#A27 INTEGERS 22 (2022)

FINDING ALMOST SQUARES VII

Tsz Ho Chan
Department of Mathematics, Kennesaw State University, Marietta, Georgia
tchan4@kennesaw.edu

Received: 8/6/21, Accepted: 2/16/22, Published: 2/28/22

Abstract

A positive integer n is called an almost square of type 2 if it can be factored in two different ways as $n = a_1 b_1 = a_2 b_2$ with $a_1, a_2, b_1, b_2 \approx \sqrt{n}$. We are interested in how short an interval around x, say $[x - x^\phi, x + x^\phi]$, contains such kinds of integers. In this note, we improve upon previous upper bounds on ϕ using exponential sum techniques including recent breakthroughs on Vinogradov's mean value theorem.

1. Introduction and Main Results

In a series of papers, [3], [4], [5], [6], [7], [8], the author studied short intervals containing *almost squares*, which are positive integers n that can be factored as $n = ab$ with a, b close to $\sqrt{n}$. For example, $n = 9999 = 99 \times 101$ is an almost square. We say that a positive integer n is an *almost square of type* 2 if it has two different such factorizations (i.e., $n = a_1 b_1 = a_2 b_2$ with $a_1, a_2, b_1, b_2 \approx \sqrt{n}$). For example, $n = 99990000$ is an almost square of type 2 as $n = 99990000 = 9999 \times 10000 = 9900 \times 10100$. More precisely, for $0 \le \theta \le 1/2$ and $C > 0$, we make the following definition.

Definition 1. An integer n is a (θ, C)-*almost square of type* 2 if $n = a_1 b_1 = a_2 b_2$ for some integers $a_2 < a_1 \le b_1 < b_2$ in the interval $[n^{1/2} - Cn^\theta, n^{1/2} + Cn^\theta]$.

Given $0 \le \theta \le 1/2$, we say that ϕ is an *admissible exponent* if there exist constants $C_\theta, D_\theta > 0$ such that the interval $[x - D_\theta x^\phi, x + D_\theta x^\phi]$ always contains a (θ, C_θ)-almost square of type 2 for all sufficiently large x. Define

$$g(\theta) := \inf \phi,$$

where the infimum is over all such admissible exponents. In [3], the author proved that no such admissible exponent ϕ exists when $0 \le \theta < 1/4$ and

$$1 - 2\theta \le g(\theta) \le \min(1 - \theta, 2/3), \quad \text{if} \quad 1/4 \le \theta \le 1/2. \tag{1}$$

Later in [6] and [7], the author improved the upper bound to

$$g(\theta) \leq \begin{cases} 5/8, & \text{if } 1/4 \leq \theta < 5/16, \\ 17/32, & \text{if } 5/16 \leq \theta \leq 743/2306, \\ 1/2, & \text{if } 743/2306 < \theta \leq 1/3, \\ 1 - 3\theta/2, & \text{if } 1/3 < \theta \leq 1/2. \end{cases} \tag{2}$$

The purpose of the current paper is to showcase the usage of exponential sum techniques to improve some of these upper bounds. Firstly, we obtain the following theorem to illustrate the basic ideas.

Theorem 1. *For* $11/34 < \theta < 3/8$*, we have* $g(\theta) \leq 25/28 - 17\theta/14$*.*

Theorem 1 is based on the classical van der Corput exponential sum estimate:

$$\sum_{N < n \leq 2N} e(f(n)) \ll \Lambda^{1/6} N + \Lambda^{-1/6} N^{1/2} \text{ with } |f'''(x)| \asymp \Lambda \text{ for } x \in (N, 2N].$$

When $f'''(x)$ is monotonic, this was improved nontrivially by Robert [14] to

$$\sum_{N < n \leq 2N} e(f(n)) \ll \Lambda^{1/6 + 1/1354} N \text{ provided } N \geq \Lambda^{-1}.$$

This already could lead to improvement to Theorem 1. In [16], Robert and Sargo considered mean values of exponential sums and proved

$$\frac{1}{H} \sum_{h=H+1}^{2H} \Big| \sum_{N < n \leq 2N} e\Big(\frac{h}{H} f(n)\Big) \Big| \ll_\epsilon N^\epsilon \Big(\frac{N \Lambda^{1/6}}{H^{1/9}} + N \Lambda^{1/5} + N^{3/4} \Big) + \Lambda^{-1/3}. \tag{3}$$

We are going to use a consequence of this to get the following slightly better result.

Theorem 2. *For* $29/90 < \theta < 3/8$*, we have* $g(\theta) \leq 67/76 - 45\theta/38 < 1/2$*.*

One can check that $67/76 - 45\theta/38 < 25/28 - 17\theta/14 < 1 - 3\theta/2$ when $\theta < 3/8$. Also, $29/90 < 11/34 < 1/3$. Hence, we push a little further on where $g(\theta) < 1/2$. Using recent breakthroughs on Vinogradov's mean value theorem by Wooley [17] and Bourgain, Demeter and Guth [2], we obtain the following upper bounds.

Theorem 3. *For* $1/4 \leq \theta \leq 1/2$*,*

$$g(\theta) \leq \begin{cases} 5/8, & \textit{if } 1/4 \leq \theta < 5/16, \\ 17/32, & \textit{if } 5/16 \leq \theta \leq 19/59, \\ 1/2, & \textit{if } 19/59 < \theta \leq 29/90, \\ 67/76 - 45\theta/38, & \textit{if } 29/90 < \theta \leq 89/250, \\ 47/48 - 35\theta/24, & \textit{if } 89/250 < \theta \leq 35/94, \\ 79/80 - 59\theta/40, & \textit{if } 35/94 < \theta \leq 67/174. \end{cases} \tag{4}$$

Note that $19/59 = 0.3220...$, $29/90 = 0.3222...$, $89/250 = 0.356$, $35/94 = 0.3723...$, *and* $67/174 = 0.3850... = s(4)$. *In general, for* $m \geq 5$, *we have the following series of inequalities:*

$$g(\theta) \leq \begin{cases} \left(1 - \frac{1}{4m(m-1)}\right) - \left(\frac{3}{2} - \frac{1}{2m(m-1)}\right)\theta, & \text{if } \frac{1}{2} - s_{m-1} \leq \theta \leq \frac{1}{2} - r_m, \\ \left(1 - \frac{m}{4(m^2-m+1)}\right) - \left(\frac{3}{2} - \frac{m+1}{2(m^2-m+1)}\right)\theta, & \text{if } \frac{1}{2} - r_m \leq \theta \leq \frac{1}{2} - s_m, \end{cases}$$

where

$$r_x := \frac{x^2 - x}{2(x^3 - x^2 - 1)} \quad \text{and} \quad s_x := \frac{x^2 + x}{2(x^3 + x^2 + 2x - 1)}.$$

Moreover, $g(\theta) \leq 1 - 3\theta/2 - (1/2 - \theta)^3$ *for* $11/34 \leq \theta < 1/2$.

So, we improve upon (2) by a magnitude of $(1/2 - \theta)^3$. Unfortunately, this improvement tends to 0 as θ goes to $1/2$ and we still cannot beat the $1/4$ upper bound for $g(1/2)$ which is the currently best known bound (see Lemma 1 for example).

Before proceeding, let us make some remarks. From (2), we may assume $\phi \leq 5/8$. If one can show that the difference between the largest and smallest factors of $n = a_1 b_1 = a_2 b_2$ satisfies

$$b_2 - a_2 \leq \frac{C}{2} x^\theta, \tag{5}$$

then, for $n \in [x - Dx^\phi, x + Dx^\phi]$,

$$n^{1/2} - Cn^\theta < b_2 - \frac{C}{2}x^\theta \leq a_2 < b_2 \leq a_2 + \frac{C}{2}x^\theta < n^{1/2} + Cn^\theta$$

because $a_2 < n^{1/2} < b_2$ and $x/2 < n < 2x$ for x sufficiently large. Thus, from now on, we simply check inequality (5) to find almost squares of type 2.

We will use the following notation. We will use $[x]$ to denote the integer part of x. The symbols $f(x) = O(g(x))$, $f(x) \ll g(x)$ and $g(x) \gg f(x)$ are equivalent to $|f(x)| \leq Cg(x)$ for some constant $C > 0$. And $f(x) \asymp g(x)$ means that $C_1 f(x) \leq g(x) \leq C_2 f(x)$ for some constants $0 < C_1 < C_2$. Finally $f(x) = O_{\lambda_1,...\lambda_k}(g(x))$, $f(x) \ll_{\lambda_1,...,\lambda_k} g(x)$, $g(x) \gg_{\lambda_1,...\lambda_k} f(x)$ or $g(x) \asymp_{\lambda_1,...\lambda_k} f(x)$ mean that the implicit constant C may depend on $\lambda_1, ..., \lambda_k$.

2. Preliminaries

Lemma 1. *For* $x \geq 1$, *one can find a difference of two squares* $n^2 - m^2$ *with* $n > m \geq 1$ *such that* $|n^2 - m^2 - x| \leq 7\sqrt[4]{x}$.

Proof. Take $k = [\sqrt{x}] \geq 1$. Then $k^2 \leq x < (k+1)^2$ which gives $0 \leq x - k^2 < 2k + 1$. This implies

$$5 \leq 2k + 3 < (k+2)^2 - x \leq 4k + 4.$$

Now, take $m = [\sqrt{(k+2)^2 - x}] \geq 2$. We have $m^2 \leq (k+2)^2 - x < (m+1)^2$ and

$$0 \leq (k+2)^2 - m^2 - x < 2m + 1 \leq 2\sqrt{4k+4} + 1 \leq 2\sqrt{8\sqrt{x}} + 1 \leq 7\sqrt[4]{x}$$

as $x \geq 1$. □

We will need to produce small fractional parts of some real-valued function. To attain this goal, we use the Erdős-Turán inequality in the following form (see [12, Corollary 1.2] for example).

Lemma 2. *Let $x_1, x_2, \cdots, x_J$ be a sequence of real numbers. Suppose that M is a positive integer chosen so that*

$$\sum_{l=1}^{M} \Big| \sum_{j=1}^{J} e(lx_j) \Big| \leq \frac{J}{10},$$

where $e(x) = e^{2\pi i x}$. Then, for every interval $[\alpha, \beta] \subset [0,1]$ of length $\beta - \alpha \geq \frac{4}{M+1}$, there are at least $\frac{1}{2} J(\beta - \alpha)$ of the numbers x_j in the sequence with fractional parts $\{x_j\} \in [\alpha, \beta]$.

Thus, we need some exponential sum estimates. The first one is a classical van der Corput estimate. See [9] or [15, Theorem 2], for example, where the second paper contains a nice survey on the latest developments of van der Corput's method.

Lemma 3. *Suppose that $f(x)$ is a real-valued function over the interval $[N, 2N]$ such that its third derivative is continuous and satisfies $|f'''(x)| \asymp \Lambda$. Then*

$$\sum_{N < n \leq 2N} e(f(n)) \ll \Lambda^{1/6} N + \Lambda^{-1/6} N^{1/2}.$$

In [16], Robert and Sargos considered mean values of exponential sums

$$S := \frac{1}{H} \sum_{h=H+1}^{2H} \Big| \sum_{N < n \leq 2N} e\Big(\frac{h}{H} f(n)\Big) \Big|,$$

and proved inequality (3) by Weyl's shift, van der Corput's method, multi-variable partial summation, and Bombieri and Iwaniec's double large sieve. They applied it to study lattice points close of curves and we summarize their Corollary 2 as follows.

Lemma 4. *Suppose that $g(x)$ is a function with continuous third derviative which satisfies $g'''(x) \asymp \Lambda$ for $1 \leq x \leq M$. Let*

$$\mathcal{R}(g, \delta) := \#\{m \in \{1, 2, ..., M\} : ||g(m)|| \leq \delta\},$$

where $||x|| := \min_{n \in \mathbb{Z}} |x - n|$ stands for the distance from x to the nearest integer. Then, for any $0 \leq \delta < 1/2$,

$$\mathcal{R}(g, \delta) = 2M\delta + O_\epsilon(M^{1+\epsilon}\Lambda^{3/19} + M^{3/4+\epsilon}) + O(\Lambda^{-1/3}).$$

It is worth mentioning that Huxley's book [11] contains detailed studies of such lattice point problems and related exponential sum techniques.

Recently, Wooley [17] and Bourgain, Demeter and Guth [2] made significant contributions to exponential sums by proving Vinogradov's mean value theorem. See [13] for a nice survey on these results. As a consequence, Heath-Brown [10, Theorem 1] obtained the following new exponential sum estimates.

Lemma 5. *Let $m \geq 3$ be an integer. Suppose that $f(x) : [0, N] \to \mathbb{R}$ has continuous derivatives of order up to m on $(0, N)$. Suppose further that*

$$0 < \lambda_m \leq f^{(m)}(x) \leq A\lambda_m, \quad x \in (0, N).$$

Then

$$\sum_{n \leq N} e(f(n)) \ll_{A,m,\epsilon} N^{1+\epsilon}(\lambda_m^{\frac{1}{m(m-1)}} + N^{-\frac{1}{m(m-1)}} + N^{-\frac{2}{m(m-1)}} \lambda_m^{-\frac{2}{m^2(m-1)}}).$$

3. Basic Strategy and Proof of Theorem 1

First, let us consider integers of the form $d(d+1)(e-k)(e+k) \in [x, x+x^{\phi}]$ with $d = [x^{\alpha}]$ for some $0 < \alpha < 1/2$. By Lemma 1, one can find a difference of two squares $e^2 - k^2$ such that

$$\Big| e^2 - k^2 - \frac{x}{d(d+1)} \Big| \ll \Big(\frac{x}{d(d+1)} \Big)^{1/4} \ll x^{1/4-\alpha/2}$$

with $e \ll \sqrt{\frac{x}{d(d+1)}} \ll x^{1/2-\alpha}$ and $k \ll \sqrt[4]{\frac{x}{d(d+1)}} \ll x^{1/4-\alpha/2}$. This implies

$$|x - d(d+1)(e-k)(e+k)| \ll x^{1/4+3\alpha/2}.$$

Observe that $n = d(d+1)(e-k)(e+k)$ has two different factorizations,

$$n = [d(e+k)] \cdot [(d+1)(e-k)] = [d(e-k)] \cdot [(d+1)(e+k)],$$

and the difference between the largest and smallest factors is

$$(d+1)(e+k) - d(e-k) = e + 2dk + k \ll x^{1/2-\alpha} + x^{1/4+\alpha/2}.$$

Pick $\alpha = 1/6$. Then $\theta = 1/2 - \alpha = 1/4 + \alpha/2 = 1/3$ and $\phi = 1/4 + 3\alpha/2 = 1/2$. Hence, we find a $(1/3, C)$-almost square of type 2, namely $n = d(d+1)(e-k)(e+k)$, in the interval $[x - Dx^{1/2}, x + Dx^{1/2}]$. This gives $g(1/3) \leq 1/2$.

Proof. Now, instead of fixing $d = [x^{\alpha}]$, we allow d to vary over an interval $(x^{\alpha}, 2x^{\alpha}]$. Define

$$F(d) := \sqrt{\frac{x}{d(d+1)}} = \frac{x^{1/2}}{d} \frac{1}{(1+1/d)^{1/2}} = x^{1/2} \Big(\frac{1}{d} - \frac{1}{2d^2} + \frac{3}{8d^3} - \cdots \Big).$$

One can easily check that, for $i = 1, 2, 3$, the i-th derivative

$$F^{(i)}(d) \asymp (-1)^i \frac{x^{1/2}}{x^{(1+i)\alpha}}$$

for $d \in (x^\alpha, 2x^\alpha]$. We apply Lemma 3 with $f(x) = -lF(x)$ and get

$$\sum_{l \le M} \Big| \sum_{d \in [x^\alpha, 2x^\alpha]} e(lF(d)) \Big| \ll M^{7/6} x^{1/12+\alpha/3} + M^{5/6} x^{7\alpha/6-1/12}$$

which is less than $x^\alpha/10$ provided

$$M \ll x^{4\alpha/7-1/14} \quad \text{and} \quad M \ll x^{1/10-\alpha/5}. \tag{6}$$

We need both exponents in (6) to be positive to ensure the existence of M. So, we require $1/8 < \alpha < 1/2$. Consequently, by Lemma 2,

$$e - \frac{5}{M} \le F(d) = \sqrt{\frac{x}{d(d+1)}} \le e - \frac{1}{M}$$

for some positive integers $d \in (x^\alpha, 2x^\alpha]$ and e. Thus,

$$e^2 - \frac{x}{d(d+1)} \asymp x^{4/7-11\alpha/7}$$

by choosing $M = [cx^{4\alpha/7-1/14}]$ for some small enough $c > 0$. We need $\alpha < 4/11$ to ensure that the exponent $4/7 - 11\alpha/7$ is positive. With $k = [\sqrt{e^2 - \frac{x}{d(d+1)}}]$, we have

$$\left| e^2 - k^2 - \frac{x}{d(d+1)} \right| \ll k \ll x^{2/7-11\alpha/14} \text{ or } |x - d(d+1)(e-k)(e+k)| \ll x^{2/7+17\alpha/14}.$$

Recall the two different factorizations $n = [d(e+k)] \cdot [(d+1)(e-k)] = [d(e-k)] \cdot [(d+1)(e+k)]$. The difference between the largest and smallest factors is

$$(d+1)(e+k) - d(e-k) = e + 2dk + k \ll x^{1/2-\alpha} + x^{2/7+3\alpha/14}. \tag{7}$$

Since our goal is to obtain a result with $\phi < 1/2$, this forces

$$\frac{2}{7} + \frac{17\alpha}{14} < \frac{1}{2} \quad \text{or} \quad \alpha < \frac{3}{17}.$$

In the range $1/8 < \alpha < 3/17$, the first upper bound in (7) dominates. Thus, with

$$\frac{11}{34} < \theta = \frac{1}{2} - \alpha < \frac{3}{8} \quad \text{and} \quad \phi = \frac{2}{7} + \frac{17\alpha}{14} = \frac{25}{28} - \frac{17\theta}{14},$$

there is a (θ, C)-almost square of type 2 in the interval $[x - Dx^\phi, x + Dx^\phi]$. This gives Theorem 1. □

4. Proof of Theorem 2

Proof. Again, we consider the function

$$F(d) = \sqrt{\frac{x}{d(d+1)}} = \frac{x^{1/2}}{d}\frac{1}{(1+1/d)^{1/2}} = x^{1/2}\Big(\frac{1}{d} - \frac{1}{2d^2} + \frac{3}{8d^3} - \cdots\Big)$$

for $d \in (x^\alpha, 2x^\alpha]$. Let $M = [x^\alpha]$ and $g(m) = F(m+M) + 2\delta$. One can check that $g'''(m) \asymp x^{1/2-4\alpha} =: \Lambda$ when $1 \le m \le M$. For $1/8 < \alpha < 8/45$ and $\epsilon > 0$, set

$$\delta := \frac{1}{x^{12\alpha/19 - 3/38 - \epsilon}}.$$

One can check that

$$M^{1+\epsilon}\Lambda^{3/19},\ M^{3/4+\epsilon},\ \text{and}\ \Lambda^{-1/3} \ll_\epsilon M^{1-\epsilon/2}\delta$$

when $1/8 < \alpha < 8/45$. By Lemma 4,

$$\#\{m \in \{1, 2, ..., M\} : n - \delta \le g(m) \le n + \delta \text{ for some integer } n > 0\} \gg_\epsilon M\delta > 0.$$

Hence, there are some integers $d \in (x^\alpha, 2x^\alpha]$ and e such that $e - 3\delta \le F(d) \le e - \delta$. Since $e \asymp F(d) \asymp x^{1/2-\alpha}$, we have

$$e^2 - \frac{x}{d(d+1)} \asymp_\epsilon x^{11/19 - 31\alpha/19 + \epsilon}.$$

Note that $11/19 - 31\alpha/19$ is positive when $\alpha < 8/45$. With $k = [\sqrt{e^2 - \frac{x}{d(d+1)}}]$, we have

$$\Big|e^2 - k^2 - \frac{x}{d(d+1)}\Big| \ll k \ll_\epsilon x^{11/38 - 31\alpha/38 + \epsilon/2}$$

or

$$|x - d(d+1)(e-k)(e+k)| \ll_\epsilon x^{11/38 + 45\alpha/38 + \epsilon/2}.$$

The difference between the largest and smallest factors of $n = [d(e+k)] \cdot [(d+1)(e-k)] = [d(e-k)] \cdot [(d+1)(e+k)]$ satisfies

$$(d+1)(e+k) - d(e-k) = e + 2dk + k \ll_\epsilon x^{1/2-\alpha} + x^{11/38 + 7\alpha/38 + \epsilon/2} \ll_\epsilon x^{1/2-\alpha}$$

when $1/8 < \alpha < 8/45 - \epsilon/2$. Thus, with

$$\frac{29}{90} + \frac{\epsilon}{2} < \theta = \frac{1}{2} - \alpha < \frac{3}{8} \quad \text{and} \quad \phi = \frac{11}{38} + \frac{45\alpha}{38} + \frac{\epsilon}{2} = \frac{67}{76} - \frac{45\theta}{38} + \frac{\epsilon}{2},$$

there is a (θ, C_ϵ)-almost square of type 2 in the interval $[x - D_\epsilon x^\phi, x + D_\epsilon x^\phi]$. This gives Theorem 2 as ϵ can be arbitrarily small. □

5. Proof of Theorem 3

Proof. First, we note that the first three inequalities of (4) basically follow from (2). The slightly smaller lower limit $15/19 = 0.3220...$ for the third inequality (instead of $743/2306 = 0.3222...$ in (2)) is due to the new exponent pair $(13/84+\epsilon, 55/84+\epsilon)$, obtained by Bourgain [1] using a decoupling technique. One can simply input this pair into the proof of Theorem 3(v) in [6] and get 15/19. The fourth inequality follows from Theorem 2.

To go beyond $\theta < 3/8$ or to improve upon $g(\theta)$, we apply Lemma 5. Recall that

$$F(d) = \sqrt{\frac{x}{d(d+1)}} = x^{1/2}\Big(\frac{1}{d} - \frac{1}{2d^2} + \frac{3}{8d^3} - \cdots\Big)$$

and $d \in (x^\alpha, 2x^\alpha]$ for some $0 < \alpha < 1/4$. Let

$$N = [x^\alpha], \quad \lambda_m = \frac{m!}{2^{m+1}} \cdot \frac{lx^{1/2}}{N^{(m+1)}}, \quad \text{and} \quad A = 2^{2m}.$$

Then the derivative condition in Lemma 5 is satisfied with $f(u) = lF(u+N)$ for x sufficiently large. Hence,

$$\begin{aligned}\sum_{l\le M}\Big|\sum_{d\in[x^\alpha,2x^\alpha]} e(lF(d))\Big| \\ \ll_{m,\epsilon}\Big(M^{1+\frac{1}{m(m-1)}}x^{\frac{1}{2m(m-1)}+(1-\frac{(m+1)}{m(m-1)})\alpha} + Mx^{(1-\frac{1}{m(m-1)})\alpha} \\ + M^{1-\frac{2}{m^2(m-1)}}x^{-\frac{1}{m^2(m-1)}+(1+\frac{2}{m^2(m-1)})\alpha}\Big)x^{\frac{\epsilon}{2}},\end{aligned}$$

which is less than $x^\alpha/10$ when

$$M \ll_{m,\epsilon} x^{\frac{(m+1)\alpha}{m^2-m+1}-\frac{1}{2(m^2-m+1)}-\epsilon}, M \ll_{m,\epsilon} x^{\frac{\alpha}{m(m-1)}-\epsilon}, \text{ and } M \ll_{m,\epsilon} x^{\frac{1-2\alpha}{m^3-m^2-2}-\epsilon}. \tag{8}$$

Notice that the first two bounds in (8) are the same when $\alpha = \frac{m^2-m+1}{2(m^3-m^2-1)}$. Thus, we make the following change of variable

$$\alpha := \frac{m^2-m}{2(m^3-m^2-1)} + \eta \quad \text{with} \quad -\frac{1}{2(m^2+m)} \le \eta \le \frac{1}{2(m^2-1)}. \tag{9}$$

For $\eta \ge 0$, the second upper bound in (8) is the smallest and we pick

$$M = \Big[c_\epsilon x^{\frac{1}{2(m^3-m^2-1)}+\frac{\eta}{m(m-1)}-\epsilon}\Big]$$

for some small enough constant $c_\epsilon > 0$. By Lemma 2,

$$e - \frac{5}{M} \le F(d) = \sqrt{\frac{x}{d(d+1)}} \le e - \frac{1}{M}$$

for some positive integers $d \in [x^\alpha, 2x^\alpha]$ and $e \asymp x^{1/2-\alpha}$. Thus,

$$e^2 - \frac{x}{d(d+1)} \asymp \frac{e}{M} \asymp_\epsilon x^{\frac{1}{2} - \frac{m^2-m+1}{2(m^3-m^2-1)} - \frac{m^2-m+1}{m(m-1)}\eta + \epsilon}.$$

Then, with $k = [\sqrt{e^2 - \frac{x}{d(d+1)}}]$, we have

$$\left| e^2 - k^2 - \frac{x}{d(d+1)} \right| \ll k \ll_\epsilon x^{\frac{1}{4} - \frac{m^2-m+1}{4(m^3-m^2-1)} - \frac{m^2-m+1}{2m(m-1)}\eta + \frac{\epsilon}{2}}$$

or

$$|x - d(d+1)(e-k)(e+k)| \ll_\epsilon x^{\frac{1}{4} + \frac{3m^2-3m-1}{4(m^3-m^2-1)} + \frac{3m^2-3m-1}{2m(m-1)}\eta + \frac{\epsilon}{2}}. \tag{10}$$

So, the difference between the largest and smallest factors of $n = [d(e+k)] \cdot [(d+1)(e-k)] = [d(e-k)] \cdot [(d+1)(e+k)]$ satisfies

$$\begin{aligned}(d+1)(e+k) - d(e-k) =& e + 2dk + k \\ &\ll_\epsilon x^{\frac{1}{2} - \frac{m^2-m}{2(m^3-m^2-1)} - \eta} + x^{\frac{1}{4} + \frac{m^2-m-1}{4(m^3-m^2-1)} + \frac{m^2-m-1}{2m(m-1)}\eta + \frac{\epsilon}{2}}. \end{aligned} \tag{11}$$

When $m \geq 4$, the first bound in (11) dominates by $\eta \leq \frac{1}{2(m^2-1)}$ in (9).

For $\eta \leq 0$, the first upper bound in (8) is the smallest and we pick

$$M = \left[c_\epsilon x^{\frac{1}{2(m^3-m^2-1)} + \frac{(m+1)\eta}{m^2-m+1} - \epsilon} \right]$$

for some small enough constant $c_\epsilon > 0$. Here, one can check that the exponent is positive by $-\frac{1}{2(m^2+m)} \leq \eta$ in (9). By Lemma 2,

$$e - \frac{5}{M} \leq F(d) = \sqrt{\frac{x}{d(d+1)}} \leq e - \frac{1}{M}$$

for some positive integers $d \in [x^\alpha, 2x^\alpha]$ and $e \asymp x^{1/2-\alpha}$. Thus,

$$e^2 - \frac{x}{d(d+1)} \asymp \frac{e}{M} \asymp_\epsilon x^{\frac{1}{2} - \frac{m^2-m+1}{2(m^3-m^2-1)} - \frac{m^2+2}{m^2-m+1}\eta + \epsilon}.$$

Then, with $k = [\sqrt{e^2 - \frac{x}{d(d+1)}}]$, we have

$$\left| e^2 - k^2 - \frac{x}{d(d+1)} \right| \ll k \ll_\epsilon x^{\frac{1}{4} - \frac{m^2-m+1}{4(m^3-m^2-1)} - \frac{m^2+2}{2(m^2-m+1)}\eta + \frac{\epsilon}{2}}$$

or

$$|x - d(d+1)(e-k)(e+k)| \ll_\epsilon x^{\frac{1}{4} + \frac{3m^2-3m-1}{4(m^3-m^2-1)} + \frac{3m^2-4m+2}{2(m^2-m+1)}\eta + \frac{\epsilon}{2}}. \tag{12}$$

So, the difference between the largest and smallest factors of $n = [d(e+k)] \cdot [(d+1)(e-k)] = [d(e-k)] \cdot [(d+1)(e+k)]$ satisfies

$$\begin{aligned}(d+1)(e+k) - d(e-k) &= e + 2dk + k \\ &\ll_\epsilon x^{\frac{1}{2} - \frac{m^2-m}{2(m^3-m^2-1)} - \eta} + x^{\frac{1}{4} + \frac{m^2-m-1}{4(m^3-m^2-1)} + \frac{m^2-2m}{2(m^2-m+1)}\eta + \frac{\epsilon}{2}}. \end{aligned} \tag{13}$$

When $\eta \le 0$, one can check that the first upper bound in (13) dominates. Summarizing from (9), (10) and (12), we have the admissible exponents

$$\phi = \begin{cases} \frac{1}{4} + \frac{3m^2-3m-1}{4(m^3-m^2-1)} + \frac{3m^2-3m-1}{2m(m-1)}\eta + \frac{\epsilon}{2}, & \text{if } 0 \le \eta \le \frac{1}{2(m^2-1)}, \\ \frac{1}{4} + \frac{3m^2-3m-1}{4(m^3-m^2-1)} + \frac{3m^2-4m+2}{2(m^2-m+1)}\eta + \frac{\epsilon}{2}, & \text{if } -\frac{1}{2(m^2+m)} \le \eta \le 0. \end{cases} \tag{14}$$

Based on (11) and (13), we set

$$\theta = \frac{1}{2} - \frac{m^2-m}{2(m^3-m^2-1)} - \eta. \tag{15}$$

Substituting (15) into (14) (with the help of Mathematica), we obtain

$$\phi = \Big(1 - \frac{1}{4m(m-1)}\Big) - \Big(\frac{3}{2} - \frac{1}{2m(m-1)}\Big)\theta + \frac{\epsilon}{2} \tag{16}$$

when $\frac{1}{2} - \frac{m^2-m}{2(m^3-m^2-1)} - \frac{1}{2(m^2-1)} \le \theta \le \frac{1}{2} - \frac{m^2-m}{2(m^3-m^2-1)}$, and

$$\phi = \Big(1 - \frac{m}{4(m^2-m+1)}\Big) - \Big(\frac{3}{2} - \frac{m+1}{2(m^2-m+1)}\Big)\theta + \frac{\epsilon}{2} \tag{17}$$

when $\frac{1}{2} - \frac{m^2-m}{2(m^3-m^2-1)} \le \theta \le \frac{1}{2} - \frac{m^2-m}{2(m^3-m^2-1)} + \frac{1}{2(m^2+m)}$. One can check that

$$\frac{1}{2} - \frac{m^2-m}{2(m^3-m^2-1)} - \frac{1}{2(m^2-1)} < \frac{1}{2} - \frac{(m-1)^2-(m-1)}{2((m-1)^3-(m-1)^2-1)}$$

and

$$\frac{1}{2} - \frac{(m+1)^2-(m+1)}{2((m+1)^3-(m+1)^2-1)} \le \frac{1}{2} - \frac{m^2-m}{2(m^3-m^2-1)} + \frac{1}{2(m^2+m)}$$

for $m \ge 4$. Consequently, with

$$r_x := \frac{x^2-x}{2(x^3-x^2-1)},$$

we have

$$g(\theta) \le \begin{cases} \big(1 - \frac{1}{4m(m-1)}\big) - \big(\frac{3}{2} - \frac{1}{2m(m-1)}\big)\theta, & \text{if } \frac{1}{2} - r_{m-1} \le \theta \le \frac{1}{2} - r_m, \\ \big(1 - \frac{m}{4(m^2-m+1)}\big) - \big(\frac{3}{2} - \frac{m+1}{2(m^2-m+1)}\big)\theta, & \text{if } \frac{1}{2} - r_m \le \theta \le \frac{1}{2} - r_{m+1} \end{cases} \tag{18}$$

by (16) and (17) as ϵ can be arbitrarily small. Replacing m by $m+1$,

$$g(\theta) \le \begin{cases} \big(1 - \frac{1}{4(m^2+m)}\big) - \big(\frac{3}{2} - \frac{1}{2(m^2+m)}\big)\theta, & \text{if } \frac{1}{2} - r_m \le \theta \le \frac{1}{2} - r_{m+1}, \\ \big(1 - \frac{m+1}{4(m^2+m+1)}\big) - \big(\frac{3}{2} - \frac{m+2}{2(m^2+m+1)}\big)\theta, & \text{if } \frac{1}{2} - r_{m+1} \le \theta \le \frac{1}{2} - r_{m+2}. \end{cases} \tag{19}$$

Notice that the second bound in (18) and the first bound in (19) share the same range of θ. So, we decide which bound is smaller and use it accordingly. One can show that

$$\Big(1-\frac{m}{4(m^2-m+1)}\Big)-\Big(\frac{3}{2}-\frac{m+1}{2(m^2-m+1)}\Big)\theta \leq \Big(1-\frac{1}{4(m^2+m)}\Big)-\Big(\frac{3}{2}-\frac{1}{2(m^2+m)}\Big)\theta$$

when

$$\theta \leq \frac{1}{2}-\frac{m^2+m}{2(m^3+m^2+2m-1)} \leq \frac{1}{2}-r_{m+1}.$$

Therefore, for $m \geq 4$,

$$g(\theta) \leq \begin{cases} \big(1-\frac{m}{4(m^2-m+1)}\big)-\big(\frac{3}{2}-\frac{m+1}{2(m^2-m+1)}\big)\theta, & \text{if } \frac{1}{2}-r_m \leq \theta \leq \frac{1}{2}-s_m, \\ \big(1-\frac{1}{4(m^2+m)}\big)-\big(\frac{3}{2}-\frac{1}{2(m^2+m)}\big)\theta, & \text{if } \frac{1}{2}-s_m \leq \theta \leq \frac{1}{2}-r_{m+1}, \end{cases} \tag{20}$$

where

$$s_x := \frac{x^2+x}{2(x^3+x^2+2x-1)}.$$

Combining (18), (19) and (20), we have

$$g(\theta) \leq \begin{cases} 47/48-35\theta/24 & \text{if } 11/34 \leq \theta \leq 35/94, \\ 79/80-59\theta/40 & \text{if } 35/94 \leq \theta \leq 67/174, \end{cases}$$

and

$$g(\theta) \leq \begin{cases} \big(1-\frac{1}{4m(m-1)}\big)-\big(\frac{3}{2}-\frac{1}{2m(m-1)}\big)\theta, & \text{if } \frac{1}{2}-s_{m-1} \leq \theta \leq \frac{1}{2}-r_m, \\ \big(1-\frac{m}{4(m^2-m+1)}\big)-\big(\frac{3}{2}-\frac{m+1}{2(m^2-m+1)}\big)\theta, & \text{if } \frac{1}{2}-r_m \leq \theta \leq \frac{1}{2}-s_m \end{cases}$$

for $m \geq 5$. This gives the rest of Theorem 3 except for the last part. From (18), we have

$$g(\theta) \leq \Big(1-\frac{3}{2}\theta\Big)-\frac{1}{2m(m-1)}\Big(\frac{1}{2}-\theta\Big) \leq \Big(1-\frac{3}{2}\theta\Big)-\Big(\frac{1}{2}-\theta\Big)^3$$

as

$$\frac{1}{2m(m-1)} \geq \Big(\frac{(m-1)^2-(m-1)}{2((m-1)^3-(m-1)^2-1)}\Big)^2 \geq \Big(\frac{1}{2}-\theta\Big)^2$$

for $1/2-r_{m-1} \leq \theta \leq 1/2-r_m$ and $m \geq 4$. The last statement of Theorem 3 follows as the union of the intervals $\cup_{m\geq 4}[1/2-r_{m-1}, 1/2-r_m]$ equals $[11/34, 1/2)$. □

Acknowledgement. The author would like to thank the anonymous referee and the managing editor for some helpful corrections.

References

[1] J. Bourgain, Decoupling, exponential sums and the Riemann zeta function, *J. Amer. Math. Soc.* **30** (2017), 205–224.

[2] J. Bourgain, C. Demeter and L. Guth, Proof of the main conjecture in Vinogradov's mean value theorem for degrees higher than three, *Ann. of Math.* (2) **184** (2016), no. 2, 633–682.

[3] T.H. Chan, Finding almost squares II, *Integers* **5** (1) (2005), A23, 11pp.

[4] T.H. Chan, Finding almost squares, *Acta Arith.* **121** (2006), no. 3, 221–232.

[5] T.H. Chan, Finding almost squares IV, *Arch. Math. (Basel)* **92** (2009), no. 4, 303–313.

[6] T.H. Chan, Finding almost squares III, *Integers* **9** (Supplement) (2009), A2, 10pp.

[7] T.H. Chan, Finding almost squares V, *Integers* **10** (1) (2010), A46, 559–563.

[8] T.H. Chan, Finding almost squares VI, *Int. J. Number Theory* **14** (2018), no. 5, 1347–1356.

[9] S.W. Graham and G. Kolesnik, *Van der Corput's Method of Exponential Sums*, Cambridge University Press, 1991.

[10] D.R. Heath-Brown, A new kth derivative estimate for a trigonometric sum via Vinogradov's integral, (Russian) English version published in *Proc. Steklov Inst. Math.* **296** (2017), no. 1, 88–103. *Tr. Mat. Inst. Steklova* **296** (2017), Analiticheskaya i Kombinatornaya Teoriya Chisel, 95–110.

[11] M.N. Huxley, *Area, Lattice Points, and Exponential Sums*, London Mathematical Society Monographs, New Series, 13. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1996.

[12] H.L. Montgomery, *Ten Lectures on the Interface Between Analytic Number Theory and Harmonic Analysis*, CBMS Regional Conference Series in Mathematics 84, published for the Conference Board of the Mathematical Sciences, Washington, DC, American Mathematical Society, Providence, RI, 1994.

[13] L.B. Pierce, The Vinogradov mean value theorem [after Wooley, and Bourgain, Demeter and Guth], *Séminaire Bourbaki.* Vol. 2016/2017. Exposés 1120–1135. *Astérisque* No. 407 (2019), Exp. No. 1134, 479–564.

[14] O. Robert, Sommes d'exponentielles avec dérivée troisième monotone (French) [Exponential sums with monotonic third derivative], *Acta Arith.* **119** (2005), no. 1, 1–11.

[15] O. Robert, On van der Corput's k-th derivative test for exponential sums, *Indag. Math. (N.S.)* **27** (2016), no. 2, 559–589.

[16] O. Robert and P. Sargos, A third derivative test for mean values of exponential sums with application to lattice point problems, *Acta Arith.* **106** (2003), no. 1, 27–39.

[17] T.D. Wooley, The cubic case of the main conjecture in Vinogradov's mean value theorem, *Adv. Math.* **294** (2016), 532–561.