



**A NOTE ON A GENERALIZATION OF TWO WELL-KNOWN  
COMBINATORIAL IDENTITIES VIA A HYPERGEOMETRIC  
SERIES APPROACH**

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**Abstract**

In this note, we aim to provide generalizations of (i) Knuth's old sums (or Reed Dawson's identities) and (ii) Riordan's identities, using a hypergeometric series approach.

**1. Introduction and Results Required**

We start with the following well-known combinatorial sums known as Knuth's old sums [3], or alternatively as Reed Dawson's identities, viz:

$$\sum_{k=0}^{2\nu} (-1)^k \binom{2\nu}{k} 2^{-k} \binom{2k}{k} = 2^{-2\nu} \binom{2\nu}{\nu} \quad (1)$$

and

$$\sum_{k=0}^{2\nu+1} (-1)^k \binom{2\nu+1}{k} 2^{-k} \binom{2k}{k} = 0. \quad (2)$$

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It is of interest to mention that Reed Dawson presented the above identities in a private communication to Riordan who recorded them in his well-known book [9, page 71].

Several different proofs of the above sums have been given in the literature; see the survey paper by Prodinger [6]. Jonassen and Knuth [4] gave an elementary demonstration using a recursion for the binomial coefficients. Gessel [3] expressed the binomial coefficients as coefficients in appropriate generating functions. Rousseau [4] showed that the sums could be expressed in terms of the constant coefficient in the expansion of  $(x^2 + x^{-2})^n$  and Prodinger [7] employed the Euler transformation and also the Zeilberger algorithm. This latter approach seeks to find a representation for the summand in  $\sum_k F(n, k)$  in the form  $F(n, k) = G(n, k + 1) - G(n, k)$ , so that the resulting sum “telescopes” and can be evaluated.

In 1974, Andrews [1, page 478] established the above sums by employing the Gauss second summation theorem [8] given by

$${}_2F_1 \left[ \begin{matrix} a, & b \\ \frac{1}{2}(a + b + 1) \end{matrix}; \frac{1}{2} \right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2})}{\Gamma(\frac{1}{2}a + \frac{1}{2})\Gamma(\frac{1}{2}b + \frac{1}{2})}. \tag{3}$$

In 2004, Choi et al. [2] utilized the following terminating hypergeometric identities recorded, for example, in [8, page 126-127]:

$${}_2F_1 \left[ \begin{matrix} -2n, & \alpha \\ 2\alpha \end{matrix}; 2 \right] = \frac{(\frac{1}{2})_n}{(\alpha + \frac{1}{2})_n} \tag{4}$$

and

$${}_2F_1 \left[ \begin{matrix} -2n - 1, & \alpha \\ 2\alpha \end{matrix}; 2 \right] = 0, \tag{5}$$

where  $(\alpha)_n$  denotes the Pochhammer symbol (or the rising factorial) for any complex number  $\alpha(\neq 0)$  defined by

$$(\alpha)_n = \begin{cases} \alpha(\alpha + 1) \dots (\alpha + n - 1), & (n \in \mathbb{N}) \\ 1, & (n = 0) \end{cases}.$$

Also, the following well-known combinatorial identities established by Riordan [9] are seen to be closely related to Equations (1) and (2):

$$\sum_{k=0}^{2\nu} (-1)^k \binom{2\nu + 1}{k + 1} 2^{-k} \binom{2k}{k} = 2^{-2\nu} (2\nu + 1) \binom{2\nu}{\nu} \tag{6}$$

and

$$\sum_{k=0}^{2\nu+1} (-1)^k \binom{2\nu + 2}{k + 1} 2^{-k} \binom{2k}{k} = 2^{-2\nu-1} (\nu + 1) \binom{2\nu}{\nu}. \tag{7}$$

Riordan [9] established Equations (6) and (7) by the method of inverse relations.

Very recently, generalizations of Equations (4) and (5) were given by Kim et al. [5], written in the following form:

$$\begin{aligned}
 {}_2F_1 \left[ \begin{matrix} -2n, & \alpha \\ 2\alpha + i \end{matrix}; 2 \right] &= \frac{2^{-2\alpha-i} \Gamma(\alpha) \Gamma(1-\alpha)}{\Gamma(\alpha+i) \Gamma(1-2\alpha-i)} \\
 &\times \sum_{r=0}^i (-1)^r \binom{i}{r} \frac{\Gamma(\frac{1}{2}-\alpha-\frac{1}{2}i+\frac{1}{2}r) (\frac{1}{2}+\frac{1}{2}i-\frac{1}{2}r)_n}{\Gamma(\frac{1}{2}-\frac{1}{2}i+\frac{1}{2}r) (\alpha+\frac{1}{2}+\frac{1}{2}i-\frac{1}{2}r)_n}
 \end{aligned} \tag{8}$$

and

$$\begin{aligned}
 {}_2F_1 \left[ \begin{matrix} -2n-1, & \alpha \\ 2\alpha + i \end{matrix}; 2 \right] &= -\frac{2^{-2\alpha-i} \Gamma(\alpha) \Gamma(1-\alpha)}{\Gamma(\alpha+i) \Gamma(1-2\alpha-i)} \\
 &\times \sum_{r=0}^i (-1)^r \binom{i}{r} \frac{\Gamma(-\alpha-\frac{1}{2}i+\frac{1}{2}r) (1+\frac{1}{2}i-\frac{1}{2}r)_n}{\Gamma(-\frac{1}{2}i+\frac{1}{2}r) (\alpha+1+\frac{1}{2}i-\frac{1}{2}r)_n},
 \end{aligned} \tag{9}$$

which are valid for  $i \in \mathbb{N}_0$ .

In this note, we aim to provide generalizations of Knuth’s old sums (or Reed Dawson’s identities), Equations (1) and (2) and Riordan’s identities, Equations (6) and (7) in the most general form for any  $i \in \mathbb{N}_0$ . In order to obtain the results in the most general form for any  $i \in \mathbb{N}_0$ , we have to construct two master formulas. The results are established with the help of Equations (8) and (9). In Section 3 we present cases of our general result that correspond to Knuth’s old sums and Riordan’s identities, together with some interesting new results.

## 2. Generalizations

The generalizations of Knuth’s old sums are given in the following theorem.

**Theorem 1.** *For  $i \in \mathbb{N}_0$ , the following results hold true:*

$$\begin{aligned}
 &\sum_{k=0}^{2\nu} (-1)^k \binom{2\nu+i}{k+i} 2^{-k} \binom{2k}{k} \\
 &= \pi(2\nu+1)_i \frac{2^{2i} i!}{(2i)!} \sum_{r=0}^i \frac{2^{-r} \binom{i}{r} (\frac{1}{2}+\frac{1}{2}(i-r))_\nu}{(i-r)! \Gamma^2(\frac{1}{2}+\frac{1}{2}(r-i)) (1+\frac{1}{2}(i-r))_\nu}
 \end{aligned} \tag{10}$$

and

$$\begin{aligned}
 &\sum_{k=0}^{2\nu+1} (-1)^k \binom{2\nu+1+i}{k+i} 2^{-k} \binom{2k}{k} \\
 &= 2\pi(2\nu+2)_i \frac{2^{2i} i!}{(2i)!} \sum_{r=0}^i \frac{2^{-r} \binom{i}{r} (1+\frac{1}{2}(i-r))_\nu}{(i-r+1)! \Gamma^2(\frac{1}{2}(r-i)) (\frac{3}{2}+\frac{1}{2}(i-r))_\nu}.
 \end{aligned} \tag{11}$$

*Proof.* The proof of Equations (10) and (11) is straightforward. For this, let us consider the sum for  $i \in \mathbb{N}_0$ :

$$S = \sum_{k=0}^n (-1)^k \binom{n+i}{k+i} 2^{-k} \binom{2k}{k}.$$

Making use of the identities

$$(n-k)! = \frac{(-1)^k n!}{(-n)_k} \quad \text{and} \quad \Gamma(2z) = \frac{2^{2z-1} \Gamma(z) \Gamma(z + \frac{1}{2})}{\sqrt{\pi}},$$

we have after some simplification,

$$\begin{aligned} S &= \frac{\Gamma(n+1+i)}{\Gamma(1+i)\Gamma(n+1)} \sum_{k=0}^n \frac{(-n)_k (\frac{1}{2})_k 2^k}{(1+i)_k k!} \\ &= \frac{\Gamma(n+1+i)}{\Gamma(1+i)\Gamma(n+1)} {}_2F_1 \left[ \begin{matrix} -n, \frac{1}{2} \\ 1+i \end{matrix}; 2 \right]. \end{aligned} \tag{12}$$

Now for  $n = 2\nu$  (even) and  $n = 2\nu + 1$  (odd), the  ${}_2F_1$  appearing in Equation (12) can be evaluated with the help of the known results Equations (8) and (9) and after some simplification, we easily arrive at Equations (10) and (11) asserted by the theorem. This completes the proof of Equations (10) and (11).  $\square$

### 3. Corollary

In this section, we shall mention the known summations presented in Section 1 as well as two new cases of our main findings.

1. If we take  $i = 0$  in Theorem 1, we obtain

$$\sum_{k=0}^{2\nu} (-1)^k \binom{2\nu}{k} 2^{-k} \binom{2k}{k} = \frac{(\frac{1}{2})_\nu}{(1)_\nu} \tag{13}$$

and

$$\sum_{k=0}^{2\nu+1} (-1)^k \binom{2\nu+1}{k} 2^{-k} \binom{2k}{k} = 0, \tag{14}$$

which are equivalent to Knuth's old sums (or Reed Dawson's identities), Equations (1) and (2).

2. If we take  $i = 1$  in Theorem 1, we obtain

$$\sum_{k=0}^{2\nu} (-1)^k \binom{2\nu+1}{k+1} 2^{-k} \binom{2k}{k} = (2\nu+1) \frac{(\frac{1}{2})_\nu}{(1)_\nu} \tag{15}$$

and

$$\sum_{k=0}^{2\nu+1} (-1)^k \binom{2\nu+2}{k+1} 2^{-k} \binom{2k}{k} = (\nu+1) \frac{(\frac{3}{2})_\nu}{(2)_\nu}, \tag{16}$$

which are equivalent to Riordan’s identities, Equations (6) and (7).

3. Finally, if we take  $i = 2$  and  $i = 3$  in Theorem 1, we find the following new results:

$$\sum_{k=0}^{2\nu} (-1)^k \binom{2\nu+2}{k+2} 2^{-k} \binom{2k}{k} = \frac{1}{3} (4\nu+3) \frac{(\frac{3}{2})_\nu}{(1)_\nu} \tag{17}$$

$$\sum_{k=0}^{2\nu+1} (-1)^k \binom{2\nu+3}{k+2} 2^{-k} \binom{2k}{k} = 2 \frac{(\frac{5}{2})_\nu}{(1)_\nu} \tag{18}$$

and

$$\sum_{k=0}^{2\nu} (-1)^k \binom{2\nu+3}{k+3} 2^{-k} \binom{2k}{k} = \frac{1}{5} (8\nu+5) \frac{(\frac{5}{2})_\nu}{(1)_\nu} \tag{19}$$

$$\sum_{k=0}^{2\nu+1} (-1)^k \binom{2\nu+4}{k+3} 2^{-k} \binom{2k}{k} = \frac{1}{5} (8\nu+15) \frac{(\frac{5}{2})_\nu}{(1)_\nu}. \tag{20}$$

Similarly, other results can be obtained.

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