#A3 INTEGERS 22 (2022)

PATROLLING THE BORDER OF A STRIKING CONJECTURE

Aviezri S. Fraenkel
Department of Computer Science and Applied Mathematics, Weizmann Institute of Science, Rehovot, Israel
fraenkel@wisdom.weizmann.ac.il

Received: 11/11/21, Accepted: 12/7/21, Published: 1/7/22

Abstract

A striking conjecture asserts that every *exact covering family* (ECF) of $\mathbb{Z}_{>0}$ into $m \geq 3$ sets with α_k and β_k real, $\alpha_k > 1$ and α_k's *distinct* for $k = 0, \ldots, m-1$ satisfies $\alpha_0, ..., \alpha_{m-1} = (2^m - 1)/2^k$, $k = 0, ..., m-1$. We prove the conjecture for the case where the α_k's are of the form T/a^k, T a common numerator, $a \geq 2$ a fixed integer, $k = 0, ..., m-1$.

– Another paper commemorating the unforgettable Ron Graham, who combined mathematical talent and brilliance with outright kindness and tolerance to everybody. He also knew how to fuse together outstanding mathematical research with mathematical administration, having been the president of both the AMS and MAA. In addition he was a champion in various physical activities, such as juggling and gymnastic trampolining. His early passing keeps me yearning and sad.

1. Introduction

This paper is about covering the positive integers disjointly. We give a proof to an important special case of the following.

Conjecture A: Every decomposition of $\mathbb{Z}_{>0}$ into $m \geq 3$ sets $\lfloor \alpha_i + \beta_i \rfloor$, with $\alpha_i > 1$ and β_i real and α_i's *distinct* for $i = 0, ..., m-1$, satisfies

$$\alpha_0, ..., \alpha_{m-1} = \left\{ \frac{2^m - 1}{2^i} \right\}, 0 \leq i \leq m-1. \tag{1}$$

See Tijdeman [25] for the history of partial results about this conjecture. Later developments include Barát and Varjú [2], Simpson [21], Schnabel and Simpson [19]. The conjecture is quoted in several papers; for this and related material see the bibliography below. It is interesting that in [14], where exercises are classified into

Warmups, Basics, Homework exercises, Exam problems, Bonus problems, Research problems, Conjecture A appears in Research problems with the extra tag 'expect this to be hard'.

2. Background

A Beatty sequence is a sequence $S(\alpha, \beta) = \lfloor n\alpha + \beta \rfloor$, $n = 1, 2, ...$ where $\alpha > 0$ and β are real numbers and $\lfloor x \rfloor$ is the integer part of x. A system of Beatty sequences $\{S(\alpha_i + \beta_i\}_{i=1}^{k}$ such that every integer belongs to exactly one Beatty sequence is an ECF (*Exactly Covering Family*), also called a *complementary* family. By density arguments, for such a system to be complementary, it is necessary that $\sum_{i=1}^{k} 1/\alpha_i = 1$. Without loss of generality, we may assume that $\alpha_1 \leq \alpha_2 \leq ... \leq \alpha_k$.

A disjoint covering system of Beatty sequences admits *multiplicity* if there exist $1 \leq i < j \leq k$ such that $\alpha_i = \alpha_j$, $i \neq j$. For the case that all α's are integers, Mirsky, Donald Newman, Davenport and Rado proved, independently, that $\alpha_{k-1} = \alpha_k$. For example, $2n, 2n-1; 2n, 4n-3, 4n-1$. So for integers, it holds even for $k = 2$. Their slick proof, quoted by Erdős [6], uses a generating function and roots of unity. A non-analytic proof was given later by Berger et al. [5]. Graham [13] showed that if one modulus is irrational, than all are. He also concluded in the final corollary of [12] that in an irrational ECF with $k \geq 3$ moduli, $\alpha_i = \alpha_j$ for some $i \neq j$ – the strongest known evidence supporting conjecture A. In fact, the case of (1) is the only *known* case where this does not hold. The conjecture is just that this is the *only* such case. Thus only the rationals stand out, stubbornly, according to Conjecture A.

Usually in math, the rationals team up, if at all, with the integers; yet here the irrationals join in with the integers. Somewhere we saw the following sentence: "Number Theory concerns the study of properties of the integers, rational numbers, and other structures that share similar features." In the present study, rational numbers do not share similar features with integers and irrationals.

We prove Conjecture A for sequences of the form $S(a^i) = T/a^i$, that is, we show that these sequences form an ECF if and only if $a = 2$. This gives a boost for proving Conjecture A, but is of interest on its own, since it is closest to the formula:

$$S(a^i) = \{\lfloor Tn/a^i + \beta_i \rfloor : n = 1, 2, \ldots\}$$

and

$$S(a^j) = \lfloor Tn/a^j + \beta_j \rfloor : n = 1, 2, \ldots\}, \quad 0 \leq i < j \leq m-1,$$

where m is the number of sequences for any integer $a \geq 2$, with the numerator $T = (a^m - 1)/(a - 1)$.

This formula, which appears again right after Formula (2) below, gave rise to Conjecture A in the first place.

3. Introducing the Theorem

We chose a unique value for the common numerator T, namely, the integer $T = (a^m - 1)/(a - 1) = 1 + a + a^2 + ... + a^{m-1}$. Why this choice of T? We want to cut the cake into slices of size T/a^i. Each slice occupies a^i/T of the cake, so in order to cut the entire cake we need $\sum a^i = T$ which holds precisely for the T we chose. These are certainly necessary requirements.

Are they also sufficient? No! We need also that the cake is cut into *distinct* portions. Our feeling is that this is achieved when all the slices T/a^i are in lowest terms. In fact, T/a^i is in lowest terms if and only if $a = 2$. However, this feeling must be proved. This is the essence of the proof of the following theorem. For example, $1, 2, 3$ satisfy $1/6 + 2/6 + 3/6 = 1$, yet they are not moduli of an ECF; $2/6$, $3/6$ are not in lowest terms.

The crux is based on the powerful JRT (Japanese Remainder Theorem motivated by conjecture A), due to Morikawa and Simpson. It cannot produce valid moduli, it can only verify their suitability; nor can it produce valid shifts. But it is strong nevertheless. The name JRT was given by Simpson to honor Morikawa.

Theorem (The Japanese Remainder Theorem [16], [17], [21]). *Given two distinct rational Beatty sequences* $\{\lfloor T_i n/q_i + \beta_i \rfloor : n = 1, 2, \ldots\}$ *and* $\{\lfloor T_j n/q_j + \beta_j \rfloor : n = 1, 2, \ldots\}$*, and* T_i, T_j, q_i, q_j *positive integers, let* $T = \gcd(T_i, T_j)$*,* $q = \gcd(q_i, q_j)$*,* $u_i = q_i/q$*,* $u_j = q_j/q$*. Then the sequences are disjoint for some shifts* β_i*,* β_j *if and only if there exist positive integers* x, y *so that*

$$xu_i + yu_j = T - 2u_iu_jq + 2u_iu_j. \tag{2}$$

We are interested in the disjointness of sequences $S(a^i) = \{\lfloor Tn/a^i + \beta_i \rfloor : n = 1, 2, \ldots\}$ and $S(a^j) = \lfloor Tn/a^j + \beta_j \rfloor : n = 1, 2, \ldots\}$, $0 \le i < j \le m-1$, where m is the number of sequences for any integer $a \ge 2$, with the numerator $T = (a^m - 1)/(a - 1)$. In terms of JRT, given any two such sequences $S(a^i), S(a^j), 0 \le i < j \le m - 1$, we have $q = \gcd(a^i, a^j) = a^i, u_i = 1, u_j = a^{j-i}$. The sequences $S(a^i)$ and $S(a^j)$ are disjoint if and only if there exist positive integers x, y such that

$$x = (a^m - 1)/(a - 1) - 2a^j + a^{j-i}(2 - y), 0 \le i < j \le m - 1. \tag{3}$$

This is Equation (2) adapted to $S(a^i)$.

We have the following theorem.

Theorem 1. *Sequences* $S(a^i), S(a^j)$ *are disjoint if and only if* $a = 2$ *for all* $0 \le i < j \le m - 1$.

Proof. Let $T = (a^m - 1)/(a - 1) = 1 + a + a^2 + ... + a^{m-1}$. For $a = 2$, we have $T = 2^m - 1$, and (3) becomes:

$$x = 2^m - 1 - 2^{j+1} + 2^{j-i}(2 - y). \tag{4}$$

For $y = 1$,

$$x = 2^m - 1 - 2^{j+1} + 2^{j-i} > 0$$

for all $0 \le i < j \le m - 1$.

So for all $0 \le i < j \le m-1$ we can choose β_i and β_j so that $S(2^i)$ and $S(2^j)$ are disjoint.

Now let $a > 2$. Recall

$$x = (a^m - 1)/(a - 1) - 2a^j + a^{j-i}(2 - y).$$

For $j = m-1, i = m-2, y = 1$, we have $x = (a^m - 1)/(a - 1) - 2a^{m-1} + a < 0$ for all $a \ge 3, m \ge 3$. So $S(a^{m-1})$ and $S(a^{m-2})$ intersect. □

Notice that for $j = 1, i = 0$ and $y = 1$ the sequences $S(a^0)$ and $S(a^1)$ are disjoint.

Corollary 1. *Sequences of the form $\{S(a^i)\}_{i=1}^m$ form an ECF if and only if $a = 2$.*

Remarks.

A. The reader is encouraged to see where the proof of the theorem fails for the m-member sequence $a, a^2, ..., a^m$.

B. In Fraenkel [10] it was shown that sequences $S(2^i)$ form an ECF. In particular, they are disjoint. Here we went the other way around: proved disjointness of sequences (2^i); the ECF property follows from the cake preamble covering discussed above. Here we showed that some sequences $S(a^i)$ intersect for $a > 2$, thus supporting Conjecture A for this important subcase.

4. Further Work

(i) We showed that $S(a^i)$ and $S(a^j)$ intersect for $a > 2$. What are other pairs, if any, of intersecting sequences for $a > 2$?

(ii) For $a > 2$, sequences intersect. For every two intersecting sequences there are missing elements and multiple elements. Study the structure of the complementary sequences of missing and multiple elements.

(iii) We proved Conjecture A for sequences $\{S(a^i)\}_{i=1}^m$. Prove it for other constructed sequences.

Acknowledgement. Jamie Simpson has made many useful comments to this paper, in addition to editing it. My deep appreciation to him.

References

[1] E. Altman, B. Gaujal and A. Hordijk, Balanced sequences and optimal routing *Journal of the ACM* **47** (2000), 752–775.

[2] J. Barát, P.P. Varjú, Partitioning the positive integers to seven Beatty sequences (English summary), *Indagationes Mathematicae* **14**(2003), 149-161.

[3] S. Beatty, Problems and Solutions, *Amer. Math. Monthly*, **34** (1927), 159-160.

[4] M. A. Berger, A. Felzenbaum and A. S. Fraenkel, Disjoint covering systems of rational Beatty sequences, *J. Combin. Theory Ser. A*, **42** (1986), 150-153.

[5] M. A. Berger, A. Felzenbaum and A. S. Fraenkel, A non-analytic proof of the Newman—Znám result for disjoint covering systems, *Combinatorica* **6** (1986), 235–243.

[6] P. Erdős, On a problem concerning systems of congruences (Hungarian; English summary), Mat. Lapok. **3** (1952), 122–128.

[7] P. Erdős, R.L. Graham, Old and new problems and results in combinatorial number theory. *Monographies de L'Enseignement Mathèmatique* **28**, Universitè de Genéve, Geneva, 1980.

[8] A. S. Fraenkel, The bracket function and complementary sets of integers, *Canad. J. Math.* **21** (1969), 6-27.

[9] A. S. Fraenkel, Complementing and exactly covering sequences, *J. Combinatorial Theory Ser. A* **14** (1973), 8-20.

[10] A. S. Fraenkel, Complementing systems of integers, *Amer. Math. Monthly* **84** (1977), 114-115.

[11] A.S. Fraenkel, R. Holzman, Gap problems for integer part and fractional part sequences, *J. Number Theory* **50** (1995), 66–86.

[12] R. L. Graham, On a theorem of Uspensky, *Amer. Math. Monthly* **70** (1963), 407-409.

[13] R. L. Graham, Covering the positive integers by disjoint sets of the form $\{[na + b] : n = 1, 2, ...\}$, *J. Com- binatorial Theory* **15** (1973), 354-358.

[14] R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete Mathematics, A Foundation for Computer Science* 2nd Edition, Addison-Wesley, 1994.

[15] R. L. Graham, S. Lin, C.S. Lin, Spectra of numbers, *Mathematics Magazine* **51** (1978), 174-176.

[16] R. Morikawa, On eventually covering families generated by the bracket function, *Bull. Fac. Liberal Arts, Nagasaki Univ.* **23** (1982), 17–22.

[17] R. Morikawa, Disjoint sequences generated by the bracket function, *Bull. Fac. Liberal Arts, Nagasaki Univ.* **26** (1985), 1-13.

[18] I. Niven,*Diophantine Approximations*, Interscience, New York, 1963.

[19] O. Schnabel, J. Simpson, A new approach to the Fraenkel conjecture for low N values, *Integers* **19** (2019), A43.

[20] J. Simpson, Disjoint covering systems of rational Beatty sequences, *Discrete Math.* **92** (1991), 361–369.

[21] J. Simpson, Disjoint Beatty sequences, *Integers* **4** (2004), A12.

[22] J. Simpson, Intersecting rational Beatty sequences, *Integers* **13**, A50.

[23] K.B. Stolarsky, Beatty sequences, continued fractions, and certain shift operators, *Canad. Math. Bull.* **19** (1976), 473-482.

[24] J. W. Strutt, third Baron Rayleigh, *The Theory of Sound*, Dover publications, New York, 1877.

[25] R. Tijdeman, Fraenkel's conjecture for six sequences, *Discrete Math.* **222** (2000), 223–234.