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SEQUENCES IN PARTITIONS

George E. Andrews[1]
Department of Mathematics, Pennsylvania State University, University Park, PA Country
gea1@psu.edu.in

Sreerupa Bhattacharjee
Department of Mathematics and Computer Science, University of Lethbridge, Canada
bhattacharjees@uleth.ca

Manosij G. Dastidar
Institute of Discrete Mathematics and Geometry, Universitát Wien, Vienna
gdmanosij@gmail.com

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Abstract

This paper is devoted to the study of a new concept in the theory of partitions called k-measure. It is proved that the number of partitions of n with 2-measure m equals the number of partitions of n with Durfee square of side m.

1. Introduction

Sylvester was the first to prove a theorem of the type considered in this paper [5], [1], [3], namely the following.

Theorem 1. *The number of partitions of n into distinct parts containing m maximal sequences of consecutive integers as parts equals the number of partitions of n into m different odd parts (repetitions allowed).*

Example. For $n = 13$, and $m = 3$ there are five partitions of the first type: $9+3+1, 8+4+1, 7+5+1, 7+4+2, 6+4+2+1$ and five of the second type: $9+3+1, 7+5+1, 7+3+1+1+1, 5+3+3+1+1, 5+3+1+1+1+1+1$.

Definition 1. We say that the *k-measure* of a partition is the length of the largest subsequence of parts of a partition wherein the difference between any two parts of a subsequence is at least k.

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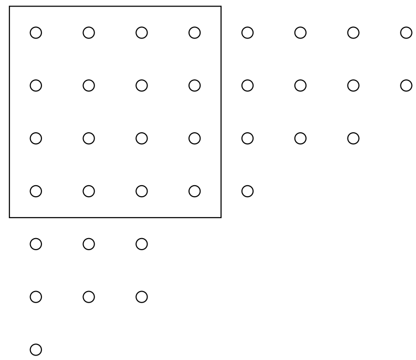

Figure 1

Clearly, the 1-measure of a partition is the number of distinct parts in the partition. Thus Sylvester's theorem could be rewritten where the odd parts have 1-measure m. We should also note that when $k > 1$, there may be more than one maximal subsequence.

Example. The partition $9+8+5+4+4+2+2+1$ has 2-measure 3 and several maximal subsequences, e.g., $(9,4,2), (9,5,1), (8,5,2)$, etc.

The object of this paper is to prove the following theorem.

Theorem 2. *The number of partitions of n with 2-measure m equals the number of partitions of n with Durfee square of side m.*

The concept of a Durfee square is most easily explained by an example.

Example. Figure 1 shows the Ferrer's graph of the partition $8+8+7+5+3+3+1$. The largest square of nodes in a Ferrer's graph is called its Durfee square. In this case, the partition has a Durfee square (as indicated) of side 4.

Section 2 will be devoted to collecting the background results necessary for proving Theorem 2. Section 3 will be devoted to the actual proof. We will close with some general observations and questions.

2. Background

We require the q-binomial coefficients:

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{cases} \frac{(q;q)_A}{(q;q)_B(q;q)_{A-B}} & \text{if } 0 \leq B \leq A \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

where $(A;q)_N = (1-A)(1-Aq)\ldots(1-Aq^{n-1})$.

Theorem 3. *Let $0 \le B \le A$ be integers. The q-binomial is a polynomial of degree $B(A-B)$ in q satisfying the following relations:*

$$\begin{bmatrix} A \\ 0 \end{bmatrix} = \begin{bmatrix} A \\ A = 1 \end{bmatrix} = 1, \tag{2.2}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A \\ A-B \end{bmatrix}, \tag{2.3}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A-1 \\ B \end{bmatrix} + q^{A-B}\begin{bmatrix} A-1 \\ B-1 \end{bmatrix}, \tag{2.4}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A-1 \\ B-1 \end{bmatrix} + q^{B}\begin{bmatrix} A-1 \\ B \end{bmatrix}. \tag{2.5}$$

Lemma 1. *For $R > 0$,*

$$\sum_{n=0}^{R} \frac{(A;q)_n (B;q)_n q^n}{(q;q)_n (ABq;q)_n} = \frac{(Aq;q)_R (Bq;q)_R}{(ABq;q)_R (q;q)_R}. \tag{2.6}$$

Proof. This is the q-analog of a result by J.H.C Searle [4]. It can be easily seen that

$$\begin{aligned}
&\frac{(Aq;q)_R (Bq;q)_R}{(ABq;q)_R (q;q)_R} - \frac{(Aq;q)_{R-1} (Bq;q)_{R-1}}{(ABq;q)_{R-1} (q;q)_{R-1}} \\
&\quad = \frac{(Aq;q)_{R-1} (Bq;q)_{R-1}}{(ABq;q)_R (q;q)_R} (1-Aq^R)(1-Bq^R) - (1-ABq^R)(1-q^R) \\
&\quad = \frac{(Aq;q)_{R-1} (Bq;q)_{R-1}}{(ABq;q)_R (q;q)_R} (q^R(1-A)(1-B)) \\
&\quad = q^R \frac{(A;q)_R (B;q)_R}{(ABq;q)_R (q;q)_R}.
\end{aligned}$$

The result now follows by telescoping the above series. □

3. Proof

We begin by studying partitions according to the length of the sequence of consecutive integer parts that contain the largest part. For example, the partition $8+8+7+6+6+4$ has $8, 7, 6$ as this sequence. We call this sequence the *top sequence* and we call its length the *top number*. In the above example, the top number is 3.

We let $e(n)$ (resp. $o(n)$) denote the generating function for partitions with even (resp. odd) top number and the largest part being at most n. This will be a two variable generating function where the exponent on z counts the 2-measure and the

exponent on q counts the number being partitioned.
Thus,

$$e(0) = e(1) = 1 \text{ (for the empty partition of 0)}, \tag{3.1}$$

$$e(2) = 1 + \frac{zq^{1+2}}{(1-q)(1-q^2)} = 1 + \frac{zq^3}{(1-q)(1-q^2)}, \tag{3.2}$$

$$o(0) = 0, \tag{3.3}$$

$$o(1) = \frac{zq}{(1-q)}, \tag{3.4}$$

$$o(2) = \frac{zq}{1-q} + \frac{zq^2}{1-q^2}. \tag{3.5}$$

We now define

$$E(n) = e(n) - e(n-1), \tag{3.6}$$

and

$$O(n) = o(n) - o(n-1). \tag{3.7}$$

So $E(n)$ (resp. $O(n)$) is the generating function for those partitions counted by $e(n)$ (resp. $o(n)$) where the largest part is exactly n. Clearly,

$$e(n) = \sum_{j=0}^{n} E(j), \tag{3.8}$$

and

$$o(n) = \sum_{j=0}^{n} O(j). \tag{3.9}$$

Lemma 2. *The following recurrences hold in addition to Equations* (3.8) *and* (3.9)*:*

$$E(n) = \frac{q^n}{1-q^n} O(n-1), \tag{3.10}$$

$$O(n) = \frac{zq^n}{1-q^n}(e(n-2) + o(n-2)) + \frac{zq^n}{1-q^n} E(n-1). \tag{3.11}$$

Proof. Note that when the largest part of a partition is $n-1$ and the top number is odd, then adding instances of n does not change the 2-measure, because if the original top sequence is $(n{-}1, n{-}2, \ldots n{-}2j{+}1)$ then exactly j terms of this sequence (namely, $n-1$, $n-3$, $n-5$,...,$n-2j+1$) contribute to the 2-measure. By inserting n we see that still only j terms of the sequence $(n, n-1,\ldots,n-2j+1)$ contribute to the 2-measure (although there are now two possible ways of contributing j terms: $(n, n-2, n-4, \ldots n-2j+2)$ or $(n-1, n-3, n-5,\ldots,n-2j+1)$). If, on the other hand, the top number is even, say $2j$, and the top sequence is $(n-1, n-2, n-3, \ldots n-2j)$, then inserting n adds 1 to the 2-measure because now $j+1$ terms can contribute to the 2-measure (namely, n, $n-2$, $n-4$,... $n-2j$). This accounts for the non-obvious appearances or non-appearances of z in the following identities.

Verification of Equation (3.10) and Equation (3.11) is straightforward. First, consider Equation (3.10). In order to produce partitions with top number even and largest part n, one must attach the n's to a partition where the largest part is $n-1$ and the top number is odd. This establishes Equation (3.10). For Equation (3.11), there are two possibilities. First if n's appear with no $(n-1)$'s, then all other parts are allowed to be less than or equal to $n-2$. Hence,

$$\frac{zq^n}{1-q^n}(e(n-2)+o(n-2)).$$

If $n-1$ appears, then it has to be part of an even top sequence (since the addition of n increases the top number by 1). Hence in this case, the partitions are generated by

$$\frac{zq^n}{1-q^n}E(n-1).$$

Thus Equation (3.11) is established. $\square$

Note that the initial conditions Equations (3.1),(3.3) and (3.4), together with Equations (3.8), (3.9), (3.10) and (3.11), uniquely define $E(n)$, $O(n)$, $e(n)$ and $o(n)$. This is easily proved by mathematical induction on n. We now define

$$\mathcal{E}(n)=\begin{cases}1 & \text{if } n=0\\ \sum_{j=1}^{\lfloor\frac{n}{2}\rfloor}\frac{z^j q^{2n+j^2-2j}\begin{bmatrix}n-j-1\\ j-1\end{bmatrix}}{(q;q)_{j-1}(q^{n-j};q)_{j+1}} & \text{if } n>0\end{cases}, \tag{3.12}$$

$$\Omega(n)=\sum_{j=1}^{\lfloor\frac{n+1}{2}\rfloor}\frac{z^j q^{n+(j-1)^2}\begin{bmatrix}n-j\\ j-1\end{bmatrix}}{(q;q)_{j-1}(q^{n-j+1};q)_j}, \tag{3.13}$$

$$\epsilon(n)=\sum_{j=0}^{n}\mathcal{E}(j), \tag{3.14}$$

$$\omega(n)=\sum_{j=0}^{n}\Omega(j). \tag{3.15}$$

Theorem 4. *For $n\geq 0$,*

$$\mathcal{E}(n)=E(n), \tag{3.16}$$

$$\Omega(n)=O(n), \tag{3.17}$$

$$\epsilon(n)=e(n), \tag{3.18}$$

$$\omega(n)=o(n). \tag{3.19}$$

Proof. We have already remarked that Equations (3.1), (3.3), (3.4), (3.8). (3.9), (3.10) and (3.11) uniquely define $E(n)$, $O(n)$, $e(n)$, $o(n)$. Hence if we show that

$\mathcal{E}(n)$, $\Omega(n)$, $\epsilon(n)$ and $\omega(n)$ satisfy the same assertions, then we will have proved our theorem.

Now Equation (3.1) is immediate from Equation (3.12), and Equation (3.3) follows from Equation (3.13) with $n = 0$. Equation (3.4) follows from Equations (3.15) and (3.13) with $n = 1$. Also, Equation (3.8) and Equation (3.9) are immediate from Equation (3.14) and Equation (3.15), respectively. Thus, all that remains are the proofs of Equation (3.10) and Equation (3.11). First, we verify Equation (3.16). We consider the definition of $E(n)$ from Equation (3.10) and see that

$$\begin{aligned}\frac{q^n}{1-q^n}\Omega(n-1) =& \frac{q^n}{1-q^n}\sum_{j=1}^{\lfloor \frac{n}{2}\rfloor} \frac{z^j q^{n-1+j^2-2j+1}\begin{bmatrix} n-j-1 \\ j-1\end{bmatrix}}{(q;q)_{j-1}(q^{n-j};q)_j} \\ =& \sum_{j=1}^{\lfloor \frac{n}{2}\rfloor} \frac{z^j q^{2n+j^2-2j}\begin{bmatrix} n-j-1 \\ j-1\end{bmatrix}}{(q;q)_{j-1}(q^{n-j};q)_{j+1}} \\ =& \mathcal{E}(n),\end{aligned}$$

as desired.

Finally, we treat the following equivalent version of Equation (3.11):

$$\frac{1-q^n}{zq^n}O(n) = e(n-2) + o(n-2) + E(n-1). \tag{3.20}$$

For the left-hand side,

$$\begin{aligned}\frac{1-q^n}{zq^n}\Omega(n) =& \sum_{j=1}^{\lfloor \frac{n+1}{2}\rfloor} \frac{z^{j-1} q^{j-1^2}\begin{bmatrix} n-j \\ j-1\end{bmatrix}}{(q;q)_{j-1}(q^{n-j+1};q)_{j-1}} \\ =& \sum_{j=0}^{\lfloor \frac{n-1}{2}\rfloor} \frac{z^j q^{j^2}\begin{bmatrix} n-j-1 \\ j\end{bmatrix}}{(q;q)_j(q^{n-j};q)_j}.\end{aligned} \tag{3.21}$$

To simplify the computation for the right-hand side, we consider the coefficient of z^j in $(\epsilon(N) + \omega(N))$:

$$\begin{aligned}&= \sum_{n=2j}^{N} \frac{q^{2n+j^2-2j}\begin{bmatrix} n-j-1 \\ j-1\end{bmatrix}}{(q;q)_{j-1}(q^{n-j};q)_{j+1}} + \sum_{n=2j-1}^{N} \frac{q^{n+(j-1)^2}\begin{bmatrix} n-j \\ j-1\end{bmatrix}}{(q;q)_{j-1}(q^{n-j+1};q)_j} \\ &= \sum_{n=0}^{N-2j} \frac{q^{2(n+2j)+j^2-2j}\begin{bmatrix} n+j-1 \\ j-1\end{bmatrix}}{(q;q)_{j-1}(q^{n+j};q)_{j+1}} + \sum_{n=0}^{N-2j+1} \frac{q^{n+j^2}\begin{bmatrix} n+j-1 \\ j-1\end{bmatrix}}{(q;q)_{j-1}(q^{n+j};q)_j} \\ &= \frac{q^{N+(j-1)^2}\begin{bmatrix} N-j \\ j-1\end{bmatrix}}{(q;q)_{j-1}(q^{N-j+1};q)_j} + \frac{q^{j^2}}{(q;q)_{j-1}} \sum_{n=0}^{N-2j} \frac{q^n\begin{bmatrix} n+j-1 \\ j-1\end{bmatrix}}{(q^{n+j};q)_{j+1}}(q^{n+2j} + (1-q^{n+2j}))\end{aligned}$$

$$
\begin{aligned}
&= \frac{q^{N+(j-1)^2}\begin{bmatrix}N-j\\j-1\end{bmatrix}}{(q;q)_{j-1}(q^{N-j+1};q)_j} + \frac{q^{j^2}}{(q;q)_{j-1}}\sum_{n=0}^{N-2j}\frac{q^n\begin{bmatrix}n+j-1\\j-1\end{bmatrix}}{(q^{n+j};q)_{j+1}}\\
&= \frac{q^{N+(j-1)^2}\begin{bmatrix}N-j\\j-1\end{bmatrix}}{(q;q)_{j-1}(q^{N-j+1};q)_j} + \frac{q^{j^2}}{(q;q)_{2j}}\sum_{n=0}^{N-2j}\frac{(q^j;q)_n^2 q^n}{(q)_n(q^{2j+1};q)_n}\\
&= \frac{q^{N+(j-1)^2}\begin{bmatrix}N-j\\j-1\end{bmatrix}}{(q;q)_{j-1}(q^{N-j+1};q)_j} + \frac{q^{j^2}}{(q;q)_{2j}}\frac{(q^{j+1};q)_{N-2j}^2}{(q;q)_{N-2j}(q^{2j+1};q)_{N-2j}} \text{ (by Lemma 1)}\\
&= \frac{q^{N+(j-1)^2}\begin{bmatrix}N-j\\j-1\end{bmatrix}}{(q;q)_{j-1}(q^{N-j+1};q)_j} + \frac{q^{j^2}(q;q)_{N-j}^2}{(q;q)_j^2(q;q)_N(q;q)_{N-2j}}\\
&= \frac{q^{N+(j-1)^2}\begin{bmatrix}N-j\\j-1\end{bmatrix}}{(q;q)_{j-1}(q^{N-j+1};q)_j} + \frac{q^{j^2}\begin{bmatrix}N-j\\j\end{bmatrix}}{(q;q)_j(q^{N-j+1};q)_j}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\epsilon(N)+\omega(N)\\
&= \sum_{j\geq 0}\frac{z^j q^{N+(j-1)^2}\begin{bmatrix}N-j\\j-1\end{bmatrix}}{(q;q)_{j-1}(q^{N-j+1};q)_j} + \sum_{j\geq 0}\frac{z^j q^{j^2}\begin{bmatrix}N-j\\j\end{bmatrix}}{(q;q)_j(q^{N-j+1};q)_j}\\
&= \sum_{j\geq 0}\frac{z^j q^{j^2}}{(q;q)_j(q^{N-j+1};q)_j}\left\{q^{N-2j+1}(1-q^j)\begin{bmatrix}N-j\\j-1\end{bmatrix} + \begin{bmatrix}N-j\\j\end{bmatrix}\right\}\\
&= \sum_{j\geq 0}\frac{z^j q^{j^2}}{(q;q)_j(q^{N-j+1};q)_j}\left\{\begin{bmatrix}N-j+1\\j\end{bmatrix} - q^{N-j+1}\begin{bmatrix}N-j\\j-1\end{bmatrix}\right\} \text{ (by Equation (2.5)).}
\end{aligned}
\tag{3.22}
$$

Hence, for the right side of Equation (3.20),

$$
\begin{aligned}
&\epsilon(n-2)+\omega(n-2)+\mathcal{E}(n-1)\\
&= \sum_{j\geq 0}\frac{z^j q^{j^2}}{(q;q)_j(q^{n-j-1};q)_j}\left(\begin{bmatrix}n-j-1\\j\end{bmatrix} - q^{n-j-1}\begin{bmatrix}n-j-2\\j-1\end{bmatrix}\right)\\
&\quad + \sum_{j\geq 1}\frac{z^j q^{2n+j^2-2j-2}}{(q;q)_{j-1}(q^{n-j-1};q)_{j+1}}\begin{bmatrix}n-j-2\\j-1\end{bmatrix}\\
&= \sum_{j\geq 0}\frac{z^j q^{j^2}}{(q;q)_j(q^{n-j-1};q)_{j+1}}\left\{\left(\begin{bmatrix}n-j-1\\j\end{bmatrix} - q^{n-j-1}\begin{bmatrix}n-j-2\\j-1\end{bmatrix}\right)(1-q^n)\right.\\
&\quad \left. + q^{2n-2j-2}(1-q^j)\begin{bmatrix}n-j-2\\j-1\end{bmatrix}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j \geq 0} \frac{z^j q^{j^2}}{(q;q)_j (q^{n-j-1};q)_{j+1}} \left\{ \begin{bmatrix} n-j-1 \\ j \end{bmatrix} (1-q^{n-1}) \right. \\
&\quad \left. - \, q^{n-j-1}(1-q^{n-j-1}) \begin{bmatrix} n-j-2 \\ j-1 \end{bmatrix} \right\} \\
&= \sum_{j \geq 0} \frac{z^j q^{j^2}}{(q;q)_j (q^{n-j-1};q)_{j+1}} \frac{(q;q)_{n-j-1}}{(q;q)_j (q;q)_{n-2j-1}} \big((1-q^{n-1}) - q^{n-j-1}(1-q^j) \big) \\
&= \sum_{j \geq 0} \frac{z^j q^{j^2}}{(q;q)_j (q^{n-j-1};q)_{j+1}} \begin{bmatrix} n-j-1 \\ j \end{bmatrix} (1-q^{n-j-1}) \\
&= \sum_{j \geq 0} \frac{z^j q^{j^2} \begin{bmatrix} n-j-1 \\ j \end{bmatrix}}{(q;q)_j (q^{n-j};q)_j}.
\end{aligned}
$$

and this is identical to the right-hand side of Equation (3.21), thus establishing

$$
\frac{(1-q^n)\Omega(n)}{zq^n} = \epsilon(n-2) + \omega(n-2) + \mathcal{E}(n-1),
$$

which is equivalent to Equation (3.20).

Hence, all the required recurrences have been established and Theorem 4 is proved. □

Proof of Theorem 2. The generating function for all partitions in which the exponent on z counts the 2-measure and the exponent of q counts the number being partitioned is given by

$$
\begin{aligned}
\lim_{N \to \infty} (e(N) + o(N)) = \\
&= \lim_{N \to \infty} (\epsilon(N) + \omega(N)) \text{ (by Theorem 4)} \\
&= \lim_{N \to \infty} \left(\sum_{j \geq 0} \frac{z^j q^{j^2}}{(q;q)_j (q^{N-j+1};q)_j} \left\{ \begin{bmatrix} N-j+1 \\ j \end{bmatrix} \right. \right. \\
&\quad \left. \left. - \, q^{N-j+1} \begin{bmatrix} N-j-1 \\ j-1 \end{bmatrix} \right\} \right) \\
&= \sum_{j \geq 0} \frac{z^j q^{j^2}}{(q;q)_j^2}.
\end{aligned} \tag{3.23}
$$

Now the generating function for partitions with Durfee square of side j is given by

$$
\frac{q^{j^2}}{(q;q)_j^2}.
$$

Comparing the last observation with the first expression in Equation (3.23) we see that Theorem 2 is established.

4. Conclusion

It is obvious that a theorem as simply stated as Theorem 2 should have a bijective proof. In addition, the expressions for $E(n)(=\mathcal{E}(n))$ and $O(n)(=\Omega(n))$ are sufficiently elegant that further investigation of partitions with top number even (or odd) are merited. Finally, it is conceivable that k-measures with $k > 2$ may have interesting properties.

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