ON SOME GENERALIZATIONS TO FLOOR FUNCTION IDENTITIES OF RAMANUJAN

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Abstract
We give some generalizations to three identities of Srinivasa Ramanujan involving the greatest integer function.

1. Introduction

Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Ramanujan [8] proposed three interesting identities involving the greatest integer function as a problem to the *Journal of the Indian Mathematical Society. If n is any positive integer, prove that*

$$\left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+2}{6} \right\rfloor + \left\lfloor \frac{n+4}{6} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+3}{6} \right\rfloor \tag{1}$$

$$\left\lfloor \frac{1}{2} + \sqrt{n + \frac{1}{2}} \right\rfloor = \left\lfloor \frac{1}{2} + \sqrt{n + \frac{1}{4}} \right\rfloor \tag{2}$$

$$\left\lfloor \sqrt{n} + \sqrt{n+1} \right\rfloor = \left\lfloor \sqrt{4n+2} \right\rfloor. \tag{3}$$

This problem involving (1)–(3) appears as Problem 723 in Ramanujan's third notebook [7, vol. 2, page 361], and proofs of (1)–(3) can be found in Berndt's book [3, pp. 76–78]. A.A. Krishnaswami Aiyangar [1] posed a problem giving analogues, one involving fourth roots and one involving fifth roots, of all three parts of Problem 723. In [2], he established results generalizing the results in his problem. Chen in [4] gave some extensions and conjectures to (2) and (3), such as

$$\begin{aligned} &\left\lfloor \sqrt{n} + \sqrt{n+1} \right\rfloor = \left\lfloor \sqrt{n} + \sqrt{n+2} \right\rfloor \\ &= \left\lfloor \sqrt{4n+1} \right\rfloor = \left\lfloor \sqrt{4n+2} \right\rfloor = \left\lfloor \sqrt{4n+3} \right\rfloor \end{aligned}$$

and

$$\left\lfloor \frac{1}{2} + \sqrt{n + \frac{1}{4}} \right\rfloor = \left\lfloor \frac{1}{2} + \sqrt{n + \frac{1}{2}} \right\rfloor = \left\lfloor \frac{1}{2} + \sqrt{n + \frac{3}{4}} \right\rfloor.$$

We give a few generalizations of all the identities (1), (2), (3). In this article, by natural numbers, we mean positive integers. In Section 2, we will show that (1) is a direct consequence of Hermite's identity (see [9]). It states that for any real number x and natural number m, we have

$$\sum_{i=0}^{m-1} \left\lfloor x + \frac{i}{m} \right\rfloor = \lfloor mx \rfloor. \tag{4}$$

We give some corollaries to Hermite's identity in Section 2.

Next, in Section 3, we prove identities similar to (2). We prove that for any natural numbers a, b, and $n \geq \frac{b}{a}$, we have

$$\left\lfloor \frac{b + \sqrt{4n+1}}{a} \right\rfloor = \left\lfloor \frac{b + \sqrt{4n+2}}{a} \right\rfloor = \left\lfloor \frac{b + \sqrt{4n+3}}{a} \right\rfloor,$$

and for any natural number m, we have

$$\left\lfloor \frac{b + \sqrt[3]{9m+1}}{a} \right\rfloor = \left\lfloor \frac{b + \sqrt[3]{9m+2}}{a} \right\rfloor = \cdots = \left\lfloor \frac{b + \sqrt[3]{9m+6}}{a} \right\rfloor = \left\lfloor \frac{b + \sqrt[3]{9m+7}}{a} \right\rfloor.$$

We prove these identities from the following theorem (which is also proved in Section 3).

Theorem 1. *Let S be any set containing all natural numbers and let $f : S \to \mathbb{R}$ be a strictly increasing (resp. decreasing) function whose restriction to natural numbers is in integers. Let a, b, and m be any three natural numbers such that $a \leq b$. If*

$$f(x) \equiv y \mod m$$

has no solutions in $x \in \mathbb{N}$ for all $y \in \{a, a+1, \cdots, b-1, b\}$, then

$$\left\lfloor f^{-1}(nm + a - 1) \right\rfloor = \left\lfloor f^{-1}(nm + a) \right\rfloor = \cdots = \left\lfloor f^{-1}(nm + b) \right\rfloor$$

$$\left(\textit{resp.} \ \left\lfloor f^{-1}(nm + a) \right\rfloor = \left\lfloor f^{-1}(nm + a + 1) \right\rfloor = \cdots = \left\lfloor f^{-1}(nm + b + 1) \right\rfloor \right)$$

for all natural numbers n such that f^{-1} exists for all elements in $[nm + a - 1, nm + b] \cap \mathbb{Z}$ (resp. $[nm + a, nm + b + 1] \cap \mathbb{Z}$).

In Section 4, we consider two identities of the form (3) and prove Theorems 2 and 3.

Theorem 2. *For all natural numbers $n, k > 1$, except for finitely many exceptions of the form $n = \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor$, we have*

$$\lfloor \sqrt[k]{n} + \sqrt[k]{n+1} \rfloor = \left\lfloor 2\sqrt[k]{n + \frac{1}{2}} \right\rfloor .$$

For proving that there are only finitely many exceptions, we use a result of Mahler [6].

Theorem 3. *Let $l, k, x_1, x_2, \ldots, x_l$ be natural numbers, and let p be a prime number such that $p|l, p^k \nmid l$, and $p \nmid (x_1 + x_2 + \cdots + x_l)$. If $n \geq \frac{l^{k-1}(x_1^2 + \cdots + x_l^2)}{2}$, then*

$$\lfloor \sqrt[k]{n+x_1} + \sqrt[k]{n+x_2} + \cdots + \sqrt[k]{n+x_l} \rfloor = \left\lfloor l\sqrt[k]{n + \frac{x_1 + x_2 + \cdots + x_l}{l}} \right\rfloor .$$

2. Identities in the Spirit of (1)

Now let us prove Ramanujan's identity (1) using Hermite's identity.

2.1. Proof of (1) using Hermite's Identity

Proof. Set $x = \frac{n}{6},\ m = 3$ and $x = \frac{n}{6},\ m = 2$ in (4) to get:

$$\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n}{6} \right\rfloor + \left\lfloor \frac{n+2}{6} \right\rfloor + \left\lfloor \frac{n+4}{6} \right\rfloor ,$$
$$\left\lfloor \frac{n}{6} \right\rfloor + \left\lfloor \frac{n+3}{6} \right\rfloor = \left\lfloor \frac{n}{3} \right\rfloor .$$

Adding both the equations and canceling out $\lfloor \frac{n}{6} \rfloor$ on both sides, we get (1). □

2.2. Corollaries of Hermite's Identity

We have the following corollary of Hermite's identity.

Corollary 4. *For any $x \in \mathbb{R}$ and natural number $n > 1$, we have*

$$\sum_{j=0}^{\infty} \sum_{i=1}^{n-1} \left\lfloor \frac{x}{n^{j+1}} + \frac{i}{n} \right\rfloor = \begin{cases} \lfloor x \rfloor & \text{for } x \geq 0 \\ \lfloor x \rfloor + 1 & \text{for } x < 0. \end{cases}$$

Proof. From (4), we have

$$\sum_{i=1}^{n-1} \left\lfloor \frac{x}{n^{j+1}} + \frac{i}{n} \right\rfloor = \sum_{i=0}^{n-1} \left\lfloor \frac{x}{n^{j+1}} + \frac{i}{n} \right\rfloor - \left\lfloor \frac{x}{n^{j+1}} \right\rfloor = \left\lfloor \frac{x}{n^{j}} \right\rfloor - \left\lfloor \frac{x}{n^{j+1}} \right\rfloor .$$

Hence

$$\sum_{j=0}^{\infty}\sum_{i=1}^{n-1}\left\lfloor \frac{x}{n^{j+1}}+\frac{i}{n}\right\rfloor = \lim_{k\to\infty}\sum_{j=0}^{k}\left(\left\lfloor\frac{x}{n^j}\right\rfloor - \left\lfloor\frac{x}{n^{j+1}}\right\rfloor\right) = \lim_{k\to\infty}\left(\lfloor x\rfloor - \left\lfloor\frac{x}{n^{k+1}}\right\rfloor\right).$$

Since $\lim_{k\to\infty}\lfloor\frac{x}{n^{k+1}}\rfloor = 0$ if $x \geq 0$ and $\lim_{k\to\infty}\lfloor\frac{x}{n^{k+1}}\rfloor = -1$ otherwise, we obtain the claim. □

By setting $n = 2$ and $x = m$ in Corollary 4, we can prove Question 6 of the International Math Olympiad held in 1968.

Corollary 5. *For every natural number m, we have*

$$\sum_{i=0}^{\infty}\left\lfloor\frac{m+2^i}{2^{i+1}}\right\rfloor = \left\lfloor\frac{m+1}{2}\right\rfloor + \left\lfloor\frac{m+2}{4}\right\rfloor + \left\lfloor\frac{m+4}{8}\right\rfloor + \cdots = m.$$

3. Generalizations of (2)

Let us now prove Theorem 1.

Proof of Theorem 1. Let us assume that the function f is strictly increasing (the proof of the decreasing case is analogous). Let n be a natural number such that f has an inverse for all elements in $[nm+a-1, nm+b]\cap\mathbb{Z}$ and let $c = \lfloor f^{-1}(nm+a-1)\rfloor$. The equation

$$f(x) \equiv y \mod m$$

has no solutions in $x \in \mathbb{N}$ for $y \in \{a, a+1, \ldots, b-1, b\}$. This implies

$$f(c) \leq nm + a - 1 < nm + a < \cdots < nm + b < f(c+1).$$

Applying f^{-1} on all expressions, we get

$$c \leq f^{-1}(nm+a-1) < f^{-1}(nm+a) < \cdots < f^{-1}(nm+b) < c+1,$$

which implies the claim. □

Let us prove Ramanujan's identity (2) using Theorem 1.

3.1. Proof of (2) Using Theorem 1

Proof. Let us take a function $f(x) := (2x-1)^2$ defined on the interval $[1, +\infty)$. This function is increasing and is invertible in $[1, \infty)$. As

$$f(x) = (2x-1)^2 \equiv 2 \mod 4$$

has no solutions in integers, applying Theorem 1 for $m = 4$ and $a = b = 2$, we have

$$\lfloor f^{-1}(4n+1) \rfloor = \lfloor f^{-1}(4n+2) \rfloor .$$

The inverse of f is $f^{-1}(x) = \frac{1+\sqrt{x}}{2}$, and therefore $\left\lfloor \frac{1+\sqrt{4n+1}}{2} \right\rfloor = \left\lfloor \frac{1+\sqrt{4n+2}}{2} \right\rfloor$ or

$$\left\lfloor \frac{1}{2} + \sqrt{n + \frac{1}{2}} \right\rfloor = \left\lfloor \frac{1}{2} + \sqrt{n + \frac{1}{4}} \right\rfloor .$$

□

Let us look at some corollaries of Theorem 1.

3.2. Corollaries of Theorem 1

Corollary 6. *Let a and b be any two natural numbers and $n \geq \frac{b}{a}$ be any other natural number. We have*

$$\left\lfloor \frac{b + \sqrt{4n+1}}{a} \right\rfloor = \left\lfloor \frac{b + \sqrt{4n+2}}{a} \right\rfloor = \left\lfloor \frac{b + \sqrt{4n+3}}{a} \right\rfloor .$$

Proof. Let $f(x) := (ax - b)^2$ be a function defined on $[\frac{b}{a}, \infty)$ (here $f^{-1}(x) = \frac{b+\sqrt{x}}{a}$). As a square can never be equivalent to 2 or 3 modulo 4, the claim follows from Theorem 1. □

Corollary 7. *Let a, b, and n be any three natural numbers. We have*

$$\left\lfloor \frac{b + \sqrt[3]{9n+1}}{a} \right\rfloor = \left\lfloor \frac{b + \sqrt[3]{9n+2}}{a} \right\rfloor = \cdots = \left\lfloor \frac{b + \sqrt[3]{9n+6}}{a} \right\rfloor = \left\lfloor \frac{b + \sqrt[3]{9n+7}}{a} \right\rfloor .$$

Proof. Let $f(x) := (ax - b)^3$ be a function defined on $\mathbb{R}$ (here $f^{-1}(x) = \frac{b+\sqrt[3]{x}}{a}$). As a cube can never be equivalent to $2, 3, \ldots, 7$ modulo 9, the claim follows from Theorem 1. □

4. Generalizations of (3)

4.1. Proof of Theorem 2

We will prove it using three lemmas. For proving one of these lemmas, we use an inequality known as the generalized mean inequality. If p is a non-zero real number, and $x_1, \ldots, x_n$ are positive real numbers, then the generalized mean with exponent p of these positive real numbers is:

$$M_p(x_1, \ldots, x_n) = \left(\frac{1}{n} \sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} .$$

For $p = 0$, we set the generalized mean equal to the geometric mean:

$$M_0(x_1, \ldots, x_n) = \left(\prod_{i=1}^{n} x_i \right)^{\frac{1}{n}}.$$

The generalized mean inequality states that, for all real numbers $p, q, x_1, \ldots, x_n$, if $p < q$ and $x_1 > 0, \ldots, x_n > 0$, then

$$M_p(x_1, \ldots, x_n) \leq M_q(x_1, \ldots, x_n).$$

The two means are equal if and only if $x_1 = x_2 = \ldots = x_n$.

The inequality of arithmetic and geometric means, or more briefly the AM–GM inequality, states that the arithmetic mean of a list of non-negative real numbers is greater than or equal to the geometric mean of the same list. Further, equality holds if and only if every number in the list is the same. For a list of n non-negative real numbers $x_1, \ldots, x_n$, we have

$$\left(\prod_{i=1}^{n} x_i \right)^{\frac{1}{n}} \leq \frac{1}{n} \sum_{i=1}^{n} x_i,$$

with equality if and only if $x_1 = x_2 = \ldots = x_n$.

Lemma 8. *For natural numbers $n, k > 1$, we have*

$$\left\lfloor \sqrt[k]{2^k n + 2^{k-1} - 1} \right\rfloor = \left\lfloor \sqrt[k]{2^k n + 2^{k-1}} \right\rfloor.$$

Proof. If $\left\lfloor \sqrt[k]{2^k n + 2^{k-1}} \right\rfloor \neq \left\lfloor \sqrt[k]{2^k n + 2^{k-1} - 1} \right\rfloor$, then there exists an integer m such that $\sqrt[k]{2^k n + 2^{k-1}} \geq m > \sqrt[k]{2^k n + 2^{k-1} - 1}$ or $2^k n + 2^{k-1} \geq m^k > 2^k n + 2^{k-1} - 1$. This implies $2^k n + 2^{k-1} = m^k$.

As $2^{k-1} | (2^k n + 2^{k-1})$ and $2^k \nmid (2^k n + 2^{k-1})$, we can see that $2^k n + 2^{k-1}$ cannot be a perfect kth power and therefore cannot be equal to m^k. Hence $\left\lfloor \sqrt[k]{2^k n + 2^{k-1} - 1} \right\rfloor = \left\lfloor \sqrt[k]{2^k n + 2^{k-1}} \right\rfloor$. □

Lemma 9. *For natural numbers $n, k > 1$ such that $n \geq 2^{k-3}$, we have*

$$\sqrt[k]{2^k n + 2^{k-1} - 1} < \sqrt[k]{n} + \sqrt[k]{n+1}.$$

Proof. From the AM-GM inequality, we have $\frac{\sqrt[k]{n} + \sqrt[k]{n+1}}{2} > \sqrt[2k]{n^2 + n}$. As $\sqrt[2k]{n^2 + n} \geq \sqrt[k]{n + \frac{2^{k-1} - 1}{2^k}}$ when $n \geq 2^{k-3}$, we have $\frac{\sqrt[k]{n} + \sqrt[k]{n+1}}{2} > \sqrt[k]{n + \frac{2^{k-1} - 1}{2^k}}$. By multiplying both sides by 2, we get the desired inequality. □

Lemma 10. *For natural numbers $n, k > 1$, we have*

$$\sqrt[k]{n} + \sqrt[k]{n+1} < \sqrt[k]{2^k n + 2^{k-1}}.$$

Proof. From the generalized mean inequality, we have

$$M_1(\sqrt[k]{n}, \sqrt[k]{n+1}) < M_k(\sqrt[k]{n}, \sqrt[k]{n+1}).$$

Hence

$$\frac{\sqrt[k]{n} + \sqrt[k]{n+1}}{2} < \sqrt[k]{\frac{n+n+1}{2}} = \sqrt[k]{n+\frac{1}{2}}.$$

By multiplying both sides by 2, we get the desired inequality. □

Let us prove Theorem 2.

Proof of Theorem 2. If $n \geq 2^{k-3}$, then from Lemma 9 and Lemma 10, we have $\sqrt[k]{2^k n + 2^{k-1} - 1} < \sqrt[k]{n} + \sqrt[k]{n+1} < \sqrt[k]{2^k n + 2^{k-1}}$. From Lemma 8, we get

$$\left\lfloor \sqrt[k]{n} + \sqrt[k]{n+1} \right\rfloor = \left\lfloor \sqrt[k]{2^k n + 2^{k-1}} \right\rfloor = \left\lfloor 2\sqrt[k]{n+\frac{1}{2}} \right\rfloor.$$

If $n < 2^{k-3}$, then as $2 < \sqrt[k]{n} + \sqrt[k]{n+1} < 4$ and $2 \leq \left\lfloor 2\sqrt[k]{n+\frac{1}{2}} \right\rfloor < 4$, we have

$$\left\lfloor \sqrt[k]{n} + \sqrt[k]{n+1} \right\rfloor, \ \left\lfloor 2\sqrt[k]{n+\frac{1}{2}} \right\rfloor \in \{2, 3\}.$$

The only possible exceptions to the theorem would occur when

$$\left\lfloor \sqrt[k]{n} + \sqrt[k]{n+1} \right\rfloor = 2 \quad \text{and} \quad \left\lfloor 2\sqrt[k]{n+\frac{1}{2}} \right\rfloor = 3. \tag{5}$$

If $\left\lfloor 2\sqrt[k]{n+\frac{1}{2}} \right\rfloor = 3$ then $n \geq \left(\frac{3}{2}\right)^k - \frac{1}{2}$, and if $\left\lfloor \sqrt[k]{n} + \sqrt[k]{n+1} \right\rfloor = 2$ then $n < \left(\frac{3}{2}\right)^k$. Therefore (5) implies $n \in \left[\left(\frac{3}{2}\right)^k - \frac{1}{2}, \left(\frac{3}{2}\right)^k\right)$ and $n = \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor$.

Suppose $n = \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor$ is an exception. From (5), we have $\sqrt[k]{n} + \sqrt[k]{n+1} < 3$. Applying the AM-GM inequality, we get $2\sqrt[2k]{n(n+1)} < 3$ or $n(n+1) < (\frac{3}{2})^{2k}$ or $(2n+1)^2 < 4(\frac{3}{2})^{2k} + 1$. Hence

$$\left(\frac{3}{2}\right)^k - \frac{1}{2} \leq n < \sqrt{\left(\frac{3}{2}\right)^{2k} + \frac{1}{4}} - \frac{1}{2} < \left(\frac{3}{2}\right)^k + \frac{1}{8\left(\frac{3}{2}\right)^k} - \frac{1}{2}.$$

Multiplying both sides by 2 and adding 1 on both sides, we get

$$2\left(\frac{3}{2}\right)^k \leq 2n+1 < 2\left(\frac{3}{2}\right)^k + \frac{1}{4\left(\frac{3}{2}\right)^k}.$$

Hence

$$\left|2\left(\frac{3}{2}\right)^k - (2n+1)\right| < \frac{1}{4\left(\frac{3}{2}\right)^k}. \tag{6}$$

Now we use (5) of Mahler's [6] paper. For all $u > v \geq 2, \epsilon > 0$, and ϑ any positive algebraic number, the following holds for all but a finite number of values of k

$$\left|\vartheta\left(\frac{u}{v}\right)^k - (\text{nearest integer})\right| > e^{-\epsilon k}. \tag{7}$$

Put $u = 3,\ v = 2,\ \vartheta = 2,\ \varepsilon = \log\frac{3}{2}$ in (7) to see that (6) can only hold for finitely many pairs (n, k). Therefore there can be at most finitely many exceptions (n, k) such that $k > 1$ and

$$\lfloor \sqrt[k]{n} + \sqrt[k]{n+1} \rfloor \neq \left\lfloor 2\sqrt[k]{n+\frac{1}{2}} \right\rfloor.$$

□

The method used in Theorem 2 allows us to show that there can only be finitely many exceptions. It would be an interesting research problem to study the properties of the exceptional set

$$\left\{(n,k) \in \mathbb{N}\times\mathbb{N} : k > 1, n = \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor \text{ and } \lfloor \sqrt[k]{n} + \sqrt[k]{n+1} \rfloor \neq \left\lfloor 2\sqrt[k]{n+\frac{1}{2}} \right\rfloor \right\}.$$

4.2. Proof of (3) Using Theorem 2

Proof. By setting $k = 2$ in Theorem 2, we obtain (3) for all $n \neq \lfloor (\frac{3}{2})^2 \rfloor = 2$. Note that (3) is true for $n = 2$, as $\lfloor \sqrt{2} + \sqrt{3} \rfloor = \lfloor \sqrt{10} \rfloor = 3$. Hence (3) is true for all natural numbers n. □

Let us now prove Theorem 3.

4.3. Proof of Theorem 3

Proof of Theorem 3. We will prove this by proving the following inequalities:

(8) $\lfloor \sqrt[k]{n+x_1} + \sqrt[k]{n+x_2} + \cdots + \sqrt[k]{n+x_l} \rfloor \leq \left\lfloor l\sqrt[k]{n + \frac{x_1+x_2+\cdots+x_l}{l}} \right\rfloor$

(9) $\sqrt[k]{n+x_1} + \sqrt[k]{n+x_2} + \cdots + \sqrt[k]{n+x_l} \geq l\sqrt[lk]{(n+x_1)(n+x_2)\cdots(n+x_l)}$

(10) for $n \geq \frac{l^{k-1}(x_1^2+\cdots+x_l^2)}{2}$, we have

$$l\sqrt[lk]{(n+x_1)(n+x_2)\cdots(n+x_l)} \geq l\sqrt[k]{n + \frac{x_1+x_2+\cdots+x_l}{l} - \frac{1}{l^k}}$$

(11) $\left\lfloor l\sqrt[k]{n+\frac{x_1+x_2+\cdots+x_l}{l}}\right\rfloor = \left\lfloor l\sqrt[k]{n+\frac{x_1+x_2+\cdots+x_l}{l}-\frac{1}{l^k}}\right\rfloor.$

We claim that the above inequalities imply Theorem 3. If $n \geq \frac{l^{k-1}(x_1^2+\cdots+x_l^2)}{2}$, then

$$\begin{aligned}\left\lfloor \sqrt[k]{n+x_1}+\sqrt[k]{n+x_2}+\cdots+\sqrt[k]{n+x_l}\right\rfloor &\overset{(9)}{\geq} \left\lfloor l\sqrt[k]{(n+x_1)(n+x_2)\cdots(n+x_l)}\right\rfloor \\ &\overset{(10)}{\geq} \left\lfloor l\sqrt[k]{n+\frac{x_1+x_2+\cdots+x_l}{l}-\frac{1}{l^k}}\right\rfloor \\ &\overset{(11)}{=} \left\lfloor l\sqrt[k]{n+\frac{x_1+x_2+\cdots+x_l}{l}}\right\rfloor.\end{aligned}$$

Therefore

$$\left\lfloor \sqrt[k]{n+x_1}+\sqrt[k]{n+x_2}+\cdots+\sqrt[k]{n+x_l}\right\rfloor \geq \left\lfloor l\sqrt[k]{n+\frac{x_1+x_2+\cdots+x_l}{l}}\right\rfloor. \qquad (12)$$

Now (8) and (12) imply the desired result

$$\left\lfloor \sqrt[k]{n+x_1}+\sqrt[k]{n+x_2}+\cdots+\sqrt[k]{n+x_l}\right\rfloor = \left\lfloor l\sqrt[k]{n+\frac{x_1+x_2+\cdots+x_l}{l}}\right\rfloor.$$

Proofs of (8) to (11) are given:

(8) Consider $f(x) = \sqrt[k]{n+x}$. As $f''(x) \leq 0$, from the finite form of Jensen's inequality, we have

$$w_1 f(x_1) + \cdots + w_l f(x_l) \leq f(w_1x_1 + \cdots + w_lx_l)$$

for all real w_i satisfying $w_1 \geq 0, \ldots, w_l \geq 0$ and $w_1 + \cdots + w_l = 1$. Put $w_1 = \ldots = w_l = \frac{1}{l}$ in the above inequality to get

$$\sqrt[k]{n+x_1}+\cdots+\sqrt[k]{n+x_l} \leq l\sqrt[k]{n+\frac{x_1+x_2+\cdots+x_l}{l}}.$$

Therefore

$$\left\lfloor \sqrt[k]{n+x_1}+\cdots+\sqrt[k]{n+x_l}\right\rfloor \leq \left\lfloor l\sqrt[k]{n+\frac{x_1+x_2+\cdots+x_l}{l}}\right\rfloor.$$

(9) This follows from the AM-GM inequality.

(10) Now for $n \geq \frac{l^{k-1}(x_1^2+\cdots+x_l^2)}{2}$, we have

$$-\frac{(x_1^2+\cdots+x_l^2)}{2n^2} \geq -\frac{1}{l^{k-1}n}.$$

Adding $\frac{x_1+\cdots+x_l}{n}$ on both sides, we get

$$\frac{x_1+\cdots+x_l}{n}-\frac{(x_1^2+\cdots+x_l^2)}{2n^2}\geq\frac{x_1+\cdots+x_l}{n}-\frac{1}{l^{k-1}n}.$$

Therefore

$$\left(\frac{x_1}{n}-\frac{x_1^2}{2n^2}\right)+\left(\frac{x_2}{n}-\frac{x_2^2}{2n^2}\right)+\cdots+\left(\frac{x_l}{n}-\frac{x_l^2}{2n^2}\right)\geq l\left(\frac{\frac{x_1+\cdots+x_l}{l}-\frac{1}{l^k}}{n}\right).$$

Using elementary inequality $x-\frac{x^2}{2}\leq\log(1+x)\leq x$ for $0<x\leq 1$, we have

$$\begin{aligned}\log\left(1+\frac{x_1}{n}\right)+\cdots+\log\left(1+\frac{x_l}{n}\right)&\geq\left(\frac{x_1}{n}-\frac{x_1^2}{2n^2}\right)+\cdots+\left(\frac{x_l}{n}-\frac{x_l^2}{2n^2}\right)\\&\geq l\left(\frac{\frac{x_1+\cdots+x_l}{l}-\frac{1}{l^k}}{n}\right)\\&\geq l\log\left(1+\frac{\frac{x_1+\cdots+x_l}{l}-\frac{1}{l^k}}{n}\right).\end{aligned}$$

Adding $l\log n$ on both sides of the above inequality and exponentiating we get

$$(n+x_1)(n+x_2)\cdots(n+x_l)\geq\left(n+\frac{x_1+\cdots+x_l}{l}-\frac{1}{l^k}\right)^l.$$

Raising both sides of the above inequality to the $\frac{1}{lk}$th power, and multiplying both sides of the resulting inequality by l, we get the desired result.

(11) We claim that $l\sqrt[k]{n+\frac{x_1+x_2+\cdots+x_l}{l}}$ is never an integer. Note that

$$l\sqrt[k]{n+\frac{x_1+x_2+\cdots+x_l}{l}}=\sqrt[k]{l^{k-1}(ln+x_1+x_2+\cdots+x_l)}.$$

Now let $\nu_p(a)$ denote the highest power of p that divides a, and let $\nu_p(l)=m$. Since $p|l$ and $p^k\nmid l$, we have $1\leq m<k$. As $p\nmid(x_1+x_2+\cdots+x_l)$ and $p|ln$, we have $p\nmid(ln+x_1+x_2+\cdots+x_l)$. Hence

$$\nu_p(l^{k-1}(ln+x_1+x_2+\cdots+x_l))=(k-1)m.$$

Now, $k\nmid(k-1)m$ as $k|(k-1)m$ implies $k|m$, which is impossible as $1\leq m<k$. Hence

$$k\nmid\nu_p(l^{k-1}(ln+x_1+x_2+\cdots+x_l)).$$

Therefore $l^{k-1}(ln+x_1+x_2+\cdots+x_l)$ cannot be a perfect kth power, and $l\sqrt[k]{n+\frac{x_1+x_2+\cdots+x_l}{l}}$ cannot be an integer.

For the sake of contradiction, assume (11) is false. If (11) is false, then there exists an integer a such that

$$l\sqrt[k]{n+\frac{x_1+x_2+\cdots+x_l}{l}-\frac{1}{l^k}} < a \leq l\sqrt[k]{n+\frac{x_1+x_2+\cdots+x_l}{l}},$$

but as the right-hand side cannot be an integer, we have

$$l\sqrt[k]{n+\frac{x_1+x_2+\cdots+x_l}{l}-\frac{1}{l^k}} < a < l\sqrt[k]{n+\frac{x_1+x_2+\cdots+x_l}{l}}.$$

Now raise all expressions to the kth power to get

$$l^k n + l^{k-1}(x_1+x_2+\cdots+x_l) - 1 < a^k < l^k + l^{k-1}(x_1+x_2+\cdots+x_l). \quad (13)$$

In (13), the left-hand side and the right-hand side are consecutive integers, and a^k is an integer. As there cannot be an integer between two consecutive integers, (13) is impossible. Hence (11) is true.

□

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