TWO COMPLEMENTARY RELATIONS FOR THE ROGERS-RAMANUJAN CONTINUED FRACTION

Sumit Kumar Jha
G-2, C-185, Ramprastha, Ghaziabad, U. P., India
kumarjha.sumit@research.iiit.ac.in

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Abstract

Let $R(q)$ be the Rogers-Ramanujan continued fraction. We give different proofs of two complementary relations for $R(q)$ given by Ramanujan and proved by Watson and Ramanathan. Our proofs only use product expansions for classical Jacobi theta functions.

1. Introduction

The Rogers-Ramanujan continued fraction, denoted by $R(q)$, is defined by

$$R(q) := \cfrac{q^{1/5}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\ddots}}}} \qquad |q|<1.$$

We assume $|q|<1$ hereon. Ramanujan [5] proved that

$$R(q) = q^{1/5}\,\frac{\sum_{\lambda=-\infty}^{\infty}(-1)^{\lambda}\,q^{\frac{\lambda}{2}(5\,\lambda+3)}}{\sum_{\lambda=-\infty}^{\infty}(-1)^{\lambda}\,q^{\frac{\lambda}{2}(5\,\lambda+1)}} = q^{1/5}\prod_{n=1}^{\infty}\frac{(1-q^{5n-1})(1-q^{5n-4})}{(1-q^{5n-2})(1-q^{5n-3})} \qquad (1)$$

where the second equality is due to the Jacobi's triple product identity [1].

Ramanujan in his notebooks states the following results:

Theorem 1 ([2, p. 58]). *If a and b are positive and $ab=1$, then*

$$\left\{\frac{\sqrt{5}+1}{2}+R\left(e^{-2\pi a}\right)\right\}\left\{\frac{\sqrt{5}+1}{2}+R\left(e^{-2\pi b}\right)\right\} = \frac{5+\sqrt{5}}{2}. \qquad (2)$$

Theorem 2 ([2, p. 91])**.** *If a and b are positive and $ab = 1/5$, then*

$$\left\{\left(\frac{\sqrt{5}+1}{2}\right)^5 + R^5\left(e^{-2\pi a}\right)\right\}\left\{\left(\frac{\sqrt{5}+1}{2}\right)^5 + R^5\left(e^{-2\pi b}\right)\right\} = 5\sqrt{5}\left(\frac{\sqrt{5}+1}{2}\right)^5. \tag{3}$$

The above results were proved by Watson [7] and Ramanathan [4]. We show that these results directly follow from four identities due to Ramanujan in his lost notebook and the product expansion of one of Jacobi's theta functions.

2. Jacobi's Theta Functions

The second Jacobi theta function, denoted by $\vartheta_2(z;q)$, and the fourth Jacobi theta function, denoted by $\vartheta_4(z;q)$ [3, p. 166], are defined by

$$\vartheta_2(z;q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}\exp((2n+1)iz), \tag{4}$$

$$\vartheta_4(z;q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}\exp(2niz). \tag{5}$$

Let $q = e^{\pi i \tau}$. The Jacobi's identity [3, p. 177]

$$\vartheta_2\left(\frac{-z}{\tau};\frac{-1}{\tau}\right) = \sqrt{-i\tau}\; e^{\frac{iz^2}{\pi\tau}}\;\vartheta_4(z,\tau)$$

gives us the expression

$$\vartheta_2(z;\tau) = \frac{1}{\sqrt{-\tau i}}\sum_{n=-\infty}^{\infty}(-1)^n\exp\left\{\frac{\pi}{i\tau}\left(n-\frac{z}{\pi}\right)^2\right\}. \tag{6}$$

In (1), we replace q by q^2 and complete the square in exponents of q to get

$$R(q^2) = \frac{\sum_{\lambda=-\infty}^{\infty}(-1)^\lambda q^{5(\lambda+3/10)^2}}{\sum_{\lambda=-\infty}^{\infty}(-1)^\lambda q^{5(\lambda+1/10)^2}} = \frac{\vartheta_2\left(\frac{3\pi}{10};\frac{-1}{5\tau}\right)}{\vartheta_2\left(\frac{\pi}{10};\frac{-1}{5\tau}\right)}. \tag{7}$$

The above equation was obtained in [6]. By replacing τ by $-1/\tau$ in the above equation gives us

$$R\left(e^{-\frac{2\pi i}{\tau}}\right) = \frac{\vartheta_2\left(\frac{3\pi}{10};q^{\frac{1}{5}}\right)}{\vartheta_2\left(\frac{\pi}{10};q^{\frac{1}{5}}\right)}. \tag{8}$$

We also recall the product expansion for θ_2 [3, p. 171]

$$\vartheta_2(z;q) = 2q^{1/4}\cos z\prod_{n=1}^{\infty}\left(1-q^{2n}\right)\left(1+2q^{2n}\cos(2z)+q^{4n}\right). \tag{9}$$

3. Product Expansions Due to Ramanujan

In the following, let

$$\alpha = \frac{1-\sqrt{5}}{2} \qquad \text{and} \qquad \beta = \frac{1+\sqrt{5}}{2}.$$

Ramanujan gave the following product expansions in his lost notebook:

Theorem 3 ([2, p. 21]). *Let $t = R(q)$. Then*

$$\frac{1}{\sqrt{t}} - \alpha\sqrt{t} = \frac{1}{q^{1/10}}\sqrt{\frac{f(-q)}{f(-q^5)}}\prod_{n=1}^{\infty}\frac{1}{1+\alpha\, q^{n/5}+q^{2n/5}}, \tag{10}$$

$$\frac{1}{\sqrt{t}} - \beta\sqrt{t} = \frac{1}{q^{1/10}}\sqrt{\frac{f(-q)}{f(-q^5)}}\prod_{n=1}^{\infty}\frac{1}{1+\beta\, q^{n/5}+q^{2n/5}}, \tag{11}$$

$$\left(\frac{1}{\sqrt{t}}\right)^5 - \left(\alpha\sqrt{t}\right)^5 = \frac{1}{q^{1/2}}\sqrt{\frac{f(-q)}{f(-q^5)}}\prod_{n=1}^{\infty}\frac{1}{(1+\alpha\, q^{n}+q^{2n})^5}, \tag{12}$$

$$\left(\frac{1}{\sqrt{t}}\right)^5 - \left(\beta\sqrt{t}\right)^5 = \frac{1}{q^{1/2}}\sqrt{\frac{f(-q)}{f(-q^5)}}\prod_{n=1}^{\infty}\frac{1}{(1+\beta\, q^{n}+q^{2n})^5}. \tag{13}$$

Here $f(-q) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}$ [2, p. 11].

4. Proof of Theorem 1

Proof. Replacing q by q^2 in (10) and (11) and then dividing one by another gives

$$\frac{1-\beta R(q^2)}{1-\alpha R(q^2)} = \prod_{n=1}^{\infty}\frac{1+\alpha\, q^{2n/5}+q^{4n/5}}{1+\beta\, q^{2n/5}+q^{4n/5}}. \tag{14}$$

Now using (9) we have

$$\frac{\vartheta_2\left(\frac{3\pi}{10};q^{\frac{1}{5}}\right)}{\vartheta_2\left(\frac{\pi}{10};q^{\frac{1}{5}}\right)} = \frac{\cos(3\pi/10)}{\cos(\pi/10)}\prod_{n=1}^{\infty}\frac{1+\alpha\, q^{2n/5}+q^{4n/5}}{1+\beta\, q^{2n/5}+q^{4n/5}}.$$

Thus (8) becomes

$$R\left(e^{-\frac{2\pi i}{\tau}}\right) = (-\alpha)\,\frac{1-\beta R(e^{2\pi i\tau})}{1-\alpha R(e^{2\pi i\tau})}.$$

Letting $b = -i\tau$ and $a = 1/b = i/\tau$ in the above gives us (2). $\square$

5. Proof of Theorem 2

Proof. Replacing q by q^2 in (12) and (13) and then dividing one by another gives

$$\frac{1-\beta^5\,R^5(q^2)}{1-\alpha^5 R^5(q^2)} = \prod_{n=1}^{\infty}\left(\frac{1+\alpha\,q^{2n}+q^{4n}}{1+\beta\,q^{2n}+q^{4n}}\right)^5. \tag{15}$$

Now using (9) we have

$$\left(\frac{\vartheta_2\left(\frac{3\,\pi}{10};q\right)}{\vartheta_2\left(\frac{\pi}{10};q\right)}\right)^5 = \frac{\cos^5(3\,\pi/10)}{\cos^5(\pi/10)}\prod_{n=1}^{\infty}\left(\frac{1+\alpha\,q^{2n}+q^{4n}}{1+\beta\,q^{2n}+q^{4n}}\right)^5.$$

Thus

$$\left(\frac{\vartheta_2\left(\frac{3\,\pi}{10};q\right)}{\vartheta_2\left(\frac{\pi}{10};q\right)}\right)^5 = (-\alpha^5)\,\frac{1-\beta^5\,R^5(q^2)}{1-\alpha^5 R^5(q^2)}.$$

Then replacing q by $q^{1/5}$ in above and using (8) gives us

$$R^5\left(e^{-\frac{2\,\pi\,i}{\tau}}\right) = (-\alpha)^5\,\frac{1-\beta^5\,R^5(q^{2/5})}{1-\alpha^5 R^5(q^{2/5})} = (-\alpha)^5\,\frac{1-\beta^5\,R^5(e^{(2\,\pi\,i\,\tau)/5})}{1-\alpha^5 R^5(e^{(2\,\pi\,i\,\tau)/5})}.$$

Letting $b=\frac{-i\tau}{5}$ and $a=\frac{1}{5b}=i/\tau$ in the above gives us (3). □

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