ON THE DIOPHANTINE EQUATION $\mathbf{7^x + 32^y = z^2}$ AND ITS GENERALIZATION

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Abstract

In this paper, we show that $(x, y, z) = (2, 1, 9)$ is the unique solution of the exponential Diophantine equation $7^x + 32^y = z^2$ in non-negative integers. We also study a generalized version of the above equation, viz. $2^x + 7^y = z^2$, and characterize its non-negative integral solutions for $x \neq 1$. Finally, we leave an open problem to explore.

1. Introduction

An exponential Diophantine equation is one in which unknowns appear as exponents. Diophantine problems have fewer equations than unknowns and involve finding integers that simultaneously solve all the equations. Such equations have drawn researchers for a long time. Diophantine equations can also be viewed over the ring of integers $\mathbb{Z}$, the field of rational numbers $\mathbb{Q}$, the finite field F_q, etc. Some Diophantine equations have infinitely many integer solutions; others have only a finite number of integer solutions. Many researchers have studied Diophantine equations that do not have integer solutions [5, 9].

In 1844, Catalan formulated a conjecture that the Diophantine equation $a^x - b^y = 1$, where $a, b, x, y \in \mathbb{Z}$ with $min\{a, b, x, y\} > 1$, has a unique solution $(a, b, x, y) = (3, 2, 2, 3)$[6]. Since then many mathematicians have tried to settle it with partial success. The conjecture was eventually proved by Mihăilescu in 2002 [11]. The proof makes extensive use of the theory of cyclotomic fields and Galois modules. In 2005, Mihăilescu published a simplified proof [12].

Between the years 1995 and 2001, Luca [8], Cao [13], and Beukers [7] did considerable work on various aspects of the Diophantine equations $x^2 + 3^m = y^n, a^x + b^y =$

$c^z, Ax^p + By^q = Cz^r$ and others. In 2007, Acu proved that the Diophantine equation $2^x + 5^y = z^2$ has only two non-negative integral solutions, viz. $(3, 0, 3)$ and $(2, 1, 3)$ [4]. In the period 2010-2016, Suvarnamani [1], Sroysang [2], and others did extensive work on different types of Diophantine equations, such as $4^x + 7^y = z^2, 4^x + 11^y = z^2, 4^x + 13^y = z^2, 4^x + 17^y = z^2, A^x + B^y = C^z, 3^x + 5^y = z^2, 8^x + 19^y = z^2, 31^x + 32^y = z^2$, and $7^x + 8^y = z^2$. Recently, Burshtein [10] did related work.

2. Preliminaries

Conjecture 1 ([11]). (Catalan's Conjecture) The unique solution for the Diophantine equation $a^x - b^y = 1$ where $a, b, x, y \in \mathbb{Z}$ with $min\{a, b, x, y\} > 1$ is $(3, 2, 2, 3)$.

Lemma 1. *The equation $7^x + 1 = z^2$ has no solution in non-negative integers.*

Proof. This is a direct consequences of Chao Ko's result [3]. □

Lemma 2. *The equation $1 + 32^y = z^2$ has no solution in non-negative integers.*

Proof. This too is a direct consequences of Chao Ko's result [3]. □

In this paper, we will use Catalan's conjecture and the above two lemmas for solving the exponential Diophantine equations $7^x + 32^y = z^2$, and a generalized version of it.

3. Main Results

Theorem 1. *The exponential Diophantine equation $7^x + 32^y = z^2$ has the unique solution $(2, 1, 9)$ in non-negative integers (x, y, z).*

Proof. From Lemma 1 and Lemma 2, we have $x \geq 1$ and $y \geq 1$. Now, the given equation is

$$7^x + 32^y = z^2. \tag{1}$$

Equation (1) implies that z is odd, which implies $z^2 = 4l + 1$ for some $l \in \mathbb{N}$. Also, Equation (1) implies $7^x \equiv z^2 \equiv 1 (\text{mod } 4)$, implying that x is even. Let $x = 2k, k \in \mathbb{N}$. Therefore, Equation (1) implies $2^{5y} = z^2 - (7^k)^2 = (z - 7^k)(z + 7^k)$, which in turn implies $z - 7^k = 2^u, z + 7^k = 2^{5y-u}$, where $u \in \mathbb{N}, 5y - u > u$, i.e., $5y > 2u$. Now, $(z + 7^k) + (z - 7^k) = 2^{5y-u} + 2^u$. This means, $2z = 2^u + 2^{5y-u}$, which implies $2z = 2^u(1 + 2^{5y-2u})$. Thus,

$$z = 2^{u-1}(1 + 2^{5y-2u}), \tag{2}$$

Since z is odd, by Equation (1), therefore Equation (2) implies $u - 1 = 0$ implying $u = 1$. Therefore,

$$z = 1 + 2^{5y-2}, \tag{3}$$

Also, $(z + 7^k) - (z - 7^k) = 2^{5y-u} - 2^u$ implies $2.7^k = 2^u(2^{5y-2u} - 1)$. Since $u = 1$, so $7^k = 2^{5y-2} - 1$, implying $2^{5y-2} - 7^k = 1$, which in turn implies $k = 0$ or 1 by Catalan's Conjecture. This means that $2^{5y-2} = 2$ or 2^3, implying $5y - 2 = 1$ or 3. From this we get that $y = 1$ is the only admissible integral solution. Also, $k = 1$ implies $x = 2k = 2$, and Equation (3) implies $z = 1 + 2^{5.1-2} = 9$. Therefore $(x, y, z) = (2, 1, 9)$ is the only non-negative integral solution of the exponential Diophantine equation $7^x + 32^y = z^2$. □

Theorem 2. *The only solutions of the exponential Diophantine equation* $2^x + 7^y = z^2$ *in non-negative integers are* $(3, 0, 3)$ *and* $(5, 2, 9)$ *for* $x \neq 1$.

Proof. We will apply a similar method of proof to that used in Theorem 1, by splitting the exponent of 2 into two terms, according to the given factorization of $z^2 - 7^{2k}$.

`Case 1:` $x = 0$. In this case, $2^x + 7^y = z^2$ becomes $1 + 7^y = z^2$. This implies $z^2 - 7^y = 1$, and the Catalan conjecture gives $y = 0$ or $y = 1$. If $y = 0$, then $z^2 = 2$ implies that no integer solution exists. If $y = 1$, then $z^2 = 8$ also implies that no integer solution exists.

`Case 2:` $x > 1$. In this case, the given equation implies that z is odd, which in turn implies $z^2 \equiv 1 (\mod 4)$. Also, $x \geq 2$, which implies that $4 \mid 2^x$. Therefore, taking Equation (4) modulo 4 we get that $0 + (-1)^y \equiv 1$ implies $y = 2k$, with k being a non-negative integer. We note that $k = 0$ corresponds to $y = 0$, and this case is a consequence of Chao Ko's result [3], but still we show it explicitly here. From Equation (4), $2^x + 7^{2k} = z^2$. This implies $2^x = z^2 - (7^k)^2 = (z - 7^k)(z + 7^k)$, which in turn implies $2^u = z - 7^k, 2^{x-u} = z + 7^k$, where $x - u > u$, i.e., $x > 2u$. Then, $2^{x-u} - 2^u = (z + 7^k) - (z - 7^k)$, implying $2^u(2^{x-2u} - 1) = 2^1.7^k$, which implies $u = 1, 2^{x-2.1} - 1 = 7^k$. This gives $2^{x-2} - 7^k = 1$, and Catalan's conjecture implies $k = 0, 1$ or $x - 2 = 0, 1$. Now, $k = 0$ implies $2^{x-2} = 2^1$, rendering $x = 3$. Also, $y = 2k = 0$ and $z^2 = 2^3 + 7^0 = 3^2$, which implies $z = 3$. Therefore $(x, y, z) = (3, 0, 3)$ is a solution of Equation (4). Again, $k = 1$ implies $2^{x-2} = 1 + 7 = 8 = 2^3$, rendering $x = 2 + 3 = 5$. Also, $y = 2k = 2$ and $z^2 = 2^5 + 7^2 = 9^2$, which implies $z = 9$. Therefore, $(5, 2, 9)$ is another solution of Equation (4). Also, $x = 2$ gives $z^2 = 2^0 + 7^{2k}$, which implies $z^2 - 7^{2k} = 1$, and Catalan's conjecture implies $k = 0$, implying $z^2 = 2$, which is impossible. Finally, for $x = 3$,

$$z^2 = 2^3 + 7^{2k}. \tag{4}$$

From this equation we get $2^3 = z^2 - (7^k)^2$, which implies $2^1.2^2 = (z - 7^k)(z + 7^k)$. Since z is odd, we have $z - 7^k = 2^1, z + 7^k = 2^2$, giving $2z = 2 + 2^2 = 6$, implying

$z = 3$. Then, Equation (5) gives $3^2 = 2^3 + 7^{2k}$, which implies $1 = 7^y$ (since $y = 2k$), i.e., $y = 0$. Thus, we again get $(x, y, z) = (3, 0, 3)$ as a solution. Therefore, there are exactly two solutions of Equation (4) for $x \neq 1$; these are $(3, 0, 3)$ and $(5, 2, 9)$. □

We observe that the solution $(5, 2, 9)$ precisely corresponds to the solution $(2, 1, 9)$ of Theorem 1.

Remark 1. The exponential Diophantine equation $2 + 7^y = z^2$ has $(y, z) = (1, 3)$ as one of the solutions. This equation is a special case of the Pilai conjecture, which says that $a^x - b^y = 2$ has only the solution $(a, b; x, y) = (3, 5; 3, 2)$ in positive integers with $x, y > 1$. If the Pilai conjecture is true, $(y, z) = (1, 3)$ is the unique solution of $2 + 7^y = z^2$. It is still an open problem to determine if it has any other solution.

4. Conclusion

The Diophantine equation has been of interest to mathematicians since antiquity. Many Diophantine equations can be solved with a finite or an infinite number of variables. In this article, we have shown that the exponential Diophantine equation $7^x + 32^y = z^2$ has a unique solution, viz. $(x, y, z) = (2, 1, 9)$. We have further shown that for $x \neq 1$, the generalized exponential Diophantine equation $2^x + 7^y = z^2$ has exactly two solutions: $(3, 0, 3)$ and $(5, 2, 9)$. It remains open to investigation if it has a solution other than (1, 1, 3) for $x = 1$.

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