



**ON A FAMILY OF CUBIC THUE EQUATIONS INVOLVING
FIBONACCI AND LUCAS NUMBERS**

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Abstract

Let F_n denote the n -th Fibonacci number and L_n the n -th Lucas number. We completely solve the family of cubic Thue equations

$$(X - F_n Y)(X - L_n Y)X - Y^3 = \pm 1$$

and show that there are no non-trivial solutions for $n \neq 1, 3$.

1. Introduction

There is a vast amount of research on the resolution of Diophantine equations of the form

$$F(X, Y) = m,$$

where $F \in \mathbb{Z}[X, Y]$ is an irreducible form of degree at least 3 and $m \neq 0$ a fixed integer. A. Thue [16] proved that there are only finitely many solutions, but used non-effective methods and did not give bounds for the size of possible solutions. Based on his theory of linear forms in logarithms, A. Baker [1] was able to give such effective bounds, which have been refined many times since.

E. Thomas [14] studied parametrized families of Thue equations of the form

$$X \prod_{i=2}^N (X - p_i(a)Y) - Y^N = \pm 1,$$

with monic polynomials $p_i \in \mathbb{Z}[a]$. He conjectured that if $0 < \deg p_2 < \deg p_3 < \dots < \deg p_N$, then there is a constant a_0 such that for all integers $a \geq a_0$ the

equation has only solutions with $|y| \leq 1$. Thomas proved his conjecture in the case $N = 3$ under some technical hypothesis. Heuberger [8] proved the conjecture in the general case, again under some technical hypothesis. Indeed, at least some of the hypotheses are necessary, as Ziegler [17] provided a counterexample to Thomas' conjecture in the cubic case.

In this paper, we consider a specific family of Thue equations. While Thomas and Heuberger considered polynomially parametrized families of Thue equations, we consider Thue equations that are parametrized by linear recurrence sequences. More specifically, we consider the Fibonacci sequence defined by $F_0 = 0, F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$, and the Lucas sequence defined by $L_0 = 2, L_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 0$, and we prove the following theorem.

Theorem 1. *Let F_n denote the n -th Fibonacci number and L_n the n -th Lucas number. The family of cubic Thue equations*

$$(X - F_n Y)(X - L_n Y)X - Y^3 = \pm 1 \tag{1}$$

has only the trivial solutions

$$\pm\{(1, 0), (0, 1), (F_n, 1), (L_n, 1)\},$$

except for $n \in \{1, 3\}$. In those cases, one has the additional solutions $\pm\{(2, 1), (7, 4)\}$ and $\pm\{(7, 4), (38, 273)\}$ respectively.

For a fixed integer n , we will call a solution $(x, y) \in \mathbb{Z}^2$ to (1) *trivial* if $|y| \leq 1$.

The remainder of this paper is structured as follows. First, we state preliminary results in Section 2, which we use during our proof, including standard results on linear forms in logarithms, as per Baker-Wüstholz [2] or Bugeaud-Györy [3]. We then study properties of Equation (1). Assuming the existence of non-trivial solutions (x, y) , we construct linear forms in logarithms in Section 3. Using Baker's method, we get a lower bound on $\log|y|$, which is contradictory to Bugeaud's and Györy's upper bound. This will lead to a huge bound for n , which we then reduce using the LLL-algorithm in Section 4.

2. Preliminary Results and Notation

We start with Binet's formula for the n -th Fibonacci and Lucas number,

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \quad L_n = \alpha^n + \beta^n,$$

where

$$\alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}.$$

Note that

$$\beta = -\alpha^{-1}.$$

Furthermore, the inequality

$$\alpha^{n-2} \leq F_n \pm 11\alpha^{-n} \leq \alpha^{n-1} \tag{2}$$

holds for all $n \geq 6$, and we also use the relations

$$L_n = F_{n-1} + F_{n+1} = F_n + 2F_{n-1} \tag{3}$$

a couple of times throughout the paper.

To associate the roots of the polynomial $F(X, 1)$ with the respective Fibonacci and Lucas numbers, we will use the following result due to Kriegl, Losik, and Michor [9, Theorem 5.1]. We state a reformulated version due to Heuberger [7, Theorem 14, Corollary 15].

Theorem 2. *Let U be a neighborhood of 0 in \mathbb{R} , $a_i(t)$ be real analytic functions on U for $1 \leq i \leq N$, and*

$$P(t)(x) = x^N - a_1(t)x^{N-1} + \dots + (-1)^N a_N(t),$$

such that for all $t \in U$ the roots of $P(t)(x) = 0$ are real. Then there are real analytic functions $x_i(t)$, $i = 1, \dots, N$, on U such that these functions are roots of $P(t)$.

We will also use a very well known and simple result a couple of times throughout the paper:

Lemma 1. *If $x \in \mathbb{C}$ such that $|x - 1| \leq 1/2$, then $\log|x| \leq 2|x - 1|$.*

Proof. Use the Taylor series expansion of $\log x$ at 1. □

Definition 1. Let f, g be two complex-valued functions. We write $f = L(g)$ if $|f(x)| \leq |g(x)|$ for all $x \in \mathbb{C}$.

Lemma 2. *If $f(x) = 1 + g(x)$ with $|g(x)| \leq 1/2$, then $\log|f| = L(2g)$.*

Proof. This follows immediately from Lemma 1. □

Before we state lower bounds on linear forms in logarithms, we first recall the definition of the logarithmic (Weil) height.

Definition 2. Let γ be an algebraic number over \mathbb{Q} with minimal polynomial

$$a_d X^d + \dots + a_1 X + a_0 = a_d \prod_{i=1}^d (X - \gamma_i),$$

with relatively prime integers a_i and conjugates $\gamma_1 = \gamma, \gamma_2, \dots, \gamma_d$. The (absolute) logarithmic height of γ is defined by

$$h(\gamma) := \frac{1}{d} \left(\log|a_d| + \sum_{i=1}^d \log \max\{1, |\gamma_i|\} \right).$$

Theorem 3 (Baker, Wüstholz [2]). *Let $\gamma_1, \dots, \gamma_t$ be algebraic numbers not 0 or 1 in $K = \mathbb{Q}(\gamma_1, \dots, \gamma_t)$ of degree D , let $b_1, \dots, b_t \in \mathbb{Z}$, and let*

$$\Lambda = b_1 \log \gamma_1 + \dots + b_t \log \gamma_t$$

be non-zero. Then

$$\log|\Lambda| \geq -18(t+1)!t^{t+1}(32D)^{t+2} \log(2tD)h_1 \cdots h_t \log B,$$

where

$$B \geq \max\{|b_1|, \dots, |b_t|\},$$

and

$$h_i \geq \max\{h(\gamma_i), |\log \gamma_i|D^{-1}, 0.16D^{-1}\} \text{ for } 1 \leq i \leq t.$$

Finally, we also state the following result due to Bugeaud and Györy [3].

Theorem 4 (Bugeaud, Györy [3]). *Let $B \geq \max\{|m|, e\}$, α be a root of $F(X, 1)$, $K := \mathbb{Q}(\alpha)$, $R := R_K$ the regulator of K , and r the unit rank of K . Let $H \geq 3$ be an upper bound for the absolute values of the coefficients of F and N its degree.*

Then all solutions $(x, y) \in \mathbb{Z}^2$ of the Thue equation $F(X, Y) = m$ satisfy

$$\max\{\log|x|, \log|y|\} \leq 3^{r+27}(r+1)^{7r+19} N^{2N+6r+14} \cdot R \cdot \max\{\log R, 1\} \cdot (R + \log(HB)).$$

3. Auxiliary Results

We start by taking a look at the polynomial of Equation (1) at $Y = 1$,

$$f_n(X) := (X - F_n)(X - L_n)X - 1, \tag{4}$$

and its three roots $\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}$.

Lemma 3. *Let $\alpha = (1 + \sqrt{5})/2$, then the roots of f_n are given by*

$$\begin{aligned} \alpha^{(1)} &= \frac{1}{\sqrt{5}}\alpha^n - (-1)^n \frac{1}{\sqrt{5}}\alpha^{-n} + \frac{5}{1 - \sqrt{5}}\alpha^{-2n} + L(12\alpha^{-3n}) \\ &= F_n + L(6\alpha^{-2n}), \\ \alpha^{(2)} &= \alpha^n + (-1)^n \alpha^{-n} + L(4\alpha^{-2n}) = L_n + L(4\alpha^{-2n}), \\ \alpha^{(3)} &= \sqrt{5}\alpha^{-2n} + L(\alpha^{-4n}). \end{aligned}$$

Proof. The form of each $\alpha^{(i)}$ follows from Theorem 2. As real analytic functions, they can each be expressed by a Laurent series

$$\alpha^{(i)} = \sum_{k=-N}^{\infty} d_k^{(i)} \alpha^{-kn}.$$

Comparing the coefficients on both sides of the equation

$$(\alpha^{(i)} - F_n)(\alpha^{(i)} - L_n)\alpha^{(i)} = 1, \tag{5}$$

which follows from $f_n(\alpha^{(i)}) = 0$, yields the coefficients $d_k^{(i)}$ to arbitrary depth.

Assume, for a moment, that $N > 1$. We only consider the highest order terms of the left hand side of Equation (5),

$$(d_{-N}^{(i)}\alpha^{Nn} + \dots)(d_{-N}^{(i)}\alpha^{Nn} + \dots)(d_{-N}^{(i)}\alpha^{Nn} + \dots) = (d_{-N}^{(i)})^3 \alpha^{3Nn} + \dots.$$

Since the right hand side of Equation (5) only has a non-zero coefficient 1 of α^0 , it immediately follows that $d_{-N}^{(i)} = 0$.

Thus we may assume $N = 1$. Equation (5) then becomes

$$\left(\left(d_{-1}^{(i)} - \frac{1}{\sqrt{5}} \right) \alpha^n + \dots \right) \left(\left(d_{-1}^{(i)} - 1 \right) \alpha^n + \dots \right) \left(d_{-1}^{(i)} \alpha^n + \dots \right) = 1$$

and thus

$$\left(d_{-1}^{(i)} - \frac{1}{\sqrt{5}} \right) \left(d_{-1}^{(i)} - 1 \right) d_{-1}^{(i)} = 0.$$

It follows that the coefficient of α^n in $\alpha^{(i)}$ is $\frac{1}{\sqrt{5}}, 1$ or 0 . Similarly, we get the coefficient of α^0 being 0 and of α^{-n} being $(-1)^{n+1} \frac{1}{\sqrt{5}}, (-1)^n$ or 0 respectively.

We calculate $d_2 = d_2^{(1)}$ for $\alpha^{(1)}$ next, already knowing that the first and second coefficients are the same as for F_n . So we look at Equation (5), which is now

$$(d_2\alpha^{-2n} + \dots) \left(\left(\frac{1}{\sqrt{5}} - 1 \right) \alpha^n + \dots \right) \left(\frac{1}{\sqrt{5}} \alpha^n + \dots \right) = 1.$$

This yields

$$d_2 \left(\frac{1}{\sqrt{5}} - 1 \right) \frac{1}{\sqrt{5}} = 1,$$

and thus

$$d_2 = \frac{5}{1 - \sqrt{5}}.$$

We have now calculated sufficiently many coefficients. The rest of the Laurent series we hide inside the L -notation. Note, for example, that $f_n(F_n) = -1 < 0$. If either $f_n(F_n + \kappa\alpha^{-2n}) > 0$ or $f_n(F_n - \kappa\alpha^{-2n}) > 0$, then the root $\alpha^{(1)}$ must lie in the interval $(F_n \pm \kappa\alpha^{-2n})$ due to the intermediate value theorem, from which $\alpha^{(1)} = F_n + L(\kappa\alpha^{-2n})$ follows. Indeed, we have $f_n(F_n - 6\alpha^{-2n}) > 0$ and thus $\alpha^{(1)} = F_n + L(6\alpha^{-2n})$. The other assertions regarding the L -notation hold analogously. \square

As an immediate consequence, we also get the following lemma.

Lemma 4. *We have*

$$\begin{aligned} \log|\alpha^{(1)}| &= n \log \alpha - \log \sqrt{5} + L(3\alpha^{-2n}), \\ \log|\alpha^{(1)} - F_n| &= -2n \log \alpha + \log\left(\frac{5}{\sqrt{5} - 1}\right) + L(6\alpha^{-n}), \\ \log|\alpha^{(2)}| &= n \log \alpha + L(4\alpha^{-2n}), \\ \log|\alpha^{(2)} - F_n| &= n \log \alpha + \log\left(1 - \frac{1}{\sqrt{5}}\right) + L(6\alpha^{-2n}), \\ \log|\alpha^{(3)}| &= -2n \log \alpha + \log \sqrt{5} + L(\alpha^{-2n}), \\ \log|\alpha^{(3)} - F_n| &= n \log \alpha - \log \sqrt{5} + L(3\alpha^{-2n}). \end{aligned}$$

Proof. This follows from Lemma 3 and Lemma 2. For example, we have

$$\begin{aligned} \log|\alpha^{(1)}| &= \log(F_n + L(6\alpha^{-2n})) = \log\left(\frac{1}{\sqrt{5}}\alpha^n\left(1 \pm \alpha^{-2n} + L(6\sqrt{5}\alpha^{-3n})\right)\right) \\ &= n \log \alpha - \log \sqrt{5} + \log\left(1 + L\left(\frac{3}{2}\alpha^{-2n}\right)\right), \end{aligned}$$

and thus the claimed form for $\log|\alpha^{(1)}|$ after using Lemma 2. □

Lemma 5. *The only solutions $(x, y) \in \mathbb{Z}^2$ to the Thue Equation (1) with $|y| < 2$ are $\pm\{(1, 0), (0, 1), (F_n, 1), (L_n, 1)\}$.*

Proof. If $y = 0$, then Equation (1) becomes $X^3 = \pm 1$, and thus $x = \pm 1$. If $y = \pm 1$, then either

$$(X \pm F_n)(X \pm L_n)X = 0$$

or

$$(X \pm F_n)(X \pm L_n)X = \pm 2.$$

The first case immediately gives the solutions $\pm\{(F_n, 1), (L_n, 1), (0, 1)\}$. The second case gives no further solutions, unless $L_n - F_n \leq 2$, which implies $n \leq 3$ but gives no new solutions. □

Throughout the remainder of this paper, let us assume that (x, y) is a non-trivial solution of (1), i.e., $|y| \geq 2$. It is also helpful to assume $n \geq 10$. This we can do without loss of generality since we solve the Thue Equation (1) for such small n later.

For each of the $\alpha^{(i)}$ we define

$$\beta_i := x - \alpha^{(i)}y.$$

Since $\beta_1\beta_2\beta_3 = \pm 1$, they are all units. Furthermore, we have the following property for one of the β_i . Let j be defined by the β_i with minimal absolute value; that is,

$$|\beta_j| = \min_{i \in \{1,2,3\}} |\beta_i|.$$

Then we have the following result.

Lemma 6. *We have*

$$|\beta_j| \leq 2\alpha^5|y|^{-2} \cdot \alpha^{-2n}.$$

Proof. Let $i \neq j$, then

$$|y| \left| \alpha^{(i)} - \alpha^{(j)} \right| \leq \left| x - \alpha^{(i)}y \right| + \left| x - \alpha^{(j)}y \right| \leq 2 \left| x - \alpha^{(i)}y \right|,$$

and thus

$$\frac{|y|}{2} \left| \alpha^{(i)} - \alpha^{(j)} \right| \leq \left| x - \alpha^{(i)}y \right|. \tag{6}$$

For $\{j, k, l\} = \{1, 2, 3\}$, we have $\beta_j\beta_k\beta_l = 1$ and thus

$$|\beta_j| = \left| x - \alpha^{(k)}y \right|^{-1} \left| x - \alpha^{(l)}y \right|^{-1}.$$

Applying Inequality (6) on both factors on the right yields

$$|\beta_j| \leq 4|y|^{-2} \left| \alpha^{(k)} - \alpha^{(j)} \right|^{-1} \left| \alpha^{(l)} - \alpha^{(j)} \right|^{-1}. \tag{7}$$

Using case differentiation for the values of j, k, l gives us

$$\begin{aligned} \left| \alpha^{(k)} - \alpha^{(j)} \right| \left| \alpha^{(l)} - \alpha^{(j)} \right| &\geq (L_n - F_n + L(10\alpha^{-2n}))(F_n + L(9\alpha^{-2n})) \\ &= 2F_nF_{n-1} + L(11\alpha^{-n}) \end{aligned}$$

by Equation (3). We then use Inequality (2) to obtain the bound

$$\left| \alpha^{(k)} - \alpha^{(j)} \right| \left| \alpha^{(l)} - \alpha^{(j)} \right| \geq 2\alpha^{n-2}\alpha^{n-3} = 2\alpha^{-5}\alpha^{2n}.$$

Combining this with Inequality (7) yields

$$|\beta_j| \leq 2\alpha^5|y|^{-2} \cdot \alpha^{-2n}.$$

□

From now on, let $\{j, k, l\} = \{1, 2, 3\}$. We take a look at Siegel's identity, which, in our case, yields

$$\beta_j \left(\alpha^{(k)} - \alpha^{(l)} \right) + \beta_l \left(\alpha^{(j)} - \alpha^{(k)} \right) + \beta_k \left(\alpha^{(l)} - \alpha^{(j)} \right) = 0.$$

By rearranging terms, we arrive at

$$\frac{\beta_l}{\beta_k} \frac{\alpha^{(j)} - \alpha^{(k)}}{\alpha^{(j)} - \alpha^{(l)}} - 1 = \frac{\beta_j}{\beta_k} \frac{\alpha^{(l)} - \alpha^{(k)}}{\alpha^{(j)} - \alpha^{(l)}} =: \gamma. \tag{8}$$

Lemma 7. *We have*

$$|\gamma| \leq 8\alpha^8|y|^{-3} \cdot \alpha^{-3n}.$$

Proof. First, recall that we have $|\beta_j| \leq 2\alpha^5|y|^{-2} \cdot \alpha^{-2n}$ by Lemma 6. Next, note that for $i \neq j$, we have

$$\begin{aligned} |\beta_i| &= \left| x - \alpha^{(i)}y \right| = \left| \left(x - \alpha^{(j)}y \right) + y \left(\alpha^{(j)} - \alpha^{(i)} \right) \right| \\ &= \left| \beta_j + y \left(\alpha^{(j)} - \alpha^{(i)} \right) \right| \\ &\geq |y| \left| \alpha^{(j)} - \alpha^{(i)} \right| - |\beta_j|. \end{aligned} \tag{9}$$

Moreover, we have

$$\left| \alpha^{(j)} - \alpha^{(i)} \right| \geq L_n - F_n + L(10\alpha^{-2n}) \geq 2F_{n-1} - 10\alpha^{-2n} \geq 2\alpha^{n-3}$$

by Inequality (2). Thus, using Lemma 6, we obtain from Inequality (9) that

$$\begin{aligned} |\beta_i| &\geq |y| \cdot 2\alpha^{n-3} - 2\alpha^5 \cdot |y|^{-2} \cdot \alpha^{-2n} \\ &= |y|\alpha^{n-3} \left(2 - 2\alpha^5|y|^{-3}\alpha^{-3n+3} \right) \geq |y|\alpha^n\alpha^{-3}. \end{aligned}$$

We thus have

$$\left| \frac{\beta_j}{\beta_k} \right| \leq 2\alpha^5|y|^{-2} \cdot \alpha^{-2n} \cdot \left(|y|\alpha^n\alpha^{-3} \right)^{-1} = 2\alpha^8|y|^{-3} \cdot \alpha^{-3n}. \tag{10}$$

For the second term of $|\gamma|$, we differentiate the cases for l . If $l = 1$, then

$$\left| \frac{\alpha^{(l)} - \alpha^{(k)}}{\alpha^{(l)} - \alpha^{(j)}} \right| \leq \left| \frac{\alpha^{(1)} - \alpha^{(2)}}{\alpha^{(1)} - \alpha^{(3)}} \right| = \left| \frac{F_n - L_n + L(10\alpha^{-2n})}{F_n + L(9\alpha^{-2n})} \right| < 4.$$

Analogously, if $l = 2$, then

$$\left| \frac{\alpha^{(l)} - \alpha^{(k)}}{\alpha^{(l)} - \alpha^{(j)}} \right| \leq \frac{L_n + L(7\alpha^{-2n})}{L_n - F_n + L(10\alpha^{-2n})} < 4,$$

and if $l = 3$, then

$$\left| \frac{\alpha^{(l)} - \alpha^{(k)}}{\alpha^{(l)} - \alpha^{(j)}} \right| \leq \frac{L_n + L(7\alpha^{-2n})}{F_n + L(9\alpha^{-2n})} < 4.$$

We thus have

$$\left| \frac{\alpha^{(l)} - \alpha^{(k)}}{\alpha^{(l)} - \alpha^{(j)}} \right| \leq 4$$

in all cases. This yields, combined with Inequality (10),

$$|\gamma| = \left| \frac{\beta_j}{\beta_k} \right| \left| \frac{\alpha^{(l)} - \alpha^{(k)}}{\alpha^{(l)} - \alpha^{(j)}} \right| \leq 8\alpha^8|y|^{-3} \cdot \alpha^{-3n}.$$

□

For $i \in \{1, 2, 3\}$, let $K_i = \mathbb{Q}(\alpha^{(i)})$. As per Equation (5), both

$$\epsilon_i := \alpha^{(i)} \quad \text{and} \quad \delta_i := \alpha^{(i)} - F_n$$

are units in $\mathbb{Z}[\alpha^{(i)}]$. By a result of Thomas [13, Theorem 3.9], ϵ_i and δ_i even form a system of fundamental units of $\mathbb{Z}[\alpha^{(i)}]$. The regulator can then be bounded as follows.

Lemma 8. *Let*

$$G = G_i = \langle -1, \epsilon_i, \delta_i \rangle = \left(\mathbb{Z}[\alpha^{(i)}] \right)^\times$$

and $R_G = R_{G_i}$ denote the regulator of G , then we have

$$2(\log \alpha)^2 n^2 \leq R_G \leq 2n^2.$$

Proof. Since $[K_i : \mathbb{Q}] = 3$ and K_i is a totally real number field, there are three embeddings $\sigma_1, \sigma_2, \sigma_3$ of K_i into \mathbb{R} , with $\sigma_1 = \text{id}$. The regulator R_G of G can, up to sign, be expressed as the determinant of the matrix

$$\pm R_G = \det \begin{pmatrix} \log|\epsilon_i| & \log|\delta_i| \\ \log|\sigma_2(\epsilon_i)| & \log|\sigma_2(\delta_i)| \end{pmatrix}.$$

Since the σ_i are \mathbb{Q} -automorphisms, we have $\sigma_i(t) = t$ for all rational t . Thus the expression simplifies to

$$\pm R_G = \det \begin{pmatrix} \log|\alpha^{(i)}| & \log|\alpha^{(i)} - F_n| \\ \log|\sigma_2(\alpha^{(i)})| & \log|\sigma_2(\alpha^{(i)} - F_n)| \end{pmatrix}.$$

Let us denote $\sigma_2(\alpha^{(i)}) = \alpha^{(i')}$. By Lemma 4, the determinant becomes, up to sign,

$$\pm R_G = 3(\log \alpha)^2 n^2 + L(4n)$$

for each case for (i, i') . This gives

$$2(\log \alpha)^2 n^2 \leq R_G \leq 2n^2$$

for $n \geq 5$, which proves the asserted bounds for R_G . □

Since ϵ_i and δ_i are fundamental units in $\mathbb{Z}[\alpha^{(i)}]$, the units $\beta_1, \beta_2, \beta_3$ can be expressed as

$$\beta_i = \pm \epsilon_i^{b_1} \delta_i^{b_2} \tag{11}$$

for some integers b_1, b_2 that do not depend on i , since the β_i are conjugates. We take the absolute value and the logarithm and get

$$\log|\beta_i| = b_1 \log|\epsilon_i| + b_2 \log|\delta_i|. \tag{12}$$

This will lead to a bound for b_1 and b_2 .

Lemma 9. *Let $\{j, k, l\} = \{1, 2, 3\}$. Then we have*

$$\max\{|b_1|, |b_2|\} \leq 7n^{-1} \cdot \max\{\log|\beta_k|, \log|\beta_l|\}.$$

Proof. We rewrite Equation (12) for k, l into the system of linear equations

$$\begin{pmatrix} \log|\epsilon_k| & \log|\delta_k| \\ \log|\epsilon_l| & \log|\delta_l| \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \log|\beta_k| \\ \log|\beta_l| \end{pmatrix}. \tag{13}$$

We call the above 2×2 matrix M . Its determinant is, up to sign, equal to R_G of Lemma 8. Thus we have

$$|\det M| \geq 2(\log \alpha)^2 n^2 \tag{14}$$

and especially $\det M \neq 0$. We can thus multiply Equation (13) with M^{-1} and take the matrix-infinity norm $\|\cdot\|_\infty$. This yields, in combination with the inequality $\|Ax\|_\infty \leq \|A\|_\infty \cdot \|x\|_\infty$,

$$\max\{|b_1|, |b_2|\} \leq \|M^{-1}\|_\infty \max\{\log|\beta_k|, \log|\beta_l|\}. \tag{15}$$

The inverse matrix of M is given by

$$\frac{1}{\det M} \begin{pmatrix} \log|\delta_l| & -\log|\delta_k| \\ -\log|\epsilon_l| & \log|\epsilon_k| \end{pmatrix},$$

and thus its infinity norm by

$$\|M^{-1}\|_\infty = \frac{1}{|\det M|} \max\{|\log|\delta_l|| + |\log|\delta_k||, |\log|\epsilon_l|| + |\log|\epsilon_k||\}. \tag{16}$$

By Lemma 4, all terms inside the maximum are linear in n . Furthermore, the highest coefficient of n is given by

$$|\log|\delta_1|| + |\log|\delta_3|| = 3 \log \alpha \cdot n + L(6),$$

and $3 \log \alpha \cdot n + L(6) < 3n$ for $n \geq 10$. This yields

$$\max\{|\log|\delta_l|| + |\log|\delta_k||, |\log|\epsilon_l|| + |\log|\epsilon_k||\} \leq 3n.$$

If we combine this with Equation (16) and Inequality (14), we get

$$\|M^{-1}\|_\infty \leq \frac{3}{2} (\log \alpha)^{-2} n^{-1} < 7n^{-1}.$$

Inequality (15) thus becomes

$$\max\{|b_1|, |b_2|\} \leq 7n^{-1} \max\{\log|\beta_k|, \log|\beta_l|\}.$$

□

Lemma 10. For $\{j, k, l\} = \{1, 2, 3\}$, we have

$$\max\{\log|\beta_k|, \log|\beta_l|\} \leq \log|y| + n \log \alpha + 1.$$

Proof. Let

$$(X - F_n)(X - L_n)X - 1 = (X - p_1)(X - p_2)(X - p_3) - 1,$$

that is, $p_1 = F_n, p_2 = L_n, p_3 = 0$. For $i \neq j$, we then have

$$\left| \frac{\beta_i}{y} \right| = \left| \frac{x}{y} - \alpha^{(i)} \right| \leq \left| \frac{x}{y} - \alpha^{(j)} \right| + \left| \alpha^{(j)} - p_j \right| + \left| p_j - \alpha^{(i)} \right|$$

due to the triangle inequality. Factorizing the right-hand side leads to

$$\left| \frac{\beta_i}{y} \right| \leq \left| \alpha^{(i)} - p_j \right| \left(1 + \frac{|\alpha^{(j)} - p_j|}{|\alpha^{(i)} - p_j|} + \frac{|\beta_j/y|}{|\alpha^{(i)} - p_j|} \right). \tag{17}$$

We estimate the terms inside the bracket. Using Lemma 3, we get

$$\left| \alpha^{(i)} - p_j \right| \geq F_n + L(9\alpha^{-2n}),$$

and thus, by Inequality (2),

$$\left| \alpha^{(i)} - p_j \right| \geq \alpha^{n-2} - 9\alpha^{-2n} \geq \alpha^{n-3} = \alpha^{-3}\alpha^n \tag{18}$$

on the one hand, and

$$\left| \alpha^{(i)} - p_j \right| \leq (L_n + L(4\alpha^{-2n}) - 0) \leq \alpha^n + \alpha^{n-2} + 4\alpha^{-2n}, \tag{19}$$

by Inequality (2), on the other. Moreover, the root $\alpha^{(j)}$ is close to the coefficient p_j , by Lemma 3 as close as

$$\left| \alpha^{(j)} - p_j \right| \leq |F_n + L(6\alpha^{-2n}) - F_n| \leq 6\alpha^{-2n}. \tag{20}$$

Since we assume that $|y| \geq 2$, we have $|\beta_j| \leq \frac{1}{2}\alpha^5\alpha^{-2n}$ by Lemma 6. Combining this with Inequalities (18) and (20) yields

$$\begin{aligned} \left(1 + \frac{|\alpha^{(j)} - p_j|}{|\alpha^{(i)} - p_j|} + \frac{|\beta_j/y|}{|\alpha^{(i)} - p_j|} \right) &\leq \left(1 + 6\alpha^3\alpha^{-3n} + \frac{1}{4}\alpha^8\alpha^{-3n} \right) \\ &\leq (1 + 38\alpha^{-3n}). \end{aligned} \tag{21}$$

We combine Inequalities (17) and (21), multiply with $|y|$, and arrive at

$$|\beta_i| \leq \left| \alpha^{(i)} - p_j \right| \cdot |y| \cdot (1 + 38\alpha^{-3n}).$$

Using Inequality (19) and taking logarithms yields

$$\log|\beta_i| \leq \log(\alpha^n + \alpha^{n-2} + 4\alpha^{-2n}) + \log|y| + \log(1 + 38\alpha^{-3n}).$$

We then make full use of Lemma 2 and arrive at

$$\log|\beta_i| \leq \log(\alpha^n) + 2(\alpha^{-2} + 4\alpha^{-3n}) + \log|y| + 76\alpha^{-3n},$$

and thus

$$\log|\beta_i| \leq \log|y| + n \log \alpha + 1.$$

□

4. Linear Forms in Logarithms and an Effective Bound for n

We go back to Equation (8),

$$\frac{\beta_l \alpha^{(j)} - \alpha^{(k)}}{\beta_k \alpha^{(j)} - \alpha^{(l)}} - 1 = \gamma,$$

and use Equation (11) to rewrite the expression as

$$\left(\frac{\epsilon_l}{\epsilon_k}\right)^{b_1} \left(\frac{\delta_l}{\delta_k}\right)^{b_2} \frac{\alpha^{(j)} - \alpha^{(k)}}{\alpha^{(j)} - \alpha^{(l)}} = 1 + \gamma.$$

We then define the linear form in logarithms Λ by

$$\Lambda = \log|1 + \gamma| = b_1 \log\left|\frac{\epsilon_l}{\epsilon_k}\right| + b_2 \log\left|\frac{\delta_l}{\delta_k}\right| + \log\left|\frac{\alpha^{(j)} - \alpha^{(k)}}{\alpha^{(j)} - \alpha^{(l)}}\right|.$$

Due to Lemma 1 and 7, and $|y| \geq 2$, we have

$$|\Lambda| \leq 2|\gamma| \leq 16\alpha^8|y|^{-3} \cdot \alpha^{-3n} \leq 2\alpha^8\alpha^{-3n}. \tag{22}$$

We want to immediately apply Theorem 3 to bound $|\Lambda|$ from below. The problem with this, however, lies in the expressions ϵ_l/ϵ_k and δ_l/δ_k having huge heights that would render the bound useless for our purposes. Thus we first rewrite the linear form to get the (exponential) dependency on n out of the logarithms.

To that end, we need to differentiate the cases for j . Let $j = 1$ and choose $k = 3, l = 2$.

By Lemma 3, we get

$$\begin{aligned} \frac{\epsilon_2}{\epsilon_3} &= \frac{\alpha^{(2)}}{\alpha^{(3)}} = \frac{\alpha^n \pm \alpha^{-n} + L(4\alpha^{-2n})}{\sqrt{5}\alpha^{-2n} + L(\alpha^{-4n})} = \frac{\alpha^n (1 \pm \alpha^{-2n} + L(4\alpha^{-3n}))}{\sqrt{5}\alpha^{-2n} (1 + L(\sqrt{5}^{-1}\alpha^{-2n}))} \\ &= \frac{1}{\sqrt{5}}\alpha^{3n} \left(1 + L\left(\frac{3}{2}\alpha^{-2n}\right)\right), \end{aligned}$$

and thus, combined with Lemma 2,

$$\log \left| \frac{\epsilon_2}{\epsilon_3} \right| = 3n \log \alpha - \log \sqrt{5} + L(3\alpha^{-2n}). \tag{23}$$

Similarly,

$$\begin{aligned} \frac{\delta_2}{\delta_3} &= \frac{\left(1 - \frac{1}{\sqrt{5}}\right)\alpha^n \pm \left(1 + \frac{1}{\sqrt{5}}\right)\alpha^{-n} + L(10\alpha^{-2n})}{-\frac{1}{\sqrt{5}}\alpha^n \pm \frac{1}{\sqrt{5}}\alpha^{-n} + L(3\alpha^{-2n})} \\ &= \left(1 - \sqrt{5}\right) \left(1 + L\left(\frac{5}{2}\alpha^{-2n}\right)\right), \end{aligned}$$

and thus

$$\log \left| \frac{\delta_2}{\delta_3} \right| = \log(\sqrt{5} - 1) + L(5\alpha^{-2n}). \tag{24}$$

Lastly,

$$\begin{aligned} \frac{\alpha^{(1)} - \alpha^{(3)}}{\alpha^{(1)} - \alpha^{(2)}} &= \frac{\frac{1}{\sqrt{5}}\alpha^n \pm \frac{1}{\sqrt{5}}\alpha^{-n} + L(9\alpha^{-2n})}{\left(\frac{1}{\sqrt{5}} - 1\right)\alpha^n \pm \left(\frac{1}{\sqrt{5}} + 1\right)\alpha^{-n} + L(10\alpha^{-2n})} \\ &= \frac{1}{1 - \sqrt{5}} \left(1 + L\left(\frac{3}{2}\alpha^{-2n}\right)\right), \end{aligned}$$

and thus

$$\log \left| \frac{\alpha^{(1)} - \alpha^{(3)}}{\alpha^{(1)} - \alpha^{(2)}} \right| = -\log(\sqrt{5} - 1) + L(3\alpha^{-2n}). \tag{25}$$

If we combine Equations (23)-(25) and shift the L -terms into the upper bound, the original linear form

$$|\Lambda| = \left| b_1 \log \left| \frac{\epsilon_l}{\epsilon_k} \right| + b_2 \log \left| \frac{\delta_l}{\delta_k} \right| + \log \left| \frac{\alpha^{(j)} - \alpha^{(k)}}{\alpha^{(j)} - \alpha^{(l)}} \right| \right| \leq 2\alpha^8 \alpha^{-3n}$$

transforms into a linear form

$$\xi = x_1 \log \alpha + x_2 \log \sqrt{5} + x_3 \log(\sqrt{5} - 1),$$

where

$$x_1 = 3nb_1, \quad x_2 = -b_1, \quad x_3 = b_2 - 1. \tag{26}$$

Analogously, the linear form ξ has the same logarithms in the case $j = 2$, where we choose $(k, l) = (1, 3)$, and in the case $j = 3$, where we choose $(k, l) = (1, 2)$. The coefficients are

$$x_1 = -3nb_1 + 3nb_2, \quad x_2 = 2b_1 - 3b_2 - 1, \quad x_3 = b_2 + 1 \tag{27}$$

and

$$x_1 = 3nb_2, x_2 = b_1 - 3b_2 - 1, x_3 = 2b_2 \tag{28}$$

respectively.

In all three cases, the respective L-terms can be bounded from above by the term $(3|b_1| + 5|b_2| + 5)\alpha^{-2n}$, thus

$$|\xi| \leq 2\alpha^8\alpha^{-3n} + (3|b_1| + 5|b_2| + 5)\alpha^{-2n}. \tag{29}$$

Lemma 11. *The linear form ξ fulfills*

$$\xi \neq 0$$

in all cases.

Proof. Since $\log \alpha, \log \sqrt{5}, \log(\sqrt{5} - 1)$ are \mathbb{Q} -linearly independent, $\xi = 0$ would imply $x_1 = x_2 = x_3 = 0$.

In the case $j = 1$, this implies $b_1 = 0, b_2 = 1$. By Equation (11), this means

$$x - \alpha^{(i)}y = \alpha^{(i)} - F_n$$

and thus $(x, y) = -(F_n, 1)$, which we ruled out.

In the case $j = 2$, the expression $x_1 = 0$ implies $b_1 = b_2$. It then follows from $x_2 = 0$ that $b_1 = b_2 = -1$. Thus Equation (11) implies

$$x - \alpha^{(i)}y = \left(\alpha^{(i)}\right)^{-1} \left(\alpha^{(i)} - F_n\right)^{-1}.$$

This implies

$$\left(-\alpha^{(i)}y + x\right) \left(\alpha^{(i)} - F_n\right) \left(\alpha^{(i)}\right) - 1 = 0$$

for each $\alpha^{(i)}$, which means that the $\alpha^{(i)}$ are roots of the polynomial

$$(-Xy + x)(X - F_n)X - 1.$$

The minimal polynomial of the $\alpha^{(i)}$ is given by Equation (4), however, and it follows that $(x, y) = -(L_n, 1)$, which we ruled out.

In the case $j = 3$, it follows from $x_1 = 0$ that $b_2 = 0$, and from $x_2 = 0$ that $b_1 = 1$. Equation (11) then implies

$$x - \alpha^{(i)}y = \alpha^{(i)}$$

and thus $(x, y) = -(0, 1)$, which we also ruled out. □

Before we can apply the lower bound from Theorem 3, we have to bound the coefficients x_i of ξ from above:

Lemma 12. *We have*

$$\max\{|x_1|, |x_2|, |x_3|\} \leq 42 \log|y| + 42 \log \alpha \cdot n + 43.$$

Proof. By looking at Equations (26), (27), and (28) we have

$$\max\{|x_1|, |x_2|, |x_3|\} \leq 6n \max\{|b_1|, |b_2|\} + 1$$

for $n \geq 1$ in all cases. Additionally, Lemma 9 gives us

$$\max\{|x_1|, |x_2|, |x_3|\} \leq 42 \cdot \max\{\log|\beta_k|, \log|\beta_l|\} + 1.$$

By Lemma 10, this then yields

$$\max\{|x_1|, |x_2|, |x_3|\} \leq 42 \cdot (\log|y| + n \log \alpha + 1) + 1,$$

and thus

$$\max\{|x_1|, |x_2|, |x_3|\} \leq 42 \log|y| + 42 \log \alpha \cdot n + 43.$$

□

We now use Baker’s and Wüstholz’s lower bound (Theorem 3) for the linear form ξ to derive an exponential lower bound for $\log|y|$.

Lemma 13. *One has*

$$\log|y| \geq \exp\left(\frac{2 \log \alpha}{1 + C} \cdot n - \log n - 5\right),$$

with $C = 17496 \cdot 64^5 \cdot \log(12) \cdot \log \alpha \cdot \log \sqrt{5} \cdot \log 2 \approx 1.253 \cdot 10^{13}$.

Proof. By Theorem 3, we have

$$\begin{aligned} \log|\xi| &\geq -18(3 + 1)!3^{3+1}(32 \cdot 2)^{3+2} \log(2 \cdot 3 \cdot 2)h_1 \cdots h_t \log B \\ &= -34992 \cdot 64^5 \cdot \log(12) \cdot h_1 h_2 h_3 \cdot \log \max\{|x_1|, |x_2|, |x_3|\}, \end{aligned}$$

since ξ is a linear form in the $t = 3$ logarithms $\log \alpha, \log \sqrt{5}, \log(\sqrt{5} - 1)$, and $D = [\mathbb{Q}(\alpha, \sqrt{5}, \sqrt{5} - 1) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{5}) : \mathbb{Q}] = 2$.

The minimal polynomials of $\alpha, \sqrt{5}$ and $\sqrt{5} - 1$ are given by

$$\begin{aligned} X^2 - X - 1 &= (X - \alpha)(X - \beta), \\ X^2 - 5 &= (X - \sqrt{5})(X + \sqrt{5}), \text{ and} \\ X^2 + 2X - 4 &= (X - (\sqrt{5} - 1))(X + (\sqrt{5} + 1)) \end{aligned}$$

respectively. Thus, the logarithmic heights are $\frac{1}{2} \log \alpha, \log \sqrt{5}$, and $\log 2$ respectively. Together with Lemma 12, this yields

$$\log|\xi| \geq -C \cdot \log(42 \log|y| + 42 \log \alpha \cdot n + 43),$$

with $C = 17496 \cdot 64^5 \cdot \log(12) \cdot \log \alpha \cdot \log \sqrt{5} \cdot \log 2 \approx 1.253 \cdot 10^{13}$. We estimate the last term using $\log|y| \geq \log 2$ and $n \geq 10$:

$$\begin{aligned} & \log(42 \log|y| + 42 \log \alpha \cdot n + 43) \\ &= \log \log|y| + \log\left(42 + \frac{42}{\log|y|} \log \alpha \cdot n + \frac{43}{\log|y|}\right) \\ &\leq \log \log|y| + \log(30n + 103) \leq \log n + \log \log|y| + 4. \end{aligned}$$

Thus, the lower bound for $\log|\xi|$ becomes

$$\log|\xi| \geq -C(\log n + \log \log|y| + 4). \tag{30}$$

On the other hand, recall that Inequality (29) gives us an upper bound for $|\xi|$,

$$|\xi| \leq 2\alpha^8 \alpha^{-3n} + (3|b_1| + 5|b_2| + 5)\alpha^{-2n}.$$

Plugging in the bound from Lemma 9 gives us

$$|\xi| \leq 2\alpha^8 \alpha^{-3n} + (56n^{-1} \max\{\log|\beta_k|, \log|\beta_l|\} + 5)\alpha^{-2n}.$$

By Lemma 10, and with $n \geq 10$, this then yields

$$\begin{aligned} |\xi| &\leq \alpha^{-2n} (2\alpha^8 \alpha^{-n} + 56n^{-1} \log|y| + 56 \log \alpha + 56n^{-1} + 5) \\ &\leq \alpha^{-2n} (6 \log|y| + 39). \end{aligned} \tag{31}$$

We take the logarithm of Inequality (31) and compare it with Inequality (30), which yields

$$\begin{aligned} -C(\log n + \log \log|y| + 4) &\leq -2n \log \alpha + \log(6 \log|y| + 39) \\ &\leq -2n \log \alpha + \log \log|y| + 5. \end{aligned}$$

We rewrite this into

$$\log \log|y|(1 + C) \geq 2 \log \alpha \cdot n - C \log n - (5 + 4C),$$

and thus get

$$\log|y| \geq \exp\left(\frac{2 \log \alpha}{1 + C} \cdot n - \log n - 5\right).$$

□

Next, we use Bugeaud’s and Györy’s upper bound (Theorem 4) for $\log|y|$.

Lemma 14. *We have*

$$\log|y| \leq 3^{94} \cdot 2n^2 \cdot \log(2n^2) \cdot (2n^2 + (2n - 1) \log \alpha + 1).$$

Proof. Let $K = \mathbb{Q}(\alpha^{(1)})$. By Lemma 8, we have

$$R_K \leq 2n^2$$

for the regulator R_K . The unit-rank is $r = 2$, an upper bound for the coefficients of the Thue Equation (1) is given by $H = \alpha^{2n-1} \geq L_n \cdot F_n = F_{2n}$ by Equation (2), and $B = \max\{1, e\} = e$.

Theorem 4 then yields

$$\begin{aligned} \log|y| &\leq 3^{2+27} \cdot 3^{14+19} 3^{6+12+14} \cdot (2n^2) \cdot \log(2n^2) \cdot (2n^2 + \log(\alpha^{2n-1}e)) \\ &= 3^{94} \cdot 2n^2 \cdot \log(2n^2) \cdot (2n^2 + (2n - 1) \log \alpha + 1). \end{aligned}$$

□

If we compare the lower bound from Lemma 13 and the upper bound from Lemma 14, we get the following absolute bound for n .

Lemma 15. *We have*

$$n \leq 1.144 \cdot 10^{15}.$$

5. Reducing the Bound for n

We want to reduce the bound for n using the LLL-algorithm as described in [10, Lemma VI.1].

To that end we take a look at the linear form ξ and rewrite its upper bound from Inequality (31), namely

$$\begin{aligned} |\xi| &= \left| x_1 \log \alpha + x_2 \log \sqrt{5} + x_3 \log(\sqrt{5} - 1) \right| \\ &\leq \alpha^{-2n} (6 \log|y| + 39) = c_2 e^{-c_3 \cdot n}, \end{aligned}$$

where $c_3 := 2 \log \alpha$, and c_2 is given by Lemma 14 to be

$$c_2(n) := 6 \cdot 3^{94} \cdot 2n^2 \cdot \log(2n^2) \cdot (2n^2 + (2n - 1) \log \alpha + 1) + 39.$$

This gives $c_2 = c_2(1.144 \cdot 10^{15}) \approx 2.036 \cdot 10^{108}$ when using the bound for n found in Lemma 15.

We can bound x_i for each $i \in \{1, 2, 3\}$ from above by Lemma 12, together with the bound from Lemma 14 for $\log|y|$, and Lemma 15 for n . This gives

$$|x_i| \leq X_i(n) := 42 \cdot 3^{94} \cdot 2n^2 \cdot \log(2n^2) \cdot (2n^2 + (2n - 1) \log \alpha + 1) + 42 \log \alpha \cdot n + 43,$$

and $X_i = X_i(1.144 \cdot 10^{15}) \approx 1.425 \cdot 10^{109}$. We set C to be a generous upper bound to $\max\{X_1, X_2, X_3\}^3$, in this first step we choose $C = 10^{330}$ and then form the

matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ [C \log \alpha] & [C \log \sqrt{5}] & [C \log(\sqrt{5} - 1)] \end{pmatrix},$$

where $[\cdot]$ denotes the nearest integer. We use the LLL-algorithm in Sage [11] on this matrix to get a LLL-reduced Basis in form of a matrix B , let b_1 denote its first column vector. Furthermore, let B^* be the result of the Gram-Schmidt process on B with column vectors b_i^* . We set $c_1 = \max\{\|b_1\|^2 / \|b_i^*\|^2 : b_i^* \text{ column vector of } B^*\}$, and $c_4 := \|b_1\| / \sqrt{c_1}$.

If we set S to be $S = \sum_{i=1}^2 X_i^2$, and $T = (1 + \sum_{i=1}^3 X_i) / 2$, then we have $c_4^2 \geq T^2 + S$. Lemma VI.1 of [10] then yields

$$n \leq \frac{1}{c_3} \left(\log(C \cdot c_2) - \log\left(\sqrt{c_4^2 - S} - T\right) \right),$$

and thus $n \leq 786$. We iterate the above process to get a further reduced bound each time, until we get $n \leq 403$, after which we get no further reduction.

The bound $n \leq 403$ is still too large to simply tackle the remaining cases with a brute-force approach. We let PARI/GP, with its Sage-interface, solve the equations up to $n = 48$ and only found additional solutions for $n = 1, 3$.

To tackle the remaining cases, we instead look at the original linear form in logarithms

$$|\Lambda| = \left| b_1 \log \left| \frac{\epsilon_l}{\epsilon_k} \right| + b_2 \log \left| \frac{\delta_l}{\delta_k} \right| + \log \left| \frac{\alpha^{(j)} - \alpha^{(k)}}{\alpha^{(j)} - \alpha^{(l)}} \right| \right| \leq 16\alpha^8 |y|^{-3} \alpha^{-3n}.$$

We improve the asymptotic of the upper bound by ascertaining the following property.

Lemma 16. *For $48 \leq n \leq 403$, we have*

$$|y| \geq \alpha^n.$$

Proof. It follows immediately from Lemma 6 that $\frac{x}{y}$ is a convergent to $\alpha^{(j)}$. For each n , we therefore check all convergents $\frac{p}{q}$ to either root $\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}$, with denominator $2 \leq |q| \leq \alpha^n$, and ascertained that the pair (p, q) does not provide a solution to the respective Thue Equation. \square

Using $|y| \geq \alpha^n$, instead of just $|y| \geq 2$, the upper bound for Λ in Inequality (22) becomes

$$|\Lambda| \leq 16\alpha^8 |y|^{-3} \alpha^{-3n} \leq 16\alpha^8 \alpha^{-6n}.$$

We then use a method by Baker-Davenport, similar to how it is formulated in [4, Lemma 5], on the linear form Λ . We first divide the inequality $|\Lambda| \leq 16\alpha^8\alpha^{-6n}$ by $|\log|\delta_l/\delta_k||$ and get

$$|b_1\gamma - (-b_2) + \mu| < A \cdot \alpha^{-6n},$$

where we set

$$\begin{aligned} \gamma &:= \log|\epsilon_l/\epsilon_k|/\log|\delta_l/\delta_k|, \quad \mu := \log\left|\frac{\alpha^{(j)} - \alpha^{(k)}}{\alpha^{(j)} - \alpha^{(l)}}\right|/\log|\delta_l/\delta_k|, \\ A &:= 16\alpha^8/\log|\delta_l/\delta_k|. \end{aligned}$$

We then set M to be an upper bound for both $|b_1|$ and $6n$, which is given by

$$M(n) := \frac{7}{n}(3^{94}2n^2 \log(2n^2)(2n^2 + (2n - 1)\log(\alpha) + 1) + n \log(\alpha) + 1),$$

as per Lemma 9, 10, and 14.

For each of the three cases for j , we estimate γ and μ numerically, by using a Newton-descent with high bit-precision and small error-tolerance, to find the roots $\alpha^{(i)}$. We then search the convergents $\frac{p}{q}$ of γ until the inequality

$$\epsilon := |\mu q - [\mu q]| - M|\gamma q - [\gamma q]| > 0$$

holds. We always find such a convergent. But we have the inequality

$$\begin{aligned} qA\alpha^{-6n} &> |b_1\gamma q - (-b_2)q + \mu q| = |\mu q - (-b_2 - b_1p) + b_1(\gamma q - p)| \\ &\geq |\mu q - (-b_2 - b_1p)| - |b_1||\gamma q - p| \\ &\geq |\mu q - [\mu q]| - M|\gamma q - [\gamma q]| = \epsilon, \end{aligned}$$

as $[\mu q]$ minimises the distance of μq to any integer. Now, $\epsilon > 0$ allows us to take the logarithm on both sides and thus

$$6n < \frac{\log(Aq/\epsilon)}{\log \alpha}.$$

We check for each $48 \leq n \leq 403$ that this bound is too small (for the respective n). For example, if we set $n = 403$ it would follow that $403 = n \leq 203$ in all three cases for j , which gives the contradiction.

We thus have no further cases to consider and conclude our proof of Theorem 1.

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