.*

(1) Every matrix in $M_n(R)$ is a sum of sixth powers of matrices in $M_n(R)$.

(2) Every element of R is a square $\pmod{2R}$ *and a cube* $\pmod{3R}$.

3. Criteria for Matrices as Sums of Eighth Powers

In this section, we wish to derive results for eighth powers of matrices, similar to those derived in the previous section. For this we begin with the following lemma.

Lemma 3.1. *Let $A \in M_2(R)$ be such that $A = B^8$, for some $B \in M_2(R)$. Then there exist $t, \delta \in R$ such that Tr $A = t^8 - 8t^6\delta + 20t^4\delta^2 - 16t^2\delta^3 + 2\delta^4$.*

Proof. For any $B' \in M_2(R)$, we have $B'^2 = (\text{Tr } B')B' - (\det B')I_2$.

We know that if $t = \text{Tr } B$ and $\delta = \det B$, the above equation applied with $B' = B$, gives that Tr $B^2 = t^2 - 2\delta$. Further for $B' = B^2$, we get $B^4 = (\text{Tr } B^2)B^2 - (\det B^2)I_2$. Hence, Tr $B^4 = (t^2 - 2\delta)^2 - 2\delta^2 = t^4 - 4t^2\delta + 2\delta^2$. Again the above equation for $B' = B^4$, gives $B^8 = (\text{Tr } B^4)B^4 - (\det B^4)I_2$. Hence Tr $B^8 = (t^4 - 4t^2\delta + 2\delta^2)^2 - 2\delta^4 = t^8 - 8t^6\delta + 20t^4\delta^2 - 16t^2\delta^3 + 2\delta^4$. Hence, if $A = B^8$ and $t = \text{Tr } B$ and $\delta = \det B$, we get that Tr $A = \text{Tr } B^8 = t^8 - 8t^6\delta + 20t^4\delta^2 - 16t^2\delta^3 + 2\delta^4$. □

In order to prove Theorem 1.2, Richman had used the theory of Witt vectors. For any prime p and integer $s \geq 1$, he had defined the set

$$W(p, s, R) = \left\{x_0^{p^s} + px_1^{p^{s-1}} + \cdots + p^s x_s \colon\ x_0, x_1, \ldots, x_s \in R\right\}$$

and proved the results given below.

Proposition 3.2 ([9], Proposition 3.1). *The set $W(p, s, R)$ is closed under addition and subtraction.*

Proposition 3.3 ([9], Proposition 3.2). *Let $A \in M_n(R)$ and $n \geq 2$. Then for any prime p and integer $s \geq 1$, Tr $A^{p^s} \in W(p, s, R)$.*

In particular, if $B \in M_n(R)$, where $n \geq 2$ for the particular case $p = 2, s = 3$ we get that Tr $B^8 = x_0^8 + 2x_1^4 + 4x_2^2 + 8x_3$, with $x_0, x_1, x_2, x_3 \in R$. Motivated by the result on sixth powers, here also we can prove using Proposition 3.3 that the set of traces of eighth powers (mod $8R$) is a subgroup of R.

Lemma 3.4. *Let R be a commutative ring with unity. Define the set*

$$S_1 = \{x_0^8 + 2x_1^4 + 4x_2^2 \pmod{8R}\colon\ x_0, x_1, x_2 \in R\}.$$

Then S_1 is a subgroup of R.

Proof. By Proposition 3.2, $W(2, 3, R)$ is closed under addition. Going modulo $8R$ we deduce that S_1 is closed under addition too. As $8s = 0$, for all $s \in S_1$, we get that $7s$ serves as the additive inverse of an element $s \in S_1$. □

Theorem 3.5. *Let R be a commutative ring with unity and $n \geq 2$ be an integer. Then $A \in M_n(R)$ is a sum of eighth powers in $M_n(R)$ if and only if Tr $A \pmod{8R} \in S_1$, where $S_1 = \{x_0^8 + 2x_1^4 + 4x_2^2 \pmod{8R}\colon\ x_0, x_1, x_2 \in R\}$.*

Proof. If $A \in M_n(R)$ is a sum of eighth powers of matrices, by Proposition 3.3 and using the fact that S_1 is closed under addition, we get that Tr $A \pmod{8R} \in S_1$.

We now prove the converse. For this, note that we prove the converse only for $n = 2$. This is because using the form of the trace of the eighth power of a matrix as stated above Lemma 3.4 and the same observation as in Corollary 2.5 (with six replaced by eight), helps us to prove the result also for all $n \geq 3$. For this assume that $A \in M_2(R)$ is such that Tr $A \pmod{8R} \in S_1$. Hence there exist $a, b, c, d \in R$ such that Tr $A = a^8 + 2b^4 + 4c^2 + 8d = \text{Tr} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}^8 + \text{Tr} \begin{pmatrix} 0 & b \\ -1 & 0 \end{pmatrix}^8 + 4c^2 + 8d$. It suffices to prove that $4c^2$ and $8d$ are both sums of traces of eighth powers of matrices. This is because then, $\text{Tr}(A - \sum_{i=1}^{k} B_i^8) = 0$, for suitable $B_i \in M_2(R)$. By Theorem 1.3, A itself will be a sum of eighth powers of matrices, as required.

(a) We first prove that $4c^2$ is a sum of traces of eighth powers of matrices. For this, we write down a few matrices with the traces of their eighth powers.

Matrix C_i		Trace C_i^8
C_1	$\begin{pmatrix} 1 & -c \\ 1 & -1 \end{pmatrix}$	$2c^4 - 8c^3 + 12c^2 - 8c + 2$
C_2	$\begin{pmatrix} 1 & -c^2 - 1 \\ 1 & 0 \end{pmatrix}$	$2c^8 - 8c^6 - 16c^4 - 8c^2 - 1$
C_3	$\begin{pmatrix} 1 & -c + 1 \\ -1 & 0 \end{pmatrix}$	$2c^4 + 8c^3 - 16c^2 + 8c - 1$
C_4	$\begin{pmatrix} c - 1 & c \\ 1 & 1 \end{pmatrix}$	$c^8 + 8c^6 + 20c^4 + 16c^2 + 2$

With C_i as defined above, we have that

$$\begin{aligned}
4c^2 &= 12c^2 - 8c^2 \\
&= (2c^4 - 8c^3 + 12c^2 - 8c + 2) + (2c^8 - 8c^6 - 16c^4 - 8c^2 - 1) \\
&\quad -2c^8 + 8c^6 + 14c^4 + 8c^3 + 8c - 1 \\
&= \sum_{i=1}^{2} \text{Tr } C_i^8 - 2c^8 + 8c^6 + 12c^4 + (2c^4 + 8c^3 - 16c^2 + 8c - 1) + 16c^2 \\
&= \sum_{i=1}^{3} \text{Tr } C_i^8 - 3c^8 + (c^8 + 8c^6 + 20c^4 + 16c^2 + 2) - 8c^4 - 2
\end{aligned}$$

$$= \sum_{i=1}^{4} \text{Tr } C_i^8 - 23c^8 + (1 - 8c^4 + 20c^8 - 16c^{12} + 2c^{16}) - 3 + 16c^{12} - 2c^{16}$$

$$= \sum_{i=1}^{4} \text{Tr } C_i^8 + 23 \text{ Tr} \begin{pmatrix} c & c^2 \\ -1 & 0 \end{pmatrix}^8 + \text{Tr} \begin{pmatrix} 1 & c^4 \\ -1 & 0 \end{pmatrix}^8 + 3 \text{ Tr} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^8$$

$$+8 \text{ Tr} \begin{pmatrix} 0 & c^3 \\ -1 & 0 \end{pmatrix}^8 + 2 \text{ Tr} \begin{pmatrix} c^2 & c^4 \\ -1 & 0 \end{pmatrix}^8.$$

(b) Now we find matrices in $M_2(R)$, whose sum of the traces of eighth powers is $8d$. We do this by writing a table of some useful matrices:

Matrix D_i		Trace D_i^8
D_1	$\begin{pmatrix} -d & d+1 \\ -1 & 0 \end{pmatrix}$	$d^8 - 8d^7 + 12d^6 + 24d^5 - 26d^4$ $-40d^3 - 4d^2 + 8d + 2$
D_2	$\begin{pmatrix} d+1 & -d^2 \\ 1 & 0 \end{pmatrix}$	$-d^8 + 8d^7 + 12d^6 - 24d^5 - 30d^4$ $+8d^3 + 20d^2 + 8d + 1$
D_3	$\begin{pmatrix} d+1 & d+1 \\ -1 & 0 \end{pmatrix}$	$d^8 - 8d^6 - 8d^5 + 12d^4 + 24d^3$ $+12d^2 - 1$
D_4	$\begin{pmatrix} -d+1 & d-1 \\ 1 & 0 \end{pmatrix}$	$d^8 - 8d^6 + 8d^5 + 12d^4 - 24d^3$ $+12d^2 - 1$
D_5	$\begin{pmatrix} 1 & -d \\ -1 & 0 \end{pmatrix}$	$2d^4 + 16d^3 + 20d^2 + 8d + 1$
D_6	$\begin{pmatrix} 1 & -d-1 \\ -1 & 0 \end{pmatrix}$	$2d^4 + 24d^3 + 80d^2 + 104d + 47$
D_7	$\begin{pmatrix} 1 & -2d-2 \\ 1 & 0 \end{pmatrix}$	$32d^4 - 112d^2 - 112d - 31$
D_8	$\begin{pmatrix} 1 & d+1 \\ -1 & 0 \end{pmatrix}$	$2d^4 - 8d^3 - 16d^2 - 8d - 1$
D_9	$\begin{pmatrix} 1 & -d^2-1 \\ 1 & 0 \end{pmatrix}$	$2d^8 - 8d^6 - 16d^4 - 8d^2 - 1$
D_{10}	$\begin{pmatrix} 1 & -d+1 \\ -1 & 0 \end{pmatrix}$	$2d^4 + 8d^3 - 16d^2 + 8d - 1$
D_{11}	$\begin{pmatrix} 1 & -d \\ 1 & -1 \end{pmatrix}$	$2d^4 - 8d^3 + 12d^2 - 8d + 2$

With D_i as defined above, we have:

$$\begin{aligned}
8d &= (d^8 - 8d^7 + 12d^6 + 24d^5 - 26d^4 - 40d^3 - 4d^2 + 8d + 2) \\
&\quad -d^8 + 8d^7 - 12d^6 - 24d^5 + 26d^4 + 40d^3 + 4d^2 - 2 \\
&= \text{Tr } D_1^8 + (-d^8 + 8d^7 + 12d^6 - 24d^5 - 30d^4 + 8d^3 + 20d^2 + 8d + 1) \\
&\quad -24d^6 + 56d^4 + 32d^3 - 16d^2 - 8d - 3 \\
&= \sum_{i=1}^{2} \text{Tr } D_i^8 + (d^8 - 8d^6 - 8d^5 + 12d^4 + 24d^3 + 12d^2 - 1) \\
&\quad -d^8 - 16d^6 + 8d^5 + 44d^4 + 8d^3 - 28d^2 - 8d - 2 \\
&= \sum_{i=1}^{3} \text{Tr } D_i^8 + (d^8 - 8d^6 + 8d^5 + 12d^4 - 24d^3 + 12d^2 - 1) \\
&\quad -2d^8 - 8d^6 + 32d^4 + 32d^3 - 40d^2 - 8d - 1 \\
&= \sum_{i=1}^{4} \text{Tr } D_i^8 - 2d^8 - 8d^6 + 30d^4 + (2d^4 + 16d^3 + 20d^2 + 8d + 1) \\
&\quad + 16d^3 - 60d^2 - 16d - 2 \\
&= \sum_{i=1}^{5} \text{Tr } D_i^8 + (2d^4 + 24d^3 + 80d^2 + 104d + 47) + [(32d^4 - 112d^2) \\
&\quad -(112d + 31)] + (-2d^8 - 8d^6 - 4d^4 - 8d^3 - 28d^2 - 8d - 18) \\
&= \sum_{i=1}^{7} \text{Tr } D_i^8 - 2d^8 - 8d^6 - 6d^4 + (2d^4 - 8d^3 - 16d^2 - 8d - 1) \\
&\quad -12d^2 - 17 \\
&= \sum_{i=1}^{8} \text{Tr } D_i^8 - 4d^8 + (2d^8 - 8d^6 - 16d^4 - 8d^2 - 1) + 10d^4 - 4d^2 - 16 \\
&= \sum_{i=1}^{9} \text{Tr } D_i^8 - 4d^8 + 6d^4 + (2d^4 + 8d^3 - 16d^2 + 8d - 1) \\
&\quad + (2d^4 - 8d^3 + 12d^2 - 8d + 2) - 17 \\
&= \sum_{i=1}^{11} \text{Tr } D_i^8 - 4d^8 + 6d^4 - 17 \\
&= \sum_{i=1}^{11} \text{Tr } D_i^8 + 4\text{Tr } \begin{pmatrix} d & d^2 \\ -1 & 0 \end{pmatrix}^8 + 3\text{Tr } \begin{pmatrix} 0 & d \\ -1 & 0 \end{pmatrix}^8 + 17 \text{ Tr } \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^8.
\end{aligned}$$

This completes the proof of the fact that if $A \in M_n(R)$ is such that Tr $A (\text{mod } 8R) \in S_1$, then A is a sum of eighth powers in $M_n(R)$. $\square$

We summarize our main result for eighth powers below.

Theorem 3.6. *Let R be a commutative ring with unity. Let $n \geq 2$ be an integer. The following statements are equivalent.*

(1) Every matrix in $M_n(R)$ is a sum of eighth powers of matrices in $M_n(R)$.

(2) Every element of R is a square $(\text{mod } 2R)$.

Proof. We first prove (1) implies (2). If every matrix in $M_n(R)$ is a sum of eighth powers in $M_n(R)$, then it is also a sum of squares in $M_n(R)$. Hence, by Lemma 2.6 applied with $k = 2$, we get that every element of R is a square $(\text{mod } 2R)$.

We now prove (2) implies (1). Let $A \in M_n(R)$. Then, Tr $A \in R$ and Tr A is a square modulo $2R$. Hence there exist $l, m \in R$ such that Tr $A = l^2 + 2m$. Since l, m are also squares $(\text{mod } 2R)$, write $l = l_0^2 + 2l_1$ and $m = m_0^2 + 2m_1$, to get Tr $A = (l_0^2+2l_1)^2+2(m_0^2+2m_1) = l_0^4+2m_0^2+4(l_0^2l_1+l_1^2+m_1)$. Let $n_0 = l_0^2l_1+l_1^2+m_1 \in R$. Then Tr $A = l_0^4 + 2m_0^2 + 4n_0$. Further writing $l_0 = a_0^2 + 2a_1$ and $m_0 = b_0^2 + 2b_1$, and $n_0 = c_0^2 + 2c_1$, we get that Tr $A = (a_0^2 + 2a_1)^4 + 2(b_0^2 + 2b_1)^2 + 4(c_0^2 + 2c_1)$. Simplifying and going modulo $8R$, we have Tr $A \equiv a_0^8 + 2b_0^4 + 4c_0^2 (\text{mod } 8R)$. Hence Tr $A (\text{mod } 8R)$ is in S_1. By Theorem 3.5, A is a sum of eighth powers in $M_n(R)$. □

4. Discriminant Criteria for a Matrix to Be a Sum of Sixth and Eighth Powers of Matrices

Let K be an algebraic number field and $\mathcal{O}$ be an order of K. Here we will find the condition on the discriminant of the order $\mathcal{O}$ so that every matrix over $\mathcal{O}$ is a sum of sixth (eighth) powers of matrices. We will use the following lemma.

Lemma 4.1 ([2], Theorem 3.2). *Let $\mathcal{O}$ be an order in a number field K and p be a prime. The following statements are equivalent.*

1. *Every element of $\mathcal{O}$ is a p-th power modulo $p\mathcal{O}$.*
2. *$x \in \mathcal{O}, x^p \in p\mathcal{O}$ imply $x \in p\mathcal{O}$.*
3. *$(p, disc\ \mathcal{O}) = 1$.*

Theorem 4.2. *Let $\mathcal{O}$ be an order of an algebraic number field and let $n \geq 2$ be an integer. Then every matrix in $M_n(\mathcal{O})$ is a sum of sixth powers in $M_n(\mathcal{O})$ if and only if $(6,\ disc\ \mathcal{O}) = 1$.*

Proof. If every matrix in $M_n(\mathcal{O})$ is a sum of sixth powers of matrices, then clearly every matrix in $M_n(\mathcal{O})$ is a sum of squares and also a sum of cubes of matrices in $M_n(\mathcal{O})$. Therefore by Theorem 1.4 and Theorem 1.7, we get $(2,\ \text{disc}\ \mathcal{O}) = (3,\ \text{disc}\ \mathcal{O}) = 1$. Combining we have $(6,\ \text{disc}\ \mathcal{O}) = 1$.

Now assume that $(6,\ \text{disc } \mathcal{O}) = 1$. This implies $(2,\ \text{disc } \mathcal{O}) = 1$ and $(3,\ \text{disc } \mathcal{O}) = 1$. The result now follows from Lemma 4.1 and Theorem 2.8. □

Theorem 4.3. *Let $\mathcal{O}$ be an order in a number field and let $n \geq 2$ be an integer. Then every matrix in $M_n(\mathcal{O})$ is a sum of eighth powers in $M_n(\mathcal{O})$ if and only if $(2, disc\ \mathcal{O}) = 1$.*

Proof. If every matrix in $M_n(\mathcal{O})$ is a sum of eighth powers in $M_n(\mathcal{O})$, it is also a sum of squares in $M_n(\mathcal{O})$. By Theorem 1.4, we get that $(2,\ \text{disc } \mathcal{O}) = 1$. We now assume $(2, \text{disc } \mathcal{O}) = 1$. By Lemma 4.1 and Theorem 3.6 applied with $p = 2$, we conclude that every $A \in M_n(\mathcal{O})$ is a sum of eighth powers in $M_n(\mathcal{O})$. □

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